Intro to the Trigonometric Functions: Part 1

As we noticed in the Ferris wheel example in the Intro to Periodic Functions online lecture notes, a circle rotating about its center lends itself naturally to the study of periodic functions. In fact, the two most important *trigonometric functions* are defined in terms of a unit circle: the **sine** and **cosine** functions. (A **unit circle** is a circle with a radius of 1 unit; see Figure 1.)

DEFINITION: The **cosine** and **sine** functions (denoted by $\cos(\theta)$ and $\sin(\theta)$) represent the horizontal and vertical coordinates of the point P specified by the angle θ on the circumference of a unit circle.

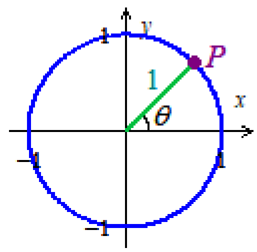
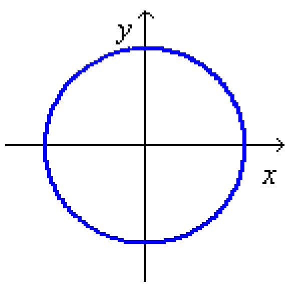
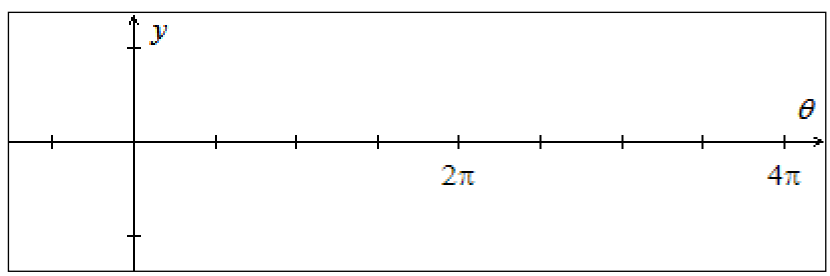


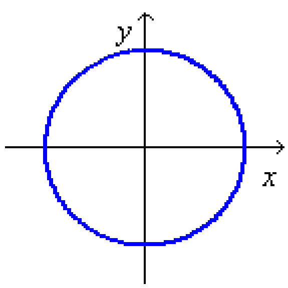
Figure 1: A Unit Circle

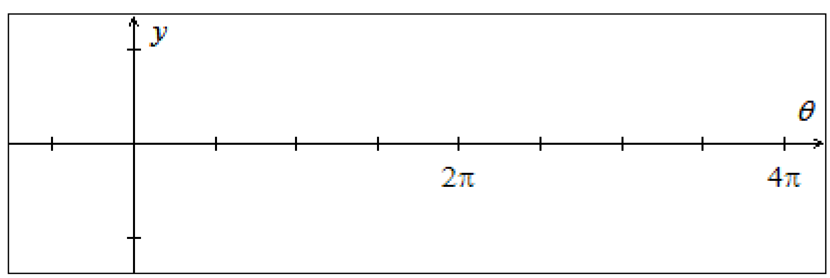
Now, let's sketch graphs of the sine and cosine functions, and then graph them in Demos.





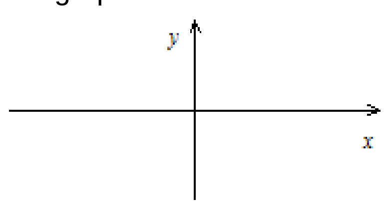
The graph of $y = \sin(\theta)$.





The graph of $y = \cos(\theta)$.

Let's determine the signs (positive or negative) of the sine and cosine functions in the four different quadrants of the coordinate plane. Notice that our analysis of the signs of the sine and cosine functions agree with the graphs of the sine and cosine functions.



DEFINITION: An **identity** is an equation that is true for all values in the domains of the involved expressions.

SOME IMPORTANT TRIG IDENTITIES

- $\cos(\theta) = \cos(\theta + 2\pi)$
- $\sin(\theta) = \sin(\theta + 2\pi)$
- $\sin(\theta) = \cos\left(\theta - \frac{\pi}{2}\right)$
- $\cos(\theta) = \sin\left(\theta + \frac{\pi}{2}\right)$
- $\cos(-\theta) = \cos(\theta)$
- $\sin(-\theta) = -\sin(\theta)$
- $\cos(\theta) = \cos(2\pi - \theta)$
- $\sin(\theta) = \sin(\pi - \theta)$

See the video linked from online lecture notes that includes explanations of these identities: https://youtu.be/j_0MflcAVYY.

Recall the Pythagorean Theorem from your previous math course-work:

THE PYTHAGOREAN THEOREM:

If the sides of a right triangle (i.e., a triangle with a 90° angle) are labeled like the one given in Figure 2, then $\boldsymbol{a}^2 + \boldsymbol{b}^2 = \boldsymbol{c}^2$.

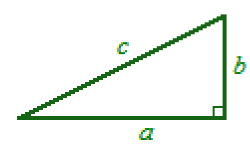


Figure 2

We can use the Pythagorean Theorem along with the definitions of sine and cosine to derive another important identity. In Figure 3, notice how the definitions of sine and cosine naturally lead us to a right triangle with side-lengths $\sin(\theta)$, $\cos(\theta)$, and 1.

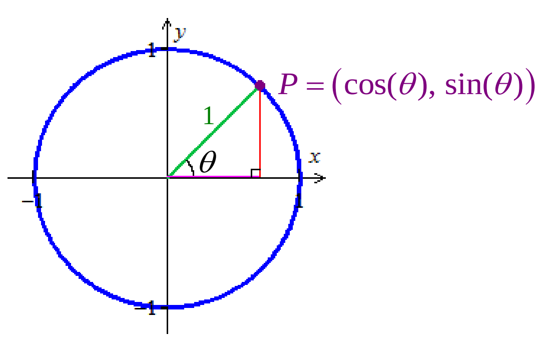


Figure 3

The Pythagorean Identity:

We can generalize the definitions of the sine and cosine functions so that they are applicable to circles of any size, rather than only for unit circles.

DEFINITION: If the point $T = (x, y)$ is specified by the angle θ on the circumference of a circle of radius r as shown in Figure 5 below, then

Notice that if $r = 1$ then this definition $\cos(\theta)$ and $\sin(\theta)$ is equivalent we saw at the beginning of this chapter:

If we solve the equations $\cos(\theta) = \frac{x}{r}$ and $\sin(\theta) = \frac{y}{r}$ for x and y, respectively, we can obtain the coordinates of a point on the circumference of a circle of any r:

If the point $T = (x, y)$ is specified by the angle θ on the circumference of a circle of radius, r, then

Let's add this information to Figure 5.

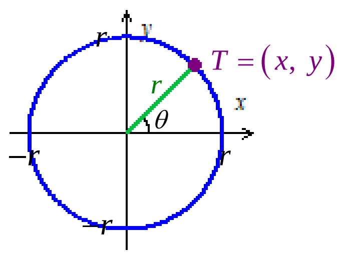


Figure 5

As mentioned at the beginning of these notes, there are four other trigonometric functions. These four functions are defined in terms of the sine and cosine functions:

DEFINITIONS: The **tangent function**, denoted $\tan(\theta)$, is defined by__________________.

The **cotangent function**, denoted $\cot(\theta)$, is defined by________________.

Consequently:

The **secant function**, denoted $\sec(\theta)$, is defined by___________________.

The **cosecant function**, denoted $\csc(\theta)$, is defined by_________________.

See Sect. I, Ch. 3, pt. 2 in online lecture notes for info about the Other Pythagorean Identities.

EXAMPLE: If $\cos(A) = -\frac{5}{7}$ and $\pi < A < \frac{3\pi}{2}$, find $\sin(A)$, $\tan(A)$, $\cot(A)$, $\sec(A)$, and $\csc(A)$.