SOLUTIONS: Week 6 Practice Worksheet

Trig Identities and Polar Coordinates

1. Use a **sum-of-angles** or **difference-of-angles identity** to calculate the *exact value* of each of the following. (These identities are included on the Identities and Formulas Reference Sheet that will be provided to you during the Final Exam so you don't need to memorize them.)

a. $\sin(165^\circ)$

$$\begin{aligned}\sin(165^\circ) &= \sin(120^\circ + 45^\circ)\\ &= \sin(120^\circ)\cos(45^\circ) + \sin(45^\circ)\cos(120^\circ)\\ &= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\cdot\left(-\frac{1}{2}\right)\\ &= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

b. $\cos\left(\frac{13\pi}{12}\right)$

$$\begin{aligned}\cos\left(\frac{13\pi}{12}\right) &= \cos\left(\frac{3\pi}{12} + \frac{10\pi}{12}\right)\\ &= \cos\left(\frac{\pi}{4} + \frac{5\pi}{6}\right)\\ &= \cos\left(\frac{\pi}{4}\right)\cos\left(\frac{5\pi}{6}\right) - \sin\left(\frac{\pi}{4}\right)\sin\left(\frac{5\pi}{6}\right)\\ &= \frac{\sqrt{2}}{2}\cdot\left(-\frac{\sqrt{3}}{2}\right) - \frac{\sqrt{2}}{2}\cdot\frac{1}{2}\\ &= -\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}\\ &= \frac{-\sqrt{6}-\sqrt{2}}{4} = -\frac{\sqrt{6}+\sqrt{2}}{4}\end{aligned}$$

c. $\tan\left(\frac{17\pi}{12}\right)$

We could use the sum/difference formula for tangent but, assuming that we were trying to memorize the identities rather than copy them off of our Identities and Formulas Reference Sheet, it's not worth trying to memorize the relatively obscure identities for tangent since we can use what we know about sine and cosine along with the fact that $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$:

$$\begin{aligned}
\tan\left(\frac{17\pi}{12}\right) &= \frac{\sin\left(\frac{17\pi}{12}\right)}{\cos\left(\frac{17\pi}{12}\right)} \\
&= \frac{\sin\left(\frac{20\pi}{12} - \frac{3\pi}{12}\right)}{\cos\left(\frac{20\pi}{12} - \frac{3\pi}{12}\right)} \\
&= \frac{\sin\left(\frac{5\pi}{3} - \frac{\pi}{4}\right)}{\cos\left(\frac{5\pi}{3} - \frac{\pi}{4}\right)} \\
&= \frac{\sin\left(\frac{5\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{5\pi}{3}\right)}{\cos\left(\frac{5\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{5\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)} \\
&= \frac{-\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2}}{\frac{1}{2}\cdot\frac{\sqrt{2}}{2} + \left(-\frac{\sqrt{3}}{2}\right)\cdot\frac{\sqrt{2}}{2}} \\
&= \frac{\frac{-\sqrt{6}-\sqrt{2}}{4}}{\frac{\sqrt{2}-\sqrt{6}}{4}} \\
&= -\frac{\sqrt{2}+\sqrt{6}}{\sqrt{2}-\sqrt{6}}\cdot\frac{\sqrt{2}+\sqrt{6}}{\sqrt{2}+\sqrt{6}}
\end{aligned}$$

(we'll use the conjugate of the denominator to rationalize the denominator; this isn'tt required)

$$\begin{aligned}
&= -\frac{2+2\sqrt{12}+6}{2-6} \\
&= -\frac{8+4\sqrt{3}}{-4} \\
&= 2+\sqrt{3}
\end{aligned}$$

2. In order to get familiar with the **sum-of-angles**, **difference-of-angles**, **double-angle** and **half-angle identities**, we'll use these identities to calculate some "friendly" sine and cosine values – so we already know these sine and cosine values and we'll verify that the identities lead us to these values.

a. Find $\sin\left(\frac{\pi}{2}\right)$ using the fact that $\frac{\pi}{2} = \frac{\pi}{6} + \frac{\pi}{3}$.

$$\begin{aligned}\sin\left(\frac{\pi}{2}\right) &= \sin\left(\frac{\pi}{6} + \frac{\pi}{3}\right)\\ &= \sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{3}\right)\\ &= \frac{1}{2}\cdot\frac{1}{2} + \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}\\ &= \frac{1}{4} + \frac{3}{4}\\ &= 1\end{aligned}$$

We know that $\sin\left(\frac{\pi}{2}\right) = 1$ so the identity gave us the correct value.

b. Find $\cos(\pi)$ using the fact that $\pi = \frac{5\pi}{6} + \frac{\pi}{6}$.

$$\begin{aligned}\cos(\pi) &= \cos\left(\frac{5\pi}{6} + \frac{\pi}{6}\right)\\ &= \cos\left(\frac{5\pi}{6}\right)\cos\left(\frac{\pi}{6}\right) - \sin\left(\frac{5\pi}{6}\right)\sin\left(\frac{\pi}{6}\right)\\ &= -\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{1}{2}\cdot\frac{1}{2}\\ &= -\frac{3}{4} - \frac{1}{4}\\ &= -1\end{aligned}$$

We know that $\cos(\pi) = -1$ so the identity gave us the correct value.

c. Find $\sin\left(\frac{2\pi}{3}\right)$ using the fact that $\frac{2\pi}{3} = 2\cdot\frac{\pi}{3}$. (hint: use a double-angle identity)

$$\begin{aligned}\sin\left(\frac{2\pi}{3}\right) &= \sin\left(2\cdot\frac{\pi}{3}\right)\\ &= 2\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}\right)\\ &= 2\cdot\frac{\sqrt{3}}{2}\cdot\frac{1}{2}\\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

We know that $\sin\left(\frac{\pi}{2}\right) = 1$ so the identity gave us the correct value.

d. Find $\cos\left(\frac{\pi}{3}\right)$ using the fact that $\frac{\pi}{3} = 2\cdot\frac{\pi}{6}$. (hint: use a double-angle identity)

$$\begin{aligned}\cos\left(\tfrac{\pi}{3}\right) &= \cos\left(2\cdot\tfrac{\pi}{6}\right)\\ &= 1 - 2\sin^2\left(\tfrac{\pi}{6}\right)\\ &= 1 - 2\cdot\left(\tfrac{1}{2}\right)^2\\ &= 1 - 2\cdot\tfrac{1}{4}\\ &= \tfrac{1}{2}\end{aligned}$$

We know that $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$ so the identity gave us the correct value.

e. Find $\sin\left(\frac{\pi}{4}\right)$ using the fact that $\frac{\pi}{4} = \frac{\pi/2}{2}$. (hint: use a half-angle identity)

$$\begin{aligned}\sin\left(\tfrac{\pi}{4}\right) &= \sin\left(\tfrac{\pi/2}{2}\right)\\ &= +\sqrt{\frac{1-\cos\left(\frac{\pi}{2}\right)}{2}}\\ &= \sqrt{\frac{1-0}{2}}\\ &= \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}\end{aligned}$$

We know that $\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ so the identity gave us the correct value.

f. Find $\cos(30^\circ)$ using the fact that $30^\circ = \frac{60^\circ}{2}$. (hint: use a half-angle identity)

$$\begin{aligned}\cos\left(30^\circ\right) &= \cos\left(\frac{60^\circ}{2}\right)\\ &= +\sqrt{\frac{1+\cos(60^\circ)}{2}}\\ &= \sqrt{\frac{1+1/2}{2}}\\ &= \sqrt{\frac{3}{4}}\\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

We know that $\cos(30^\circ) = \frac{\sqrt{3}}{2}$ so the identity gave us the correct value.

3. Suppose that $\sin(\alpha) = -\frac{\sqrt{65}}{9}$ and $\pi < \alpha < \frac{3\pi}{2}$. Calculate the *exact value* of each of the following using an appropriate **double-angle** or **half-angle identity**. (These identities are included on the Identities and Formulas Reference Sheet.)

a. $\sin(2\alpha)$. [Hint: first find $\cos(\alpha)$]

Since the double-angle identity for sine involves $\cos(\alpha)$ let's first find this using the Pythagorean identity:

$$\sin^2(\alpha) + \cos^2(\alpha) = 1$$

$$\Rightarrow \left(-\frac{\sqrt{65}}{9}\right)^2 + \cos^2(\alpha) = 1 \quad \text{(since } \sin(\alpha) = -\frac{\sqrt{65}}{9}\text{)}$$

$$\Rightarrow \cos^2(\alpha) = 1 - \frac{65}{81}$$

$$\Rightarrow \cos^2(\alpha) = \frac{16}{81}$$

$$\Rightarrow \cos(\alpha) = -\frac{4}{9}$$

(note that we take the negative square root since cosine is negative in the 3rd quadrant)

Thus,

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$$

$$= 2\left(-\frac{\sqrt{65}}{9}\right)\left(-\frac{4}{9}\right) \quad \text{(since } \sin(\alpha) = -\frac{\sqrt{65}}{9} \text{ and } \cos(\alpha) = -\frac{4}{9}\text{)}$$

$$= \frac{8\sqrt{65}}{81}$$

b. $\cos(2\alpha)$.

We could use any one of the three double-angle identities for cosine but we'll choose the identity that only involves sine since we are given the sine value in the question. (If we use given info rather than info that we've discovered in part **a.**, we avoid the possibility of using incorrect info if we made mistakes in part **a.**)

$$\cos(2\alpha) = 1 - 2\sin^2(\alpha)$$

$$= 1 - 2\cdot\left(-\frac{\sqrt{65}}{9}\right)^2 \quad \text{(since } \sin(\alpha) = -\frac{\sqrt{65}}{9}\text{)}$$

$$= 1 - 2\cdot\frac{65}{81}$$

$$= \frac{81}{81} - \frac{130}{81}$$

$$= -\frac{49}{81}$$

c. $\sin\left(\frac{\alpha}{2}\right)$.

The half-angle identity for sine is $\sin\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos(\alpha)}{2}}$, and the "$\pm$" in the identity suggests that we need to determine which sign is correct in this case. Since

$$\begin{aligned} &\pi < \alpha < \frac{3\pi}{2} \\ \Rightarrow\ &\frac{\pi}{2} < \frac{\alpha}{2} < \frac{3\pi}{2\cdot 2} \\ \Rightarrow\ &\frac{\pi}{2} < \frac{\alpha}{2} < \frac{3\pi}{4}, \end{aligned}$$

we see that $\frac{\alpha}{2}$ is in the 2nd quadrant so $\sin\left(\frac{\alpha}{2}\right) > 0$, so we need the positive value:

$$\begin{aligned} \sin\left(\frac{\alpha}{2}\right) &= +\sqrt{\frac{1-\cos(\alpha)}{2}} \\ &= \sqrt{\frac{1-\left(-4/9\right)}{2}} \qquad \text{(since } \cos(\alpha) = -\tfrac{4}{9}\text{)} \\ &= \sqrt{\frac{13}{9}\cdot\frac{1}{2}} \\ &= \sqrt{\frac{13}{18}} \\ &= \frac{\sqrt{13}}{\sqrt{9\cdot 2}} = \frac{\sqrt{13}}{3\sqrt{2}} = \frac{\sqrt{26}}{6} \end{aligned}$$

4. Suppose that $\cos(\theta) = \frac{7}{10}$ and $\frac{3\pi}{2} < \theta < 2\pi$. Calculate the *exact value* of each of the following using an appropriate **double-angle** or **half-angle identity**. (These identities are included on the Identities and Formulas Reference Sheet.)

a. $\sin(2\theta)$. [Hint: first find $\sin(\theta)$]

Since the double-angle identity for sine involves $\cos(\theta)$ let's first find this using the Pythagorean identity:

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

$$\Rightarrow \sin^2(\theta) + \left(\tfrac{7}{10}\right)^2 = 1 \qquad \text{(since } \cos(\theta) = \tfrac{7}{10}\text{)}$$

$$\Rightarrow \sin^2(\theta) = 1 - \frac{49}{100}$$

$$\Rightarrow \sin(\theta) = -\frac{\sqrt{51}}{10}$$

(note that we take the negative square root since sine is negative in the 4th quadrant)

Thus,

$$\begin{aligned}\sin(2\theta) &= 2\sin(\theta)\cos(\theta)\\ &= 2\left(-\tfrac{\sqrt{51}}{10}\right)\left(\tfrac{7}{10}\right) \qquad \text{(since } \sin(\theta) = -\tfrac{\sqrt{51}}{10} \text{ and } \cos(\theta) = \tfrac{7}{10}\text{)}\\ &= -\frac{14\sqrt{51}}{100}\end{aligned}$$

b. $\cos(2\theta)$.

We could use any one of the three double-angle identities for cosine but we'll choose the identity that only involves cosine since we are given the cosine value in the question. (If we use given info rather than info that we've discovered in part **a.**, we avoid the possibility of using incorrect info if we made mistakes in part **a.**)

$$\begin{aligned}\cos(2\theta) &= 2\cos^2(\theta) - 1\\ &= 2\cdot\left(\tfrac{7}{10}\right)^2 - 1 \qquad \text{(since } \cos(\theta) = \tfrac{7}{10}\text{)}\\ &= 2\cdot\frac{49}{100} - 1\\ &= \frac{98}{100} - \frac{100}{100}\\ &= -\frac{2}{100}\\ &= -\frac{1}{50}\end{aligned}$$

c. $\cos\left(\frac{\theta}{2}\right)$.

The half-angle identity for cosine is $\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos(\theta)}{2}}$; and the "$\pm$" in the identity suggests that we need to determine which sign is correct in this case. Since

$$\begin{aligned} &\frac{3\pi}{2} < \theta < 2\pi \\ \Rightarrow\ &\frac{3\pi}{2\cdot 2} < \frac{\theta}{2} < \frac{2\pi}{2} \\ \Rightarrow\ &\frac{3\pi}{4} < \frac{\theta}{2} < \pi, \end{aligned}$$

we see that $\frac{\theta}{2}$ is in the 2nd quadrant so $\cos\left(\frac{\theta}{2}\right) < 0$; thus, we'll use the negative value:

$$\begin{aligned} \cos\left(\frac{\theta}{2}\right) &= -\sqrt{\frac{1+\cos(\theta)}{2}} \\ &= -\sqrt{\frac{1+7/10}{2}} \qquad \text{(since } \cos(\theta) = \tfrac{7}{10}\text{)} \\ &= -\sqrt{\frac{17}{10}\cdot\frac{1}{2}} \\ &= -\sqrt{\frac{17}{20}} \\ &= -\frac{\sqrt{17}}{2\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}} = -\frac{\sqrt{85}}{2\cdot 5} = -\frac{\sqrt{85}}{10} \end{aligned}$$

5. Suppose that $\sin(x) = \frac{9}{11}$ and $\frac{\pi}{2} < x < \pi$. Calculate the *exact value* of each of the following using an appropriate **double-angle** or **half-angle identity**. (These identities are included on the Identities and Formulas Reference Sheet.)

a. $\sin(2x)$. [Hint: first find $\cos(x)$]

Since the double-angle identity for sine involves $\cos(x)$ let's first find this using the Pythagorean identity:

$$\sin^2(x) + \cos^2(x) = 1$$

$$\Rightarrow \left(\frac{9}{11}\right)^2 + \cos^2(x) = 1 \qquad \text{(since } \sin(x) = \tfrac{9}{11}\text{)}$$

$$\Rightarrow \cos^2(x) = 1 - \frac{81}{121}$$

$$\Rightarrow \cos^2(x) = \frac{40}{121}$$

$$\Rightarrow \cos(x) = -\sqrt{\frac{40}{121}} \qquad \text{(note that we take the negative square root since cosine is negative in the 2nd quadrant)}$$

$$\Rightarrow \cos(x) = -\frac{2\sqrt{10}}{11}$$

Thus,

$$\sin(2x) = 2\sin(x)\cos(x)$$

$$= 2\left(\frac{9}{11}\right)\left(-\frac{2\sqrt{10}}{11}\right) \qquad \text{(since } \sin(x) = \tfrac{9}{11} \text{ and } \cos(x) = -\tfrac{2\sqrt{10}}{11}\text{)}$$

$$= -\frac{36\sqrt{10}}{121}$$

b. $\cos(2x)$.

We could use any one of the three double-angle identities for cosine but we'll choose the identity that only involves sine since we are given the sine value in the question. (If we use given info rather than info that we've discovered in part **a.**, we avoid the possibility of using incorrect info if we made mistakes in **a.**)

$$\cos(2x) = 1 - 2\sin^2(x)$$

$$= 1 - 2\cdot\left(\frac{9}{11}\right)^2 \qquad \text{(since } \sin(x) = \tfrac{9}{11}\text{)}$$

$$= 1 - 2\cdot\frac{81}{121}$$

$$= \frac{121}{121} - \frac{162}{121}$$

$$= -\frac{41}{121}$$

c. $\sin\left(\frac{x}{2}\right)$.

The half-angle identity for sine is $\sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1-\cos(x)}{2}}$, and the "$\pm$" in the identity suggests that we need to determine which sign is correct in this case. Since

$$\begin{aligned} &\frac{\pi}{2} < x < \pi \\ \Rightarrow\ &\frac{\pi/2}{2} < \frac{x}{2} < \frac{\pi}{2} \\ \Rightarrow\ &\frac{\pi}{4} < \frac{x}{2} < \frac{\pi}{2}, \end{aligned}$$

we see that $\frac{x}{2}$ is in the 1st quadrant so $\sin\left(\frac{x}{2}\right) > 0$, so we need the positive value:

$$\begin{aligned} \sin\left(\frac{x}{2}\right) &= +\sqrt{\frac{1-\cos(x)}{2}} \\ &= \sqrt{\frac{1-\left(-\frac{2\sqrt{10}}{11}\right)}{2}} \qquad \text{(since } \cos(x) = -\frac{2\sqrt{10}}{11}\text{)} \\ &= \sqrt{\frac{1}{2}+\frac{\sqrt{10}}{11}} \\ &= \sqrt{\frac{11+2\sqrt{10}}{22}} \end{aligned}$$

6. Prove the following identities using the double-angle identities for sine and cosine included on the Identities and Formulas Reference Sheet. (Be sure to organize your proof as shown in the Online Lecture Notes and class notes videos.)

a. $\tan(2x) = \dfrac{2\tan(x)}{1 - \tan^2(x)}$

$$\begin{aligned}
\tan(2x) &= \frac{\sin(2x)}{\cos(2x)} \\
&= \frac{2\sin(x)\cos(x)}{\cos^2(x) - \sin^2(x)} && \text{(since } \sin(2x) = 2\sin(x)\cos(x) \text{ and } \cos(2x) = \cos^2(x) - \cos^2(x)\text{)} \\
&= \frac{2\sin(x)\cos(x)}{\cos^2(x) - \sin^2(x)} \cdot \frac{\frac{1}{\cos^2(x)}}{\frac{1}{\cos^2(x)}} && \text{(I'm trying this since I can predict that it will give me the denominator I need)} \\
&= \frac{\frac{2\sin(x)\cos(x)}{\cos^2(x)}}{\frac{\cos^2(x)}{\cos^2(x)} - \frac{\sin^2(x)}{\cos^2(x)}} \\
&= \frac{\frac{2\sin(x)}{\cos(x)}}{1 - \tan^2(x)} \\
&= \frac{2\tan(x)}{1 - \tan^2(x)}
\end{aligned}$$

b. $\dfrac{1 - \cos(2t)}{\sin(2t)} = \tan(t)$

Let's start with the left side of the identity since it's "more complicated" – it involves the sine and cosine of a doubled-angle so we can start by using the double angle identities for since and cosine:

$$\begin{aligned}
\frac{1 - \cos(2t)}{\sin(2t)} &= \frac{1 - \left(1 - 2\sin^2(t)\right)}{2\sin(t)\cos(t)} && \text{(since } \cos(2t) = 1 - 2\sin^2(t) \text{ and } \sin(2t) = 2\sin(t)\cos(t)\text{)} \\
&= \frac{2\sin^2(t)}{2\sin(t)\cos(t)} \\
&= \frac{\cancel{2}\sin^{\cancel{2}}(t)}{\cancel{2}\cancel{\sin(t)}\cos(t)} \\
&= \frac{\sin(t)}{\cos(t)} \\
&= \tan(t)
\end{aligned}$$

7. Translate each of the following rectangular coordinates $(x,\ y)$ into polar coordinates $(r,\ \theta)$. Your answers should be ordered pairs involving exact values, with θ in radians.

a. $(x,\ y) = (-4,\ -4)$

We can use the Pythagorean Theorem to find r:

$$\begin{aligned} r^2 &= (-4)^2 + (-4)^2 \\ \Rightarrow r^2 &= 16 + 16 \\ \Rightarrow\ r &= \sqrt{32} = 4\sqrt{2} \end{aligned}$$

We can use tangent to find θ: $\tan(\theta) = \frac{-4}{-4} = 1$. We're familiar with the fact $\tan\left(\frac{\pi}{4}\right) = 1$ so we know that the angle should be a multiple of $\frac{\pi}{4}$. Since the given point is in the third quadrant, we can conclude that $\theta = \frac{5\pi}{4}$.

Therefore, the rectangular coordinates $(-4,\ -4)$ are equivalent to the polar coordinates. $\left(4\sqrt{2},\ \frac{5\pi}{4}\right)$.

b. $(x,\ y) = \left(6,\ -6\sqrt{3}\right)$

We can use the Pythagorean Theorem to find r:

$$\begin{aligned} r^2 &= (6)^2 + \left(-6\sqrt{3}\right)^2 \\ \Rightarrow r^2 &= 36 + 36\cdot 3 \\ \Rightarrow\ r &= \sqrt{144} = 12 \end{aligned}$$

We can use tangent to find θ: $\tan(\theta) = \frac{-6\sqrt{3}}{6} = -\sqrt{3}$

We're familiar with the fact $\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$ so we know that the angle is a multiple of $\frac{\pi}{3}$. Since the given point is in the fourth quadrant, we can conclude that $\theta = \frac{5\pi}{3}$. (Note that we could also use $\theta = -\frac{\pi}{3}$.)

Therefore, the rectangular coordinates $\left(-6,\ -6\sqrt{3}\right)$ are equivalent to the polar coordinates. $\left(12,\ \frac{5\pi}{3}\right)$.

8. Translate each of the following rectangular coordinates (x, y) into polar coordinates (r, θ). Your answers should be ordered pairs involving approximations of θ in radians.

a. $(x, y) = (10, -2)$

We can use the Pythagorean Theorem to find r:

$$\begin{aligned} r^2 &= (10)^2 + (-2)^2 \\ \Rightarrow\quad r &= \sqrt{104} = 2\sqrt{26} \end{aligned}$$

We can use tangent to find θ: $\tan(\theta) = \frac{-2}{10}$. To solve this equation for θ we'll use arctangent. Since the given point is in the fourth quadrant and since the range of arctangent is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we know that the output of arctangent will be in the correct quadrant so we won't need to adjust it:

$$\theta = \tan^{-1}\left(-\tfrac{2}{10}\right) \approx -0.197$$

Thus, the rectangular coordinates $(10, -2)$ are approximately equivalent to the polar coordinates $\left(2\sqrt{26}, -0.197\right)$.

b. $(x, y) = (-3, 7)$

We can use the Pythagorean Theorem to find r:

$$\begin{aligned} r^2 &= (-3)^2 + (7)^2 \\ \Rightarrow\quad r &= \sqrt{58} \end{aligned}$$

We can use tangent to find θ: $\tan(\theta) = \frac{7}{-3}$. To solve this equation for θ we'll use arctangent. Since the given point is in the second quadrant but the range of arctangent is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we know we'll have to add π to the output of arctangent in order to put the angle into the correct quadrant:

$$\theta = \tan^{-1}\left(-\tfrac{7}{3}\right) + \pi \approx 1.98$$

Therefore, the rectangular coordinates $(-3, 7)$ are approximately equivalent to the polar coordinates $\left(\sqrt{58}, 1.98\right)$.

9. Translate each of the following polar coordinates (r, θ) into rectangular coordinates (x, y). Your answers should be ordered pairs involving exact values.

a. $(r, \theta) = \left(5, \frac{2\pi}{3}\right)$

$$x = r \cdot \cos(\theta) = 5 \cdot \cos\left(\frac{2\pi}{3}\right) = 5 \cdot \left(-\frac{1}{2}\right) = -\frac{5}{2}$$

and

$$y = r \cdot \sin(\theta) = 5 \cdot \sin\left(\frac{2\pi}{3}\right) = 5 \cdot \left(\frac{\sqrt{3}}{2}\right) = \frac{5\sqrt{3}}{2}$$

So polar coordinates $\left(5, \frac{2\pi}{3}\right)$ are equivalent to rectangular coordinates $\left(-\frac{5}{2}, \frac{5\sqrt{3}}{2}\right)$.

b. $(r, \theta) = \left(16, 210^\circ\right)$

$$x = r \cdot \cos(\theta) = 16 \cdot \cos\left(210^\circ\right) = 16 \cdot \left(-\frac{\sqrt{3}}{2}\right) = -8\sqrt{3}$$

and

$$y = r \cdot \sin(\theta) = 16 \cdot \sin\left(210^\circ\right) = 16 \cdot \left(-\frac{1}{2}\right) = -8$$

Therefore, the polar coordinates $\left(16, 210^\circ\right)$ are equivalent to the rectangular coordinates $\left(-8\sqrt{3}, -8\right)$.

10. Translate each of the following polar coordinates $(r,\ \theta)$ into rectangular coordinates $(x,\ y)$. Your answers should be ordered pairs involving approximate values.

a. $(r,\ \theta) = \left(3,\ 80^\circ\right)$

Note that here we need to use a calculator to compute an approximation of the sine and cosine values since 80° isn't "friendly."

$$\begin{aligned} x &= r \cdot \cos(\theta) \\ &= 3 \cdot \cos\left(80^\circ\right) \\ &\approx 0.52 \end{aligned} \qquad \text{and} \qquad \begin{aligned} y &= r \cdot \sin(\theta) \\ &= 3 \cdot \sin\left(80^\circ\right) \\ &\approx 2.95 \end{aligned}$$

Therefore, the polar coordinates $\left(3,\ 80^\circ\right)$ are approximately equivalent to the rectangular coordinates $(0.52,\ 2.95)$.

b. $(r,\ \theta) = \left(7,\ -\frac{\pi}{10}\right)$

Note that here we need to use a calculator to compute an approximation of the sine and cosine values since $-\frac{\pi}{10}$ isn't "friendly."

$$\begin{aligned} x &= r \cdot \cos(\theta) \\ &= 7 \cdot \cos\left(-\tfrac{\pi}{10}\right) \\ &\approx 6.66 \end{aligned} \qquad \text{and} \qquad \begin{aligned} y &= r \cdot \sin(\theta) \\ &= 7 \cdot \sin\left(-\tfrac{\pi}{10}\right) \\ &\approx -2.16 \end{aligned}$$

Therefore, the polar coordinates $\left(7,\ -\frac{\pi}{10}\right)$ are approximately equivalent to the rectangular coordinates $(6.66,\ -2.16)$.

11. The polar ordered pair $\left(5, \frac{\pi}{3}\right)$ can be plotted as a "dot" on the polar coordinate plane. List four other (different) polar ordered pairs that are plotted on the same "dot." Use "-5" for "r" in at least one of your ordered pairs.

One way that we can create different polar ordered pairs that are plotted on the same "dot" is to use conterminal angles.

- Since $\frac{\pi}{3} + 2\pi = \frac{7\pi}{3}$ is coterminal with $\frac{\pi}{3}$, the polar ordered pair $\left(5, \frac{7\pi}{3}\right)$ is plotted in the same location as $\left(5, \frac{\pi}{3}\right)$.
- Since $\frac{\pi}{3} - 2\pi = -\frac{5\pi}{3}$ is coterminal with $\frac{\pi}{3}$, the polar ordered pair $\left(5, -\frac{5\pi}{3}\right)$ is plotted in the same location as $\left(5, \frac{\pi}{3}\right)$.

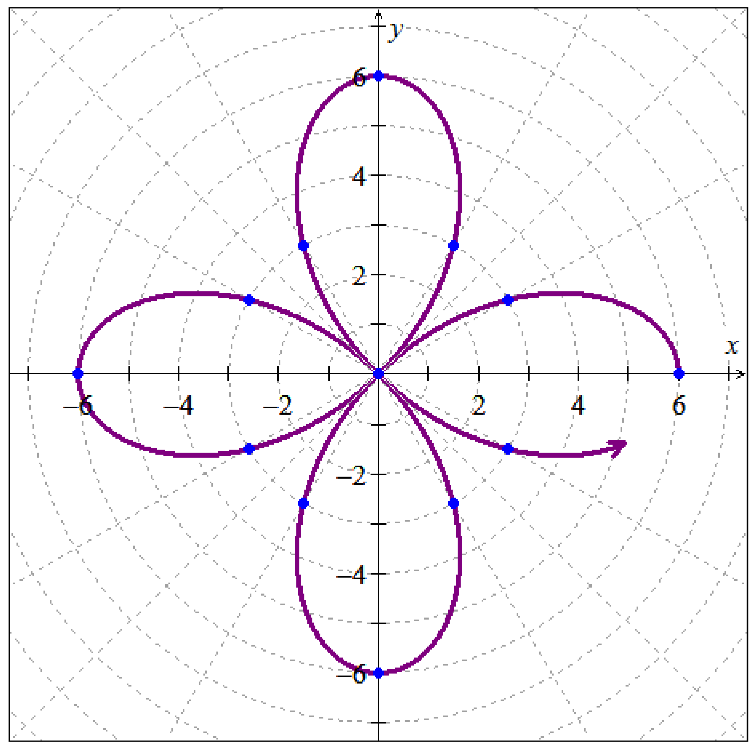
Another way that we can create different polar ordered pairs that are plotted on the same "dot" is to use a negative value for r: when r is negative, it's equivalent of adding (or subtracting) a half-revolution.

- Since $\frac{\pi}{3} + \pi = \frac{4\pi}{3}$ is a half-revolution greater than $\frac{\pi}{3}$, the polar ordered pair $\left(-5, \frac{4\pi}{3}\right)$ is plotted in the same location as $\left(5, \frac{\pi}{3}\right)$.
- Since $\frac{\pi}{3} - \pi = -\frac{2\pi}{3}$ is a half-revolution less than $\frac{\pi}{3}$, the polar ordered pair $\left(-5, -\frac{2\pi}{3}\right)$ is plotted in the same location as $\left(5, \frac{\pi}{3}\right)$.

Consider opening Desmos.com and plotting these points to see that they all land in the same location.

12. Complete the 1st column of the table below with appropriate multiples of $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$; then determine the corresponding values of r if $r = 6\cos(2\theta)$ in order to complete the 2nd column of the table; then plot the points implied by each of the 16 rows of the table on the polar plane below and connect those points in order to draw a graph of $r = 6\cos(2\theta)$.
(HINT: Graph the function in Desmos so that you can predict its shape before you draw it.)

	$\boldsymbol{\theta}$	$\boldsymbol{r}$
	0	6
	$\frac{\pi}{6}$	3
Quad. 1 angles	$\frac{\pi}{4}$	0
	$\frac{\pi}{3}$	-3
	$\pi/2$	-6
	$\frac{2\pi}{3}$	-3
Quad. 2 angles	$\frac{3\pi}{4}$	0
	$\frac{5\pi}{6}$	3
	π	6
	$\frac{7\pi}{6}$	3
Quad. 3 angles	$\frac{5\pi}{4}$	0
	$\frac{4\pi}{3}$	-3
	$3\pi/2$	-6
	$\frac{5\pi}{3}$	-3
Quad. 4 angles	$\frac{7\pi}{4}$	0
	$\frac{11\pi}{6}$	3



A graph of $r = 6\cos(2\theta)$.

13. Express the complex number $z = 10e^{i\cdot\frac{11\pi}{6}}$ in the form $z = a + bi$.

$$\begin{aligned} z &= 10e^{i\cdot\frac{11\pi}{6}} \\ &= 10\cos\left(\tfrac{11\pi}{6}\right) + 10\sin\left(\tfrac{11\pi}{6}\right)\cdot i \\ &= 10\cdot\left(\tfrac{\sqrt{3}}{2}\right) + 10\cdot\left(-\tfrac{1}{2}\right)\cdot i \\ &= 5\sqrt{3} - 5i \end{aligned}$$

14. Express the complex number $z = 8e^{i\cdot\frac{2\pi}{3}}$ in the form $z = a + bi$.

$$\begin{aligned} z &= 8e^{i\cdot\frac{2\pi}{3}} \\ &= 8\cos\left(\tfrac{2\pi}{3}\right) + 8\sin\left(\tfrac{2\pi}{3}\right)\cdot i \\ &= 8\cdot\left(-\tfrac{1}{2}\right) + 8\cdot\left(\tfrac{\sqrt{3}}{2}\right)\cdot i \\ &= -4 + 4\sqrt{3}\,i \end{aligned}$$

15. Express the complex number $z = -7e^{i\cdot\frac{5\pi}{4}}$ in the form $z = a + bi$.

$$\begin{aligned} z &= -7e^{i\cdot\frac{5\pi}{4}} \\ &= -7\cos\left(\tfrac{5\pi}{4}\right) + (-7)\sin\left(\tfrac{5\pi}{4}\right)\cdot i \\ &= -7\cdot\left(-\tfrac{\sqrt{2}}{2}\right) - 7\cdot\left(-\tfrac{\sqrt{2}}{2}\right)\cdot i \\ &= \tfrac{7\sqrt{2}}{2} + \tfrac{7\sqrt{2}}{2}i \end{aligned}$$

16. Find a polar form, $z = re^{i\theta}$, of the complex number $z = -3 - 3\sqrt{3}\,i$.

We can associate the complex number $z = -3 - 3\sqrt{3}\,i$ with the rectangular ordered pair $(-3, -3\sqrt{3})$, and then translate this ordered pair into polar coordinates (r, θ), and finally use this polar ordered pair to obtain the polar form $z = re^{i\theta}$. First, let's find r:

$$\begin{aligned} r &= \sqrt{(-3)^2 + (-3\sqrt{3})^2} \\ &= \sqrt{9 + 9 \cdot 3} \\ &= 6. \end{aligned}$$

Now, let's find θ:

$$\tan(\theta) = \frac{-3\sqrt{3}}{-3} = \sqrt{3}$$

$$\Rightarrow \qquad \theta = \tan^{-1}\left(\sqrt{3}\right) + \pi$$

(we add π since the given point is in quadrant 3 but the range of arctangent is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$)

$$\Rightarrow \qquad \theta = \frac{\pi}{3} + \pi = \frac{4\pi}{3}$$

Therefore, $z = 6e^{i \cdot \frac{4\pi}{3}}$ is a polar form of the complex number $z = -3 - 3\sqrt{3}\,i$.

17. Find a polar form, $z = re^{i\theta}$, of the complex number $z = 2 - 2i$.

We can associate the complex number $z = 2 - 2i$ with the rectangular ordered pair $(2, -2)$, and then translate this ordered pair into polar coordinates (r, θ), and finally use this polar ordered pair to obtain the polar form $z = re^{i\theta}$. First, let's find r:

$$\begin{aligned} r &= \sqrt{(2)^2 + (-2)^2} \\ &= \sqrt{4 + 4} \\ &= 2\sqrt{2}. \end{aligned}$$

Now, let's find θ:

$$\tan(\theta) = \frac{-2}{2}$$

$$\Rightarrow \quad \theta = \tan^{-1}(-1)$$

$$\Rightarrow \quad \theta = -\frac{\pi}{4}$$

Therefore, $z = 2\sqrt{2}\, e^{i\cdot\left(-\frac{\pi}{4}\right)}$ is a polar form of the complex number $z = 2 - 2i$.

18. Find a polar form, $z = re^{i\theta}$, of the complex number $z = -4\sqrt{3} + 12i$.

We can associate the complex number $z = -4\sqrt{3} + 12i$ with the rectangular ordered pair $(-4\sqrt{3}, 12)$, and then translate this ordered pair into polar coordinates (r, θ), and finally use this polar ordered pair to obtain the polar form $z = re^{i\theta}$. First, let's find r:

$$\begin{aligned} r &= \sqrt{\left(-4\sqrt{3}\right)^2 + (12)^2} \\ &= \sqrt{16\cdot 3 + 144} \\ &= \sqrt{192} \\ &= 8\sqrt{3}. \end{aligned}$$

Now, let's find θ:

$$\tan(\theta) = \frac{12}{-4\sqrt{3}} = -\frac{3}{\sqrt{3}} = -\sqrt{3}$$

$$\Rightarrow \quad \theta = \tan^{-1}\left(-\sqrt{3}\right) + \pi$$

(we add π since the given point is in quadrant 2 but the range of arctangent is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$)

$$\Rightarrow \quad \theta = -\frac{\pi}{3} + \pi = \frac{2\pi}{3}$$

Therefore, $z = 8\sqrt{3}\, e^{i\cdot\left(\frac{2\pi}{3}\right)}$ is a polar form of the complex number $z = -4\sqrt{3} + 12i$.

19. Find three different a polar forms, $z = re^{i\theta}$, of the complex number $z = 4i$. (HINT: $i = 0 + 4i$ can be associated with the point $(0, 4)$ so find three different angles that can be used to represent the "direction of this point" and use each angle to create a polar form.)

As suggested in the HINT, we can associate the complex number $z = 4i = 0 + 4i$ with the rectangular ordered pair $(0, 4)$. We're going to translate this ordered pair into polar coordinates (r, θ), so we need and then use this polar ordered pair to obtain the polar form $z = re^{i\theta}$.

First, let's find r:

$$\begin{aligned} r &= \sqrt{(0)^2 + (4)^2} \\ &= \sqrt{16} \\ &= 4. \end{aligned}$$

Now, let's find angles that we can use for θ to align with the point $(0, 4)$. This point is on the "positive y-axis": so one angle that will work is $\frac{\pi}{2}$, and we can use two other angles that are coterminal with $\frac{\pi}{2}$ for two additional polar representations: let's use $-\frac{3\pi}{2}$ and $\frac{5\pi}{2}$.

Based on the discussion above we can conclude that the following are three different polar forms of the complex number $z = 4i$:

$$z = 4e^{i\cdot\frac{\pi}{2}}, \quad z = 4e^{i\cdot\left(-\frac{3\pi}{2}\right)}, \text{ and } \quad z = 4e^{i\cdot\frac{5\pi}{2}}$$