SOLUTIONS: Week 7 Practice Worksheet

Vectors

1. Suppose that $\vec{v} = \langle -3, 7 \rangle$ and $\vec{w} = \langle 2, 10 \rangle$.

a. Express $\vec{v}$ and $\vec{w}$ using unit vectors.

$$\vec{v} = \langle -3, 7 \rangle = -3\vec{i} + 7\vec{j}$$

and

$$\vec{w} = \langle 2, 10 \rangle = 2\vec{i} + 10\vec{j}$$

b. Find $\|\vec{v}\|$ and $\|\vec{w}\|$.

$$\|\vec{v}\| = \sqrt{(-3)^2 + (7)^2} = \sqrt{9 + 49} = \sqrt{58}$$

and

$$\|\vec{w}\| = \sqrt{(2)^2 + (10)^2} = \sqrt{104} = 2\sqrt{26}$$

c. Find $2\vec{v} - 5\vec{w}$.

$$\begin{aligned} 2\vec{v} - 5\vec{w} &= 2\cdot\langle -3, 7 \rangle - 5\cdot\langle 2, 10 \rangle \\ &= \langle -6, 14 \rangle - \langle 10, 50 \rangle \\ &= \langle -6 - 10, 14 - 50 \rangle \\ &= \langle -16, -36 \rangle \end{aligned}$$

d. Find $4\vec{w} + 3\vec{v}$.

$$\begin{aligned} 4\vec{w} + 3\vec{v} &= 4\cdot\langle 2, 10 \rangle + 3\cdot\langle -3, 7 \rangle \\ &= \langle 8, 40 \rangle + \langle -9, 21 \rangle \\ &= \langle 8 + (-9), 40 + 21 \rangle \\ &= \langle -1, 61 \rangle \end{aligned}$$

3. Suppose that $\vec{m} = 7\vec{i} - 4\vec{j}$ and $\vec{n} = -5\vec{i} - 2\vec{j}$.

a. Express $\vec{m}$ and $\vec{n}$ using "pointy vector brackets" (i.e., $\langle a, b \rangle$).

$$\begin{aligned} \vec{m} &= 7\vec{i} - 4\vec{j} \\ &= \langle 7, -4 \rangle \end{aligned} \quad \text{and} \quad \begin{aligned} \vec{n} &= -5\vec{i} - 2\vec{j} \\ &= \langle -5, -2 \rangle \end{aligned}$$

b. Find $\|\vec{m}\|$ and $\|\vec{n}\|$.

$$\begin{aligned} \|\vec{m}\| &= \sqrt{(7)^2 + (-4)^2} \\ &= \sqrt{49 + 16} \\ &= \sqrt{65} \end{aligned} \quad \text{and} \quad \begin{aligned} \|\vec{n}\| &= \sqrt{(-5)^2 + (-2)^2} \\ &= \sqrt{25 + 4} \\ &= \sqrt{29} \end{aligned}$$

c. Find $\vec{m} + \vec{n}$.

$$\begin{aligned} \vec{m} + \vec{n} &= \left(7\vec{i} - 4\vec{j}\right) + \left(-5\vec{i} - 2\vec{j}\right) \\ &= 7\vec{i} + (-5\vec{i}) + (-4\vec{j}) + (-2\vec{j}) \\ &= 2\vec{i} - 6\vec{j} \end{aligned}$$

(Note that we could have used "pointy vector brackets" here but we've chosen to use unit vectors since that's how the vectors are defined in the question. Unless directed to use a specific notation, it's best to mimic the notation used in the question.)

d. Find $3\vec{m} - \vec{n}$.

$$\begin{aligned} 3\vec{m} - \vec{n} &= 3\left(7\vec{i} - 4\vec{j}\right) - \left(-5\vec{i} - 2\vec{j}\right) \\ &= 21\vec{i} - 12\vec{j} + 5\vec{i} + 2\vec{j} \\ &= 26\vec{i} - 10\vec{j} \end{aligned}$$

3. Suppose that $\vec{p} = \langle -1, 4 \rangle$ and $\vec{q} = \langle 3, -5 \rangle$.

a. Express $\vec{p}$ and $\vec{q}$ using unit vectors.

$$\begin{aligned}\vec{p} &= \langle -1, 4 \rangle \\ &= -\vec{i} + 4\vec{j}\end{aligned} \qquad \text{and} \qquad \begin{aligned}\vec{q} &= \langle 3, -5 \rangle \\ &= 3\vec{i} - 5\vec{j}\end{aligned}$$

b. Find $\|\vec{p}\|$ and $\|\vec{q}\|$.

$$\begin{aligned}\|\vec{p}\| &= \sqrt{(1)^2 + (4)^2} \\ &= \sqrt{1 + 16} \\ &= \sqrt{17}\end{aligned} \qquad \text{and} \qquad \begin{aligned}\|\vec{q}\| &= \sqrt{(3)^2 + (-5)^2} \\ &= \sqrt{9 + 25} \\ &= \sqrt{34}\end{aligned}$$

c. Find $2\vec{p} + 3\vec{q}$.

$$\begin{aligned}2\vec{p} + 3\vec{q} &= 2 \cdot \langle -1, 4 \rangle + 3 \cdot \langle 3, -5 \rangle \\ &= \langle -2, 8 \rangle + \langle 9, -15 \rangle \\ &= \langle -2 + 9, 8 - 15 \rangle \\ &= \langle 7, -7 \rangle\end{aligned}$$

d. Find $2\vec{q} - 3\vec{p}$.

$$\begin{aligned}2\vec{q} - 3\vec{p} &= 2 \cdot \langle 3, -5 \rangle - 3 \cdot \langle -1, 4 \rangle \\ &= \langle 6, -10 \rangle + \langle 3, -12 \rangle \\ &= \langle 6 + 3, -10 - 12 \rangle \\ &= \langle 9, -22 \rangle\end{aligned}$$

4. Suppose that the tail (or initial point) of $\vec{b}$ is $(2, -3)$ and the tip (or terminal point) is $(-4, 7)$. Find the components of $\vec{b}$ in order to express $\vec{b}$ using both "pointy vector brackets" and unit vectors.

$$\begin{aligned}\vec{b} &= \langle -4 - 2,\ 7 - (-3) \rangle \\ &= \langle -6,\ 10 \rangle \\ &= -6\vec{i} + 10\vec{j}\end{aligned}$$

5. Suppose that the tail (or initial point) of $\vec{r}$ is $(-5, 1)$ and the tip (or terminal point) is $(3, 6)$. Find the components of $\vec{r}$ in order to express $\vec{r}$ using both "pointy vector brackets" and unit vectors.

$$\begin{aligned}\vec{r} &= \langle 3 - (-5),\ 6 - 1 \rangle \\ &= \langle 8,\ 5 \rangle \\ &= 8\vec{i} + 5\vec{j}\end{aligned}$$

6. Suppose that $\|\vec{a}\| = 34$ and that $\vec{a}$ makes an angle of 150° with the positive x-axis. Find the components of $\vec{a}$ in order to express $\vec{a}$ using both "pointy vector brackets" and unit vectors.

$$\begin{aligned}\vec{a} &= \left\langle \|\vec{a}\|\cos(150^\circ),\ \|\vec{a}\|\sin(150^\circ) \right\rangle \\ &= \left\langle 34\cdot\left(-\tfrac{\sqrt{3}}{2}\right),\ 34\cdot\left(\tfrac{1}{2}\right) \right\rangle \\ &= \left\langle -17\sqrt{3},\ 17 \right\rangle \\ &= -17\sqrt{3}\ \vec{i} + 17\vec{j}\end{aligned}$$

7. Suppose that $\|\vec{s}\| = 18$ and that $\vec{s}$ makes an angle of -45° with the positive x-axis. Find the components of $\vec{s}$ in order to express $\vec{s}$ using both "pointy vector brackets" and unit vectors.

$$\begin{aligned}\vec{s} &= \left\langle \|\vec{s}\|\cos(-45^\circ),\ \|\vec{s}\|\sin(-45^\circ) \right\rangle \\ &= \left\langle 18\cdot\left(\tfrac{\sqrt{2}}{2}\right),\ 18\cdot\left(-\tfrac{\sqrt{2}}{2}\right) \right\rangle \\ &= \left\langle 9\sqrt{2},\ -9\sqrt{2} \right\rangle \\ &= 9\sqrt{2}\,\vec{i} + -9\sqrt{2}\,\vec{j}\end{aligned}$$

8. Suppose that $\vec{v} = \langle -3, 7 \rangle$ and $\vec{w} = \langle 2, 10 \rangle$.

a. Find $\vec{v} \cdot \vec{w}$.

$$\begin{aligned} \vec{v} \cdot \vec{w} &= \langle -3, 7 \rangle \cdot \langle 2, 10 \rangle \\ &= (-3)(2) + (7)(10) \\ &= -6 + 70 \\ &= 64 \end{aligned}$$

b. Find the angle between $\vec{v}$ and $\vec{w}$.

We can use the identity $\vec{v} \cdot \vec{w} = \|\vec{v}\| \cdot \|\vec{w}\| \cos(\theta)$, where θ is the angle between vectors $\vec{v}$ and $\vec{w}$. Above in **8.a.** we discovered that $\vec{v} \cdot \vec{w} = 64$, and in **1.b.** we determined that $\|\vec{v}\| = \sqrt{58}$ and $\|\vec{w}\| = 2\sqrt{26}$. Now we can substitute all of these values into the identity and find the angle between the vectors:

$$\begin{aligned} & \vec{v} \cdot \vec{w} = \|\vec{v}\| \|\vec{w}\| \cos(\theta) \\ \Rightarrow \quad & 64 = \sqrt{58} \cdot 2\sqrt{26} \cos(\theta) \\ \Rightarrow \quad & \cos(\theta) = \frac{64}{2\sqrt{58} \cdot \sqrt{26}} \\ \Rightarrow \quad & \theta = \cos^{-1}\left(\frac{32}{\sqrt{58} \cdot \sqrt{26}} \right) \approx 34.51^\circ \end{aligned}$$

So the angle between vectors $\vec{v}$ and $\vec{w}$ is about 34.51°.

9. Suppose that $\vec{m} = 7\vec{i} - 4\vec{j}$ and $\vec{n} = -5\vec{i} - 2\vec{j}$.

a. Find $\vec{m} \cdot \vec{n}$.

$$\begin{aligned} \vec{m} \cdot \vec{n} &= \left(7\vec{i} - 4\vec{j}\right) \cdot \left(-5\vec{i} - 2\vec{j}\right) \\ &= (7)(-5) + (-4)(-2) \\ &= -27 \end{aligned}$$

b. Find the angle between $\vec{m}$ and $\vec{n}$.

We can use the identity $\vec{m} \cdot \vec{n} = \|\vec{m}\|\|\vec{n}\|\cos(\theta)$, where θ is the angle between vectors $\vec{m}$ and $\vec{n}$. Above in **9.a.** we discovered that $\vec{m} \cdot \vec{n} = -27$, and in **2.b.** we determined that $\|\vec{m}\| = \sqrt{65}$ and $\|\vec{n}\| = \sqrt{29}$. Now we can substitute all of these values into the identity and find the angle between the vectors:

$$\begin{aligned} & \vec{m} \cdot \vec{n} = \|\vec{m}\|\|\vec{n}\|\cos(\theta) \\ \Rightarrow \quad & -27 = \sqrt{65} \cdot \sqrt{29}\,\cos(\theta) \\ \Rightarrow \quad & \cos(\theta) = \frac{-27}{\sqrt{65} \cdot \sqrt{29}} \\ \Rightarrow \quad & \theta = \cos^{-1}\left(\frac{-27}{\sqrt{65} \cdot \sqrt{29}}\right) \approx 128.45^\circ \end{aligned}$$

So the angle between vectors $\vec{m}$ and $\vec{n}$ is about 128.45°.

10. Suppose that $\vec{p} = \langle -1, 4 \rangle$ and $\vec{q} = \langle 3, -5 \rangle$.

a. Find $\vec{p} \cdot \vec{q}$.

$$\begin{aligned} \vec{p} \cdot \vec{q} &= \langle -1, 4 \rangle \cdot \langle 3, -5 \rangle \\ &= (-1)(3) + (4)(-5) \\ &= -3 - 20 \\ &= -23 \end{aligned}$$

b. Find the angle between $\vec{p}$ and $\vec{q}$.

We can use the identity $\vec{p} \cdot \vec{q} = \|\vec{p}\| \cdot \|\vec{q}\| \cos(\theta)$, where θ is the angle between vectors $\vec{p}$ and $\vec{q}$. Above in **10.a.** we discovered that $\vec{p} \cdot \vec{q} = -23$, and in **3.b.** we determined that $\|\vec{p}\| = \sqrt{17}$ and $\|\vec{q}\| = \sqrt{34}$. Now we can substitute all of these values into the identity and find the angle between the vectors:

$$\begin{aligned} & \vec{p} \cdot \vec{q} = \|\vec{p}\| \cdot \|\vec{q}\| \cos(\theta) \\ \Rightarrow \quad & -23 = \sqrt{17} \cdot \sqrt{34}\, \cos(\theta) \\ \Rightarrow \quad & \cos(\theta) = \frac{-23}{\sqrt{17} \cdot \sqrt{34}} \\ \Rightarrow \quad & \theta = \cos^{-1}\left(\frac{-23}{\sqrt{17} \cdot \sqrt{34}} \right) \approx 163.07^\circ \end{aligned}$$

So the angle between vectors $\vec{p}$ and $\vec{q}$ is about 163.07°.