SOLUTIONS: Week 5 Practice Worksheet

Triangles and Proving Trig Identities

1. a. Find the exact value of all six trig functions for the angles A and B in the triangle in Figure 1. (The triangle may not be drawn to scale.)

First let's use the Pythagorean Theorem to find x

$$\begin{aligned} x^2 + 3^2 &= 9^2 \\ \Rightarrow\quad x^2 + 9 &= 81 \\ \Rightarrow\quad x^2 &= 72 \\ \Rightarrow\quad x &= \sqrt{72} \\ \Rightarrow\quad x &= 6\sqrt{2} \end{aligned}$$

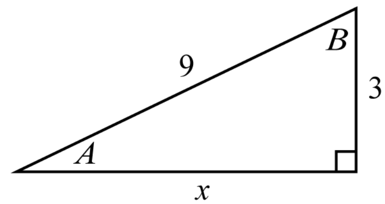


Figure 1

Now, let's find the value of all six trig functions for the angle A:

$\sin(A) = \frac{\text{OPP}}{\text{HYP}} = \frac{3}{9} = \frac{1}{3}$	$\cos(A) = \frac{\text{ADJ}}{\text{HYP}} = \frac{6\sqrt{2}}{9} = \frac{2\sqrt{2}}{3}$	$\tan(A) = \frac{\text{OPP}}{\text{ADJ}} = \frac{3}{6\sqrt{2}} = \frac{1}{2\sqrt{2}}$
$\csc(A) = \frac{1}{\sin(A)} = \frac{1}{\frac{1}{3}} = 3$	$\sec(A) = \frac{1}{\cos(A)} = \frac{1}{\frac{2\sqrt{2}}{3}} = \frac{3}{2\sqrt{2}}$	$\cot(A) = \frac{1}{\tan(A)} = \frac{1}{\frac{1}{2\sqrt{2}}} = 2\sqrt{2}$

Finally, let's find the value of all six trig functions for the angle B:

$\sin(B) = \frac{\text{OPP}}{\text{HYP}} = \frac{6\sqrt{2}}{9} = \frac{2\sqrt{2}}{3}$	$\cos(B) = \frac{\text{ADJ}}{\text{HYP}} = \frac{3}{9} = \frac{1}{3}$	$\tan(B) = \frac{\text{OPP}}{\text{ADJ}} = \frac{6\sqrt{2}}{3} = 2\sqrt{2}$
$\csc(B) = \frac{1}{\sin(B)} = \frac{1}{\frac{2\sqrt{2}}{3}} = \frac{3}{2\sqrt{2}}$	$\sec(B) = \frac{1}{\cos(B)} = \frac{1}{\frac{1}{3}} = 3$	$\cot(B) = \frac{1}{\tan(B)} = \frac{1}{2\sqrt{2}}$

b. Solve the triangle in Figure 1 by finding approximate measurements (in degrees) of angles A and B and the exact length of the side x.

In part (**a**) we discovered that $x = 6\sqrt{2}$.

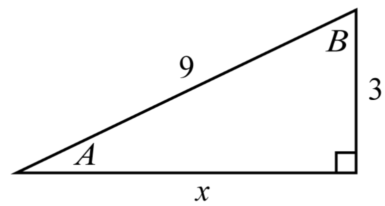


Figure 1 (again)

To find angle A we can use the given information and the sine function (I'm choosing to use sine instead of cosine or tangent since then I can use the given information, rather than information that I've derived; this way, if I've made a mistake on a previous portion of the problem, it won't impact this part of the problem):

$$\sin(A) = \frac{\text{OPP}}{\text{HYP}}$$

$$\Rightarrow \quad \sin(A) = \frac{3}{9} = \frac{1}{3}$$

$$\Rightarrow \quad \sin^{-1}\left(\sin(A)\right) = \sin^{-1}\left(\frac{1}{3}\right)$$

$$\Rightarrow \quad A = \sin^{-1}\left(\frac{1}{3}\right) \approx 19.47^\circ$$

Now we can find angle B we can use the fact that the total angle measure in a triangle is always 180°:

$$A + B + 90^\circ = 180^\circ$$

$$\Rightarrow \quad B = 180^\circ - 90^\circ - A$$

$$\Rightarrow \quad B \approx 180^\circ - 90^\circ - 19.47^\circ$$

$$\Rightarrow \quad B \approx 70.53^\circ$$

2. a. Find the exact value of all six trig functions for the angles θ and ϕ in the triangle in Figure 2. (The triangle may not be drawn to scale.)

First let's use the Pythagorean Theorem to find s:

$$
\begin{aligned}
& 4^2 + 6^2 = s^2 \\
\Rightarrow\ & 16 + 36 = s^2 \\
\Rightarrow\ & 52 = s^2 \\
\Rightarrow\ & s = \sqrt{52} = 2\sqrt{13}
\end{aligned}
$$

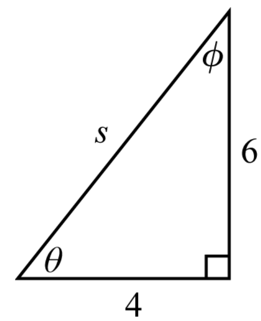


Figure 2

Now, let's find the value of all six trig functions for the angle θ:

$\sin(\theta) = \frac{\text{OPP}}{\text{HYP}} = \frac{6}{2\sqrt{13}} = \frac{3}{\sqrt{13}}$	$\cos(\theta) = \frac{\text{ADJ}}{\text{HYP}} = \frac{4}{2\sqrt{13}} = \frac{2}{\sqrt{13}}$	$\tan(\theta) = \frac{\text{OPP}}{\text{ADJ}} = \frac{6}{4} = \frac{3}{2}$
$\csc(\theta) = \frac{1}{\sin(\theta)} = \frac{1}{\frac{3}{\sqrt{13}}} = \frac{\sqrt{13}}{3}$	$\sec(\theta) = \frac{1}{\cos(\theta)} = \frac{1}{\frac{2}{\sqrt{13}}} = \frac{\sqrt{13}}{2}$	$\cot(\theta) = \frac{1}{\tan(\theta)} = \frac{1}{\frac{3}{2}} = \frac{2}{3}$

Finally, let's find the value of all six trig functions for the angle ϕ:

$\sin(\phi) = \frac{\text{OPP}}{\text{HYP}} = \frac{4}{2\sqrt{13}} = \frac{2}{\sqrt{13}}$	$\cos(\phi) = \frac{\text{ADJ}}{\text{HYP}} = \frac{6}{2\sqrt{13}} = \frac{3}{\sqrt{13}}$	$\tan(\phi) = \frac{\text{OPP}}{\text{ADJ}} = \frac{4}{6} = \frac{2}{3}$
$\csc(\phi) = \frac{1}{\sin(\phi)} = \frac{1}{\frac{2}{\sqrt{13}}} = \frac{\sqrt{13}}{2}$	$\sec(\phi) = \frac{1}{\cos(\phi)} = \frac{1}{\frac{3}{\sqrt{13}}} = \frac{\sqrt{13}}{3}$	$\cot(\phi) = \frac{1}{\tan(\phi)} = \frac{1}{\frac{2}{3}} = \frac{3}{2}$

b. Solve the triangle in Figure 2 by finding approximate measurements (in degrees) of angles θ and ϕ and the exact length of the side s.

In part (**a**) we discovered that $s = 2\sqrt{13}$.

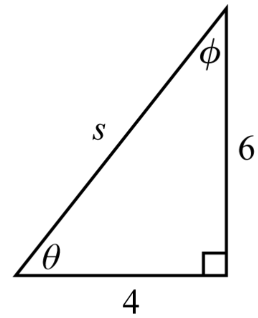


Figure 2 (again)

To find angle θ we can use the given information and the tangent function (I'm choosing to use tangent instead of sine or cosine since then I can use the given information, rather than information that I've derived; this way, if I've made a mistake on a previous portion of the problem, it won't impact this part of the problem):

$$\tan(\theta) = \frac{\text{OPP}}{\text{ADJ}}$$

$$\Rightarrow \quad \tan(\theta) = \frac{6}{4} = \frac{3}{2}$$

$$\Rightarrow \quad \tan^{-1}\left(\tan(\theta)\right) = \tan^{-1}\left(\frac{3}{2}\right)$$

$$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{3}{2}\right) \approx 56.3^\circ$$

Now we can find angle ϕ we can use the fact that the total angle measure in a triangle is always 180°:

$$\theta + \phi + 90^\circ = 180^\circ$$

$$\Rightarrow \quad \phi = 180^\circ - 90^\circ - \theta$$

$$\Rightarrow \quad \phi \approx 180^\circ - 90^\circ - 56.3^\circ$$

$$\Rightarrow \quad \phi \approx 33.7^\circ$$

3. Find the values of c, A, and B in the triangle in Figure 3. You should approximate the values (in degrees for the angles) and denote your approximations correctly.
(The triangle may not be drawn to scale.)

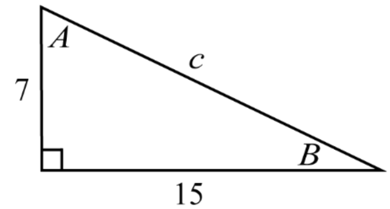


Figure 3

First let's use the Pythagorean Theorem to find the length of the side c:

$$c^2 = 7^2 + 15^2$$
$$\Rightarrow \quad c^2 = 49 + 225$$
$$\Rightarrow \quad c^2 = 274$$
$$\Rightarrow \quad c = \sqrt{274} \approx 16.55$$

Now we can use the tangent function to get an equation involving angle B and then solve the equation for B:

$$\tan(B) = \tfrac{7}{15}$$
$$\Rightarrow \qquad B = \tan^{-1}\left(\tfrac{7}{15}\right) \approx 25.02^\circ$$

Note that we've chosen to use tangent (instead of sine or cosine) to find B since that allows us to use the given side-lengths. If we use sine or cosine, we would need to use the value for c that we found above but it's possible that we made a mistake so, whenever possible, it's more sensible rely on the given information rather than on information that we've found ourselves.

Finally, we can use the rule that the sum of the angles in a triangle is always 180° to find angle A:

$$A + B + 90^\circ = 180^\circ$$
$$\Rightarrow \quad A + 25.02^\circ + 90^\circ \approx 180^\circ$$
$$\Rightarrow \quad A \approx 180^\circ - 90^\circ - 25.02^\circ$$
$$\Rightarrow \quad A \approx 64.98^\circ$$

4. Find the values of p, q, and θ in the triangle in Figure 4. You should approximate the values (in degrees for the angles) and denote your approximations correctly.
(The triangle may not be drawn to scale.)

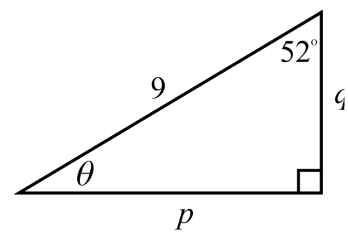


Figure 4

First we'll use the rule that the sum of the angles in a triangle is always 180° to find θ:

$$\theta + 52^\circ + 90^\circ = 180^\circ$$
$$\Rightarrow \quad \theta = 180^\circ - 90^\circ - 52^\circ$$
$$\Rightarrow \quad \theta = 38^\circ$$

Now we'll use the sine function to create an equation involving side p and then solve the equation for p:

$$\sin(52^\circ) = \tfrac{p}{9}$$
$$\Rightarrow \quad p = 9 \cdot \sin(52^\circ)$$
$$\Rightarrow \quad p \approx 7.09$$

Now we can use the cosine function to get an equation involving angle q and then solve the equation for q:

$$\cos(52^\circ) = \tfrac{q}{9}$$
$$\Rightarrow \quad q = 9 \cdot \cos(52^\circ)$$
$$\Rightarrow \quad q \approx 5.54$$

(We can check that $(7.09)^2 + (5.54)^2 \approx 9^2$ so at least this isn't an obviously-impossible result!)

5. Find the values of X and Y in the triangle in Figure 5. You should approximate the values (in degrees for the angles) and denote your approximations correctly.
(The triangle may not be drawn to scale.)

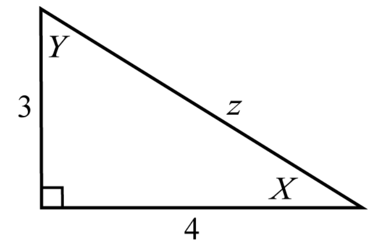


Figure 5

First let's use the Pythagorean Theorem to find the length of the side z

$$\begin{aligned} & z^2 = 3^2 + 4^2 \\ \Rightarrow\ & z^2 = 9 + 16 \\ \Rightarrow\ & z^2 = 25 \\ \Rightarrow\ & z = 5 \end{aligned}$$

There are a variety of ways find the angles X and Y; here's one option in which we only use the sine function:

First we'll use the sine function to get an equation involving angle X and then solve the equation for X:

$$\begin{aligned} & \sin(X) = \frac{\text{OPP OF } X}{\text{HYP}} \\ \Rightarrow\ & \sin(X) = \tfrac{3}{5} \\ \Rightarrow\ & X = \sin^{-1}\left(\tfrac{3}{5}\right) \\ \Rightarrow\ & X \approx 36.87^\circ \end{aligned}$$

Now we can use the sine function to get an equation involving angle Y and then solve the equation for Y:

$$\begin{aligned} & \sin(Y) = \frac{\text{OPP OF } Y}{\text{HYP}} \\ \Rightarrow\ & \sin(Y) = \tfrac{4}{5} \\ \Rightarrow\ & Y = \sin^{-1}\left(\tfrac{4}{5}\right) \\ \Rightarrow\ & Y \approx 53.13^\circ \end{aligned}$$

(We can check that $90^\circ + 36.87^\circ + 53.13^\circ = 180^\circ$ so at least this isn't an obviously-impossible result!)

6. Find the values of c, A, and B in the triangle in Figure 6. You should approximate the values (in degrees for the angles) and denote your approximations correctly.
(The triangle may not be drawn to scale.)

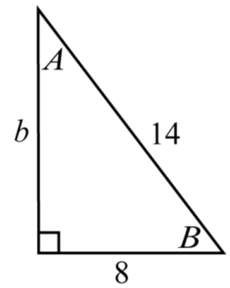


Figure 6

First let's use the Pythagorean Theorem to find the length of the side b:

$$\begin{aligned} & b^2 + 8^2 = 14^2 \\ \Rightarrow \quad & b^2 = 196 - 64 \\ \Rightarrow \quad & b^2 = 132 \\ \Rightarrow \quad & b = \sqrt{132} \approx 11.49 \end{aligned}$$

Now we can use the cosine function to get an equation involving angle B and then solve the equation for B:

$$\begin{aligned} & \cos(B) = \tfrac{8}{14} \\ \Rightarrow \quad & B = \cos^{-1}\left(\tfrac{8}{14}\right) \approx 55.15^\circ \end{aligned}$$

Note that we've chosen to use cosine (instead of sine or tangent) to find B because that allows us to use the given side-lengths. If we use sine or tangent, we would need to use the value for b that we found above but it's possible that we made a mistake so, whenever possible, it's more sensible rely on the given information rather than on information that we've found ourselves.

Finally, we can use the rule that the sum of the angles in a triangle is always 180° to find angle A:

$$\begin{aligned} & A + B + 90^\circ = 180^\circ \\ \Rightarrow \quad & A + 55.15^\circ + 90^\circ \approx 180^\circ \\ \Rightarrow \quad & A \approx 180^\circ - 90^\circ - 55.15^\circ \\ \Rightarrow \quad & A \approx 34.85^\circ \end{aligned}$$

7. a. $a = 5$, $b = 6$, $c = 7$

First, let's draw a sketch of the situation:

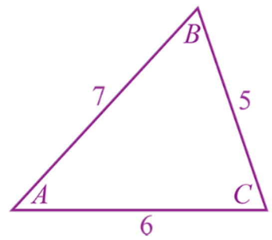


Figure 7

We aren't given any angle measures so we don't have any "angle and side opposite" pairs to work with so we cannot employ the Law of Sines, so we'll have to start with the Law of Cosines which we can use to find any of the angles but let's aim for A:

$$\begin{aligned} & 5^2 = 7^2 + 6^2 - 2\cdot 7\cdot 6\cdot \cos(A) \\ \Rightarrow\quad & 25 = 85 - 84\cdot\cos(A) \\ \Rightarrow\quad & 84\cdot\cos(A) = 60 \\ \Rightarrow\quad & \cos(A) = \tfrac{60}{84} \\ \Rightarrow\quad & A = \cos^{-1}\left(\tfrac{5}{7}\right) \\ \Rightarrow\quad & \approx 44.42^\circ \end{aligned}$$

Now that we know the measure of angle A, we have an "angle and side opposite" pair to work with so we can use the Law of Sines to find either of the remaining missing angles; let's aim for B since it's opposite a smaller side than is C. (Note that it's always better to use the Law of Sines to find the smaller angle since the inverse-sine function cannot produce angles larger than 90°.)

$$\begin{aligned} & \frac{\sin(B)}{6} = \frac{\sin(A)}{5} \\ \Rightarrow\quad & \sin(B) \approx \frac{6\cdot\sin(44.42^\circ)}{5} \\ \Rightarrow\quad & B \approx \sin^{-1}\left(\frac{6\cdot\sin(44.42^\circ)}{5}\right) \\ \Rightarrow\quad & B \approx 57.13^\circ \end{aligned}$$

Finally, we can use the fact that the sum of the angles in a triangle is 180° to find C:

$$\begin{aligned} C &= 180^\circ - A - B \\ &\approx 180^\circ - 44.42^\circ - 57.13^\circ \\ &\approx 78.45^\circ \end{aligned}$$

7. b. $B = 76^\circ$, $a = 8$, $c = 6$

First, let's draw a sketch of the situation:

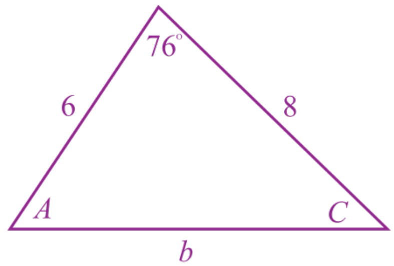


Figure 8

As with **4.a.**, we aren't given an "angle and side opposite" pair to work with so we cannot employ the Law of Sines but we are given the length of the two sides that form the angle $B = 76^\circ$ so we can use the Law of Cosines to find b:

$$\begin{aligned} & b^2 = 6^2 + 8^2 - 2\cdot 6\cdot 8\cdot \cos(76^\circ) \\ \Rightarrow\quad & b^2 = 100 - 96\cdot\cos(76^\circ) \\ \Rightarrow\quad & b = \sqrt{100 - 96\cdot\cos(76^\circ)} \\ \Rightarrow\quad & b \approx 8.76 \end{aligned}$$

Now that we know the length of side b, we have an "angle and side opposite" pair to work with so we can use the Law of Sines to find either of the remaining missing angles; let's aim for C since it's opposite a smaller side than is A. (Note that it's always better to use the Law of Sines to find the smaller angle since the inverse-sine function cannot produce angles larger than 90°.)

$$\begin{aligned} & \frac{\sin(C)}{6} = \frac{\sin(76^\circ)}{b} \\ \Rightarrow\quad & \sin(C) \approx \frac{6\cdot\sin(76^\circ)}{8.76} \\ \Rightarrow\quad & C \approx \sin^{-1}\left(\frac{6\cdot\sin(76^\circ)}{8.76}\right) \\ \Rightarrow\quad & C \approx 41.65^\circ \end{aligned}$$

Finally, we can use the fact that the sum of the angles in a triangle is 180° to find A:

$$\begin{aligned} A &= 180^\circ - 76^\circ - C \\ &\approx 180^\circ - 76^\circ - 41.65^\circ \\ &\approx 62.35^\circ \end{aligned}$$

7. c. $B = 118^\circ$, $C = 37^\circ$, $a = 5$

First, let's draw a sketch of the situation:

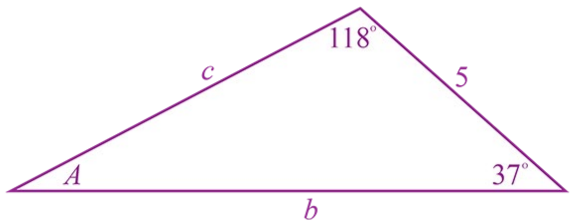


Figure 9

First, we can use the fact that the sum of the angles in a triangle is 180° to find A:

$$\begin{aligned} A &= 180^\circ - 118^\circ - 37^\circ \\ &= 25^\circ \end{aligned}$$

Now we have an "angle and side opposite" pair to work with so we can use the Law of Sines to find b and c:

$$\frac{b}{\sin(118^\circ)} = \frac{5}{\sin(A)}$$

$$\Rightarrow \qquad b = \frac{5 \cdot \sin(118^\circ)}{\sin(25^\circ)}$$

$$\Rightarrow \qquad b \approx 10.45$$

and

$$\frac{c}{\sin(37^\circ)} = \frac{5}{\sin(A)}$$

$$\Rightarrow \qquad c = \frac{5 \cdot \sin(37^\circ)}{\sin(25^\circ)}$$

$$\Rightarrow \qquad c \approx 7.12$$

7. d. $A = 62^\circ$, $B = 70^\circ$, $b = 10$

First, let's draw a sketch of the situation:

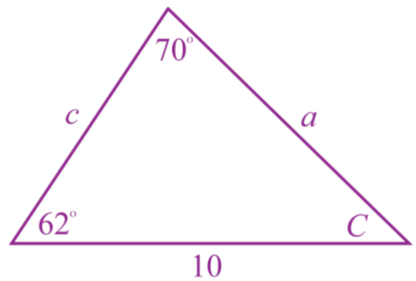


Figure 10

First, we can use the fact that the sum of the angles in a triangle is 180° to find C:

$$\begin{aligned} C &= 180^\circ - 62^\circ - 70^\circ \\ &= 48^\circ \end{aligned}$$

We have an "angle and side opposite" pair to work with so we can use the Law of Sines to find a and c:

$$\frac{a}{\sin(62^\circ)} = \frac{10}{\sin(70^\circ)}$$

$$\Rightarrow \qquad a = \frac{10 \cdot \sin(62^\circ)}{\sin(70^\circ)}$$

$$\Rightarrow \qquad a \approx 9.4$$

and

$$\frac{c}{\sin(C)} = \frac{10}{\sin(70^\circ)}$$

$$\Rightarrow \frac{c}{\sin(48^\circ)} = \frac{10}{\sin(70^\circ)}$$

$$\Rightarrow \qquad c = \frac{10 \cdot \sin(48^\circ)}{\sin(70^\circ)}$$

$$\Rightarrow \qquad c \approx 7.9$$

7. e. $A = 64^\circ$, $C = 76^\circ$, $b = 9$

First, let's draw a sketch of the situation:

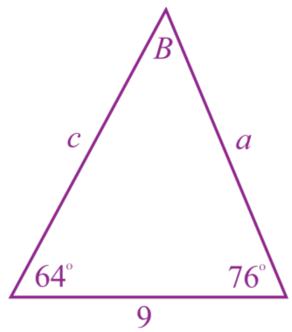


Figure 11

First, we can use the fact that the sum of the angles in a triangle is 180° to find B:

$$\begin{aligned} B &= 180^\circ - 64^\circ - 76^\circ \\ &= 40^\circ \end{aligned}$$

Now we have an "angle and side opposite" pair to work with so we can use the Law of Sines to find a and c:

$$\begin{aligned} \frac{a}{\sin(64^\circ)} &= \frac{9}{\sin(B)} \\ \frac{a}{\sin(64^\circ)} &= \frac{9}{\sin(40^\circ)} \\ \Rightarrow \qquad a &= \frac{9 \cdot \sin(64^\circ)}{\sin(40^\circ)} \\ \Rightarrow \qquad a &\approx 12.58 \end{aligned}$$

and

$$\begin{aligned} \frac{c}{\sin(76^\circ)} &= \frac{9}{\sin(B)} \\ \frac{c}{\sin(76^\circ)} &= \frac{9}{\sin(40^\circ)} \\ \Rightarrow \qquad c &= \frac{9 \cdot \sin(76^\circ)}{\sin(40^\circ)} \\ \Rightarrow \qquad c &\approx 13.59 \end{aligned}$$

7. f. $c = 40^\circ$, $a = 9$, $b = 13$

First, let's draw a sketch of the situation:

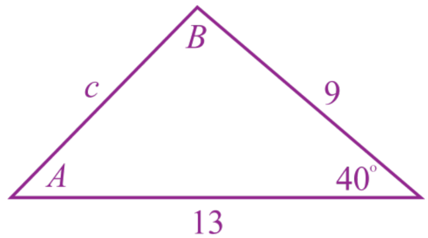


Figure 12

As with **7.b.**, we aren't given an "angle and side opposite" pair to work with so we cannot employ the Law of Sines but we are given the length of the two sides that form the angle $C = 50^\circ$ so we can use the Law of Cosines to find c:

$$\begin{aligned} & c^2 = 9^2 + 13^2 - 2\cdot 9\cdot 13\cdot\cos(40^\circ) \\ \Rightarrow\quad & c^2 = 250 - 234\cdot\cos(40^\circ) \\ \Rightarrow\quad & c = \sqrt{250 - 234\cdot\cos(40^\circ)} \\ \Rightarrow\quad & c \approx 8.41 \end{aligned}$$

Now that we know the length of side c, we have an "angle and side opposite" pair to work with so we can use the Law of Sines to find either of the remaining missing angles; let's aim for A since it's opposite a smaller side than is B. (Note that it's always better to use the Law of Sines to find the smaller angle since the inverse-sine function cannot produce angles larger than 90°. **In this case, if you try to B first using the Law of Sines, you will obtain an incorrect measure of B !!**)

$$\begin{aligned} & \frac{\sin(A)}{9} = \frac{\sin(40^\circ)}{c} \\ \Rightarrow\quad & \sin(A) \approx \frac{9\cdot\sin(40^\circ)}{8.41} \\ \Rightarrow\quad & A \approx \sin^{-1}\left(\frac{9\cdot\sin(40^\circ)}{8.41}\right) \\ \Rightarrow\quad & A \approx 43.46^\circ \end{aligned}$$

Finally, we can use the fact that the sum of the angles in a triangle is 180° to find B:

$$\begin{aligned} B &= 180^\circ - 40^\circ - A \\ &\approx 180^\circ - 40^\circ - 43.46^\circ \\ &\approx 96.54^\circ \end{aligned}$$

7. g. $A = 40^\circ$, $a = 11$, $c = 8$

First, let's draw a sketch of the situation:

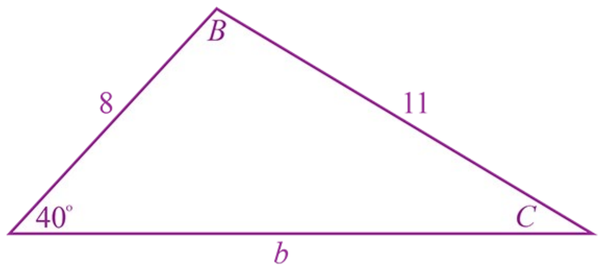


Figure 13

To solve the triangle, we can start by using the Law of Sines to find C:

$$\frac{\sin(C)}{8} = \frac{\sin(40^\circ)}{11}$$

$$\Rightarrow \quad \sin(C) = \frac{8 \cdot \sin(40^\circ)}{11}$$

$$\Rightarrow \quad C = \sin^{-1}\left(\frac{8 \cdot \sin(40^\circ)}{11}\right) \approx 27.87^\circ$$

Now we can use the fact that the sum of the angles in a triangle is 180° to find B:

$$\begin{aligned} B &= 180^\circ - 40^\circ - C \\ &\approx 180^\circ - 40^\circ - 27.87^\circ \\ &\approx 112.13^\circ \end{aligned}$$

Finally, we can use the Law of Sines to find b:

$$\frac{b}{\sin(B)} = \frac{11}{\sin(40^\circ)}$$

$$\Rightarrow \quad b = \frac{11 \cdot \sin(B)}{\sin(40^\circ)}$$

$$\Rightarrow \quad b \approx \frac{11 \cdot \sin(112.13^\circ)}{\sin(40^\circ)} \approx 15.85$$

7. h. $A = 28^\circ$, $a = 7$, $c = 12$

First, let's draw a sketch of the situation:

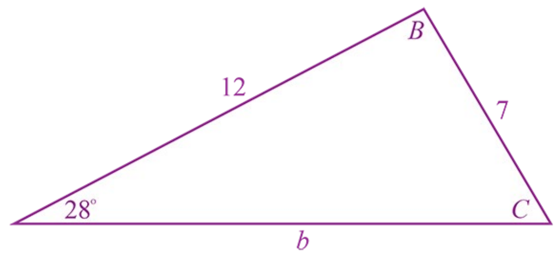


Figure 14

Since $7 < 12$, the side of length 7 units can pivot at angle B without changing any of the known info, i.e., there are two different triangles that satisfy the given information. In Figure 15, we've drawn both of these triangles, one green and one red. (Compare this situation to **1a**, **1b**, **1c**, and **1d** where any attempt to pivot any of the sides would distort some of the given information so such pivoting isn't possible and there's only one possible triangle.)

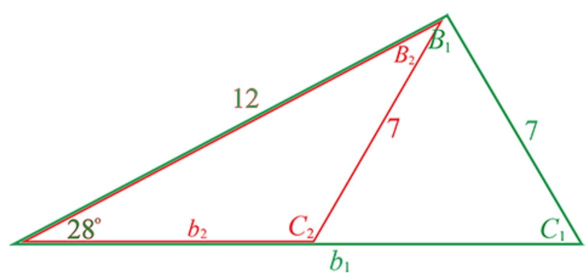


Figure 15

Now we can find the values of C_1 and C_2 using the Law of Sines. At the start, we'll represent the angle with the generic symbol "C" and wait to use subscripts until we're differentiating between the two possible values.

$$\frac{\sin(C)}{12} = \frac{\sin(28^\circ)}{7}$$

$$\Rightarrow \quad \sin(C) = \frac{12 \cdot \sin(28^\circ)}{7}$$

$$\Rightarrow \qquad C = \sin^{-1}\left(\frac{12 \cdot \sin(28^\circ)}{7}\right)$$

Now we can start distinguishing between C_1 and C_2 so that we can find these two angles. The largest angle that inverse-sine can output is 90° but C_2 in the red triangle in Figure 15 is clearly greater than 90°: to find C_2 we'll need to employ the identity $\sin(\theta) = \sin(180^\circ - \theta)$:

$$C_1 = \sin^{-1}\left(\frac{12 \cdot \sin(28^\circ)}{7}\right) \approx 53.59^\circ \qquad \text{or} \qquad C_2 = 180^\circ - \sin^{-1}\left(\frac{12 \cdot \sin(28^\circ)}{7}\right) \approx 126.41^\circ$$

To finish, we can solve for the two possible triangles:

Possibility 1: The Green Triangle

$$C_1 \approx 53.59^\circ$$

$$\begin{aligned} B_1 &= 180^\circ - 28^\circ - C_1 \\ &\approx 180^\circ - 28^\circ - 53.59^\circ \\ &\approx 98.41^\circ \end{aligned}$$

$$\frac{b_1}{\sin(B_1)} = \frac{7}{\sin(28^\circ)}$$

$$\Rightarrow \quad b_1 = \frac{7 \cdot \sin(B_1)}{\sin(28^\circ)}$$

$$\Rightarrow \quad b_1 \approx \frac{7 \cdot \sin(98.41^\circ)}{\sin(28^\circ)} \approx 14.75$$

Possibility 2: The Red Triangle

$$C_2 \approx 126.41^\circ$$

$$\begin{aligned} B_2 &= 180^\circ - 28^\circ - C_2 \\ &\approx 180^\circ - 28^\circ - 126.41^\circ \\ &\approx 25.59^\circ \end{aligned}$$

$$\frac{b_2}{\sin(B_2)} = \frac{7}{\sin(28^\circ)}$$

$$\Rightarrow \quad b_2 = \frac{7 \cdot \sin(B_2)}{\sin(28^\circ)}$$

$$\Rightarrow \quad b_2 \approx \frac{7 \cdot \sin(25.59^\circ)}{\sin(28^\circ)} \approx 6.44$$

7. i. $C = 67^\circ$, $a = 8$, $c = 5$

First, let's draw a sketch of the situation – the sketch shows an "impossible to create triangle" but below we'll use math to discover that this is in fact the case.

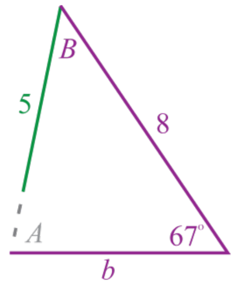


Figure 16

Approaching the situation without assuming it's impossible, we would aim for finding since we have an "angle and side opposite" pair to work with, and we know "a".

$$\frac{\sin(A)}{8} = \frac{\sin(67^\circ)}{5}$$

$$\Rightarrow \qquad A = \sin^{-1}\left(\frac{8\cdot\sin(67^\circ)}{5}\right)$$

If you ask a calculator to give you this value, it will tell you it's "**undefined**" – it turns out this is because the given conditions are impossible.

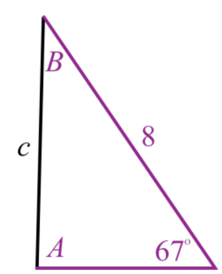
The reason $\sin^{-1}\left(\frac{8\cdot\sin(67^\circ)}{5}\right)$ is undefined is that $\frac{8\cdot\sin(67^\circ)}{5}$ is greater than 1 (so it's outside the domain of arcsine), and the reason that this fraction is "too big" is that "5" is "too small" – if the denominator of the fraction were larger, the fraction would be smaller, and then it could be less than 1 and inside the domain of arcsine.

To verify that "$c = 5$" is too short for this side in a triangle with the other given conditions, let's determine the minimum length of this side. The shortest distance between a line an any point not on the line will be in a direction *perpendicular* to the line – this means that we'll want to imagine if A were a right angle (i.e., 90°) and see how long c would be, and that is the smallest possible value for the length of c.

$$\sin(67^\circ) = \frac{c}{8}$$

$$\Rightarrow \qquad c = 8\cdot\sin(67^\circ)$$

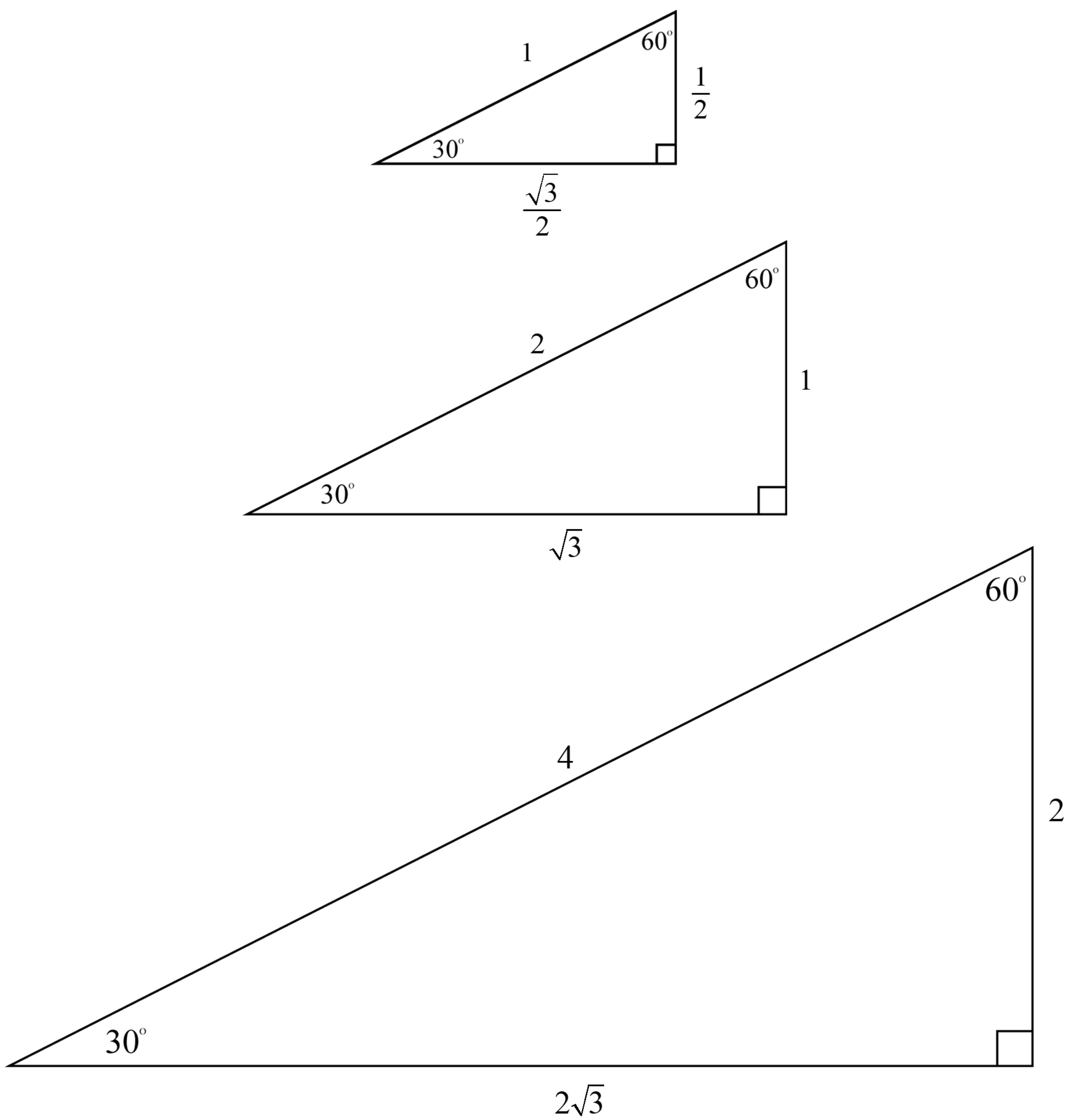
$$\Rightarrow \qquad c \approx 7.36$$



So c needs to be *at least* 7.36 or it isn't long enough, so the given "$c = 5$" results in an impossible situation

8. As you know, a triangle has three sides and three angles: these are the "six components of a triangle." Notice that in each part of the previous problem (#5) you are given the measurements of **three** of these six components. Sometimes (like in part (a)) you are given the lengths of all three sides; usually you are given a combination of sides and angles; but you aren't ever given all three angle measures: contemplate, discuss, and explain why.

The three angles in a triangle establish the *shape* of the triangle but, without knowing the length of at least one of the sides, we know nothing about the *size* of the triangle. For example, if the three facts we know about a triangle are that it has angles 30°, 60°, and 90°, the triangle could be any one of *infinitely many* triangles that have these three angles; below are three *different* triangles that all have the same three angles 30°, 60°, and 90°.



9. Prove the following identities. (Be sure to organize your proof as shown in the Online Lecture Notes. This means that you should start your proof by writing one side of the identity and then use equal signs between equivalent expressions until you obtain the other side of the identity. You should only include one step on each line and you should align your equal signs on the left of each step. Compare your proofs with those given in the solutions to make sure that you are using the correct organization and technique.)

a. $\tan(x)\sec(x) = \sin(x)\sec^2(x)$

Both sides of this equation are similarly "complicated" so it probably doesn't matter which side we start on; we'll start with the left side:

$$\begin{aligned}\tan(x)\sec(x) &= \frac{\sin(x)}{\cos(x)}\cdot\frac{1}{\cos(x)} \\ &= \sin(x)\cdot\frac{1}{\cos(x)}\cdot\frac{1}{\cos(x)} \quad \text{(reorganizing the terms to create what we need)} \\ &= \sin(x)\cdot\frac{1}{\cos^2(x)} \\ &= \sin(x)\sec^2(x)\end{aligned}$$

b. $\csc(t) - \sin(t) = \cot(t)\cos(t)$

Let's start with the left side of the identity since it involves subtraction which allows us to begin by performing the subtraction and combining the expressions being subtracted:

$$\begin{aligned}\csc(t) - \sin(t) &= \frac{1}{\sin(t)} - \sin(t)\cdot\frac{\sin(t)}{\sin(t)} \\ &= \frac{1}{\sin(t)} - \frac{\sin^2(t)}{\sin(t)} \\ &= \frac{1-\sin^2(t)}{\sin(t)} \\ &= \frac{\cos^2(t)}{\sin(t)} \\ &= \frac{\cos(t)}{\sin(t)}\cdot\frac{\cos(t)}{1} \\ &= \cot(t)\cos(t)\end{aligned}$$

c. $\dfrac{\sec(\theta)}{\sin(\theta)} - \tan(\theta) = \cot(\theta)$

Let's start with the left side of the identity since it involves addition which allows us to begin by performing the addition and combining the expressions being added:

$$\frac{\sec(\theta)}{\sin(\theta)} - \tan(\theta) = \frac{\frac{1}{\cos(\theta)}}{\sin(\theta)} - \frac{\sin(\theta)}{\cos(\theta)} \quad \text{(using definition of } \sec(\theta) \text{ and } \tan(\theta)\text{)}$$

$$= \frac{1}{\cos(\theta)\sin(\theta)} - \frac{\sin(\theta)}{\cos(\theta)} \cdot \frac{\sin(\theta)}{\sin(\theta)} \quad \text{(creating a common denominator)}$$

$$= \frac{1}{\cos(\theta)\sin(\theta)} - \frac{\sin^2(\theta)}{\cos(\theta)\sin(\theta)} \quad \text{(subtracting the fractions)}$$

$$= \frac{1 - \sin^2(\theta)}{\cos(\theta)\sin(\theta)}$$

$$= \frac{\cos^2(\theta)}{\cos(\theta)\sin(\theta)} \quad \text{(since } 1 - \sin^2(\theta) = \cos^2(\theta)\text{)}$$

$$= \frac{\cos(\theta)}{\sin(\theta)} \quad \text{(simplifying)}$$

$$= \cot(\theta) \quad \text{(since } \frac{\cos(\theta)}{\sin(\theta)} = \cot(\theta)\text{)}$$

d. $\dfrac{1}{1 - \cos(x)} - \dfrac{1}{1 + \cos(x)} = 2\cot(x)\csc(x)$

Let's start with the left side of the identity since it involves subtraction which allows us to begin by performing the subtraction and combining the expressions.

$$\frac{1}{1 - \cos(x)} - \frac{1}{1 + \cos(x)} = \frac{1}{1 - \cos(x)} \cdot \frac{1 + \cos(x)}{1 + \cos(x)} - \frac{1}{1 + \cos(x)} \cdot \frac{1 - \cos(x)}{1 - \cos(x)}$$

$$= \frac{1 + \cos(x) - \left(1 - \cos(x)\right)}{1 - \cos^2(x)}$$

$$= \frac{1 + \cos(x) - 1 + \cos(x)}{1 - \cos^2(x)}$$

$$= \frac{2\cos(x)}{\sin^2(x)}$$

$$= 2 \cdot \frac{\cos(x)}{\sin(x)} \cdot \frac{1}{\sin(x)}$$

$$= 2\cot(x)\csc(x)$$

e. $\sec(\theta) + \tan(\theta) = \dfrac{\cos(\theta)}{1 - \sin(\theta)}$

Let's start with the left side of the identity since it involves addition which allows us to begin by performing the addition and combining the expressions being added:

$$\begin{aligned}\sec(\theta) + \tan(\theta) &= \frac{1}{\cos(\theta)} + \frac{\sin(\theta)}{\cos(\theta)} \\ &= \frac{1 + \sin(\theta)}{\cos(\theta)}\end{aligned}$$

At this point, I'm stuck since there's nothing obvious that can be done to $\frac{1 + \sin(\theta)}{\cos(\theta)}$ to manipulate it. One good strategy to employ when you get stuck is to try working with the other side of the identity but, in this case, the other side of the identity presents a similar situation: there's nothing obvious that can be done to $\frac{\cos(\theta)}{1 - \sin(\theta)}$ to manipulate it. Another strategy to try when you get stuck is to "use conjugates". The conjugate of the expression $a + b$ is the expression $a - b$, and one thing that's special about conjugates is that their product is a difference of squares: $(a + b)\cdot(a - b) = a^2 - b^2$. This comes in handy when working with sine and cosine since they're related by the Pythagorean Identity which can be used to create differences of squares. To make this more meaningful, let's consider an example: Notice that "$1 + \sin(\theta)$" is in the numerator of the expression in our last step of our proof; the conjugate of that expression is "$1 - \sin(\theta)$". Let's compute the product of these conjugates:

$$\begin{aligned}\big(1 + \sin(\theta)\big)\cdot\big(1 - \sin(\theta)\big) &= 1 - \sin^2(\theta) \\ &= \cos^2(\theta)\end{aligned}$$

So, the product of these conjugates leads us to "$\cos^2(\theta)$"! Now let's try using conjugates to finish our proof; we'll re-write the first two steps:

$$\begin{aligned}\sec(\theta) + \tan(\theta) &= \frac{1}{\cos(\theta)} + \frac{\sin(\theta)}{\cos(\theta)} \\ &= \frac{1 + \sin(\theta)}{\cos(\theta)}\cdot\frac{1 - \sin(\theta)}{1 - \sin(\theta)} \\ &= \frac{1 - \sin^2(\theta)}{\cos(\theta)\cdot\big(1 - \sin(\theta)\big)} \\ &= \frac{\cos^{\not 2}(\theta)}{\cancel{\cos(\theta)}\cdot\big(1 - \sin(\theta)\big)} \\ &= \frac{\cos(\theta)}{1 - \sin(\theta)}\end{aligned}$$

f. $\cot(A) = \csc(A)\sec(A) - \tan(A)$

Let's start with the right side of the identity since it involves subtraction that we can perform to begin our proof:

$$\begin{aligned}
\csc(A)\sec(A) - \tan(A) &= \frac{1}{\sin(A)}\cdot\frac{1}{\cos(A)} - \frac{\sin(A)}{\cos(A)} \\
&= \frac{1}{\sin(A)\cos(A)} - \frac{\sin(A)}{\cos(A)}\cdot\frac{\sin(A)}{\sin(A)} \\
&= \frac{1}{\sin(A)\cos(A)} - \frac{\sin^2(A)}{\sin(A)\cos(A)} \\
&= \frac{1-\sin^2(A)}{\sin(A)\cos(A)} \\
&= \frac{\cos^{\not{2}}(A)}{\sin(A)\,\cancel{\cos(A)}} \\
&= \frac{\cos(A)}{\sin(A)} \\
&= \cot(A)
\end{aligned}$$