Parametric and Implicit Equations

EXAMPLE 1: Consider a figure skater carving a figure-eight on the ice. Figure 1 below shows the skater's path; we've embedded a coordinate plane on the ice-rink.

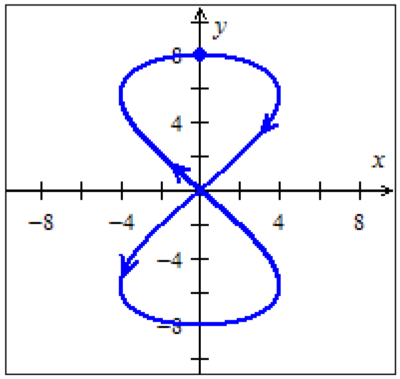


Figure 1: The skater's figure-eight.

Clearly, the y-values are not a function of the x-values (since this graph fails the "vertical line test"). But it should also be clear that at each moment in time, the skater is in *exactly* one location. In other words, the skater's location ***is*** a function of time. We can use a system of **parametric equations** to define the skater's location as a function of time. A system of parametric equations consists of a pair of functions: one that describes the x-coordinate and the other that describes the y-coordinate; typically, the input variable for these functions represents *time*. It turns out that the system of parametric equations given below describes the skater's motion that we've graphed in Figure 1 above:

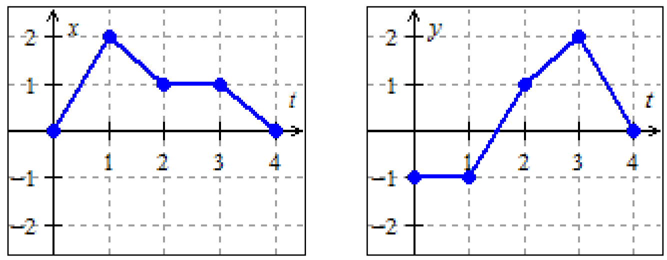
$$\begin{cases} x(t) = 4\sin\left(\frac{\pi}{2}t\right) \\ y(t) = 8\cos\left(\frac{\pi}{4}t\right) \end{cases}$$

Let's discuss how we can obtain a graph of this system of parametric equations for $0 \le t \le 8$. (Let's assume that t is measured in seconds, so we're graphing the figure skater's movement during her first 8 seconds of skating.)

Note that if you want to graph this function on your calculator, you need to change the *graph mode* of your calculator to "**parametric**".

A system of parametric equations involves **three** variables so it conveys three pieces of information (rather than just the two pieces of information conveyed by a single equation in two-variables). Typically, the parameter t represents time so that, in addition to describing location of an object, a system of parametric equations also describes when the object is at a location, so a system of parametric equations can describe both "space" and "time".

EXAMPLE 2: Suppose that a robot is moving around a coordinate plane and that $x = f(t)$ represents the x-coordinate and $y = g(t)$ represents the y-coordinate of the robot's location as functions of time, t, in minutes The graphs of $x = f(t)$ and $y = g(t)$ during the first four minutes of the robot's travels are given in Figure 2; sketch a graph of the robot's movement on the grid in Figure 3.



The graph of $x = f(t)$. The graph of $y = g(t)$.

Figure 2

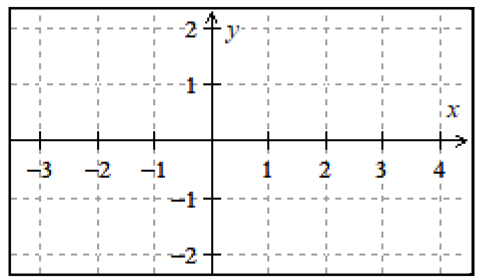


Figure 3: Sketch of $(x, y) = \big(f(t), g(t)\big)$.

EXAMPLE 3a: Suppose that the x- and y-coordinates of the movement of a robot are given by the following functions of time, t, in seconds. (We'll assume that these equations apply for all $t \geq 0$.) Sketch a graph the robot's movement on the coordinate plane given in Figure 4.

$$\begin{cases} x = 6 - 2t \\ y = -2 + 4t. \end{cases}$$

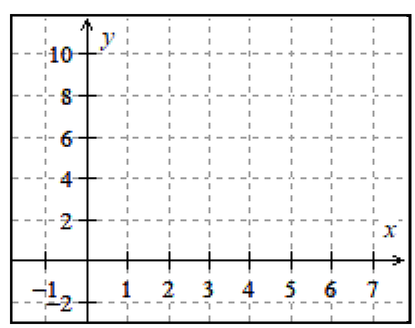


Figure 4: The robot's movement for $t \geq 0$.

Sometimes we can **eliminate the parameter** from a parametric system and obtain a single equation that describes the same curve.

EXAMPLE 3b: Eliminate the parameter from the parametric system in Example 3a:

$$\begin{cases} x = 6 - 2t \\ y = -2 + 4t. \end{cases}$$

To eliminate the parameter t, we can solve one of the equations in our system for t and then substitute the result into our other equation.

When we describe curves on the coordinate plane with algebraic equations, we can define the relationship between the x- and y-coordinates of the curve either *explicitly* or *implicitly*. To explain the difference, let's consider the equation $y = 3x + 7$.

- The equation $y = 3x + 7$ establishes a relationship between x and y. We say that y is defined ***explicitly*** in terms of x since the equation gives us a very clear description of the relationship between x and y: no matter what the x-value is, the corresponding y-value is always $3x + 7$.

- If we subtract $3x$ from each side of this equation we obtain the equation $-3x + y = 7$. This equation describes the same relationship between x and y as the equation $y = 3x + 7$, but when y isn't isolated, the relationship between the variables isn't explicit. Instead, this relationship is *implied* by the equation, so we say that it is an ***implicit equation***.

An equation involving two variables is considered *explicit* if one of the variables is isolated on one side of the equation, while an equation is considered *implicit* if neither of the variables is isolated on one side of the equation. As we've seen, equations like $y = 3x + 7$ can be written either explicitly or implicitly, but many equations can only be written implicitly.

EXAMPLE 4: The x- and y-coordinates of the movement of a particle are given by the following functions of time, t, where $0 \le t \le 2\pi$.

$$\begin{cases} x = \cos(t) \\ y = \sin(t) \end{cases}$$

Let's first graph this system on a graphing calculator, and then eliminate the parameter t from this system in order to find a single equation involving x and y that represents the path of the particle.

Since the parametric system

$$\begin{cases} x = \cos(t) \\ y = \sin(t) \end{cases}$$

describes a circle of radius 1 unit, we should expect that if we multiply *both* the x- and y-coordinate by the factor r we will stretch (or compress) the circle so that its radius will change from 1 unit to r units, and if we add constants h and k to the x- and y-coordinates, respectively, we will shift the center of the circle from the point $(0, 0)$ to the point (h, k). A generalized parameterization of a circle is given in the box below.

If $h, k, r \in \mathbb{R}$ and $r > 0$ then the system of parametric equations below defines a **circle** of radius r centered at the point (h, k).

$$\begin{cases} x = h + r\cos(t) \\ y = k + r\sin(t) \end{cases}$$