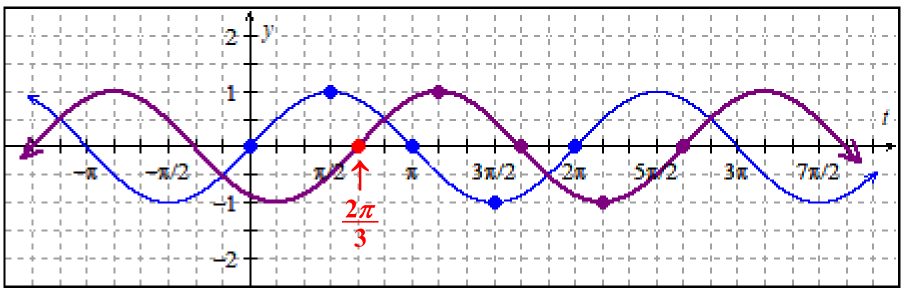
SOLUTIONS: Week 3 Practice Worksheet

Graphs of Trig Functions, Inverse Trig Functions, and Solving Trig Equations

1. The graph of $y = \sin(t)$ is given below with six key points emphasized. As we know, the graph of $y = \sin\left(t - \frac{2\pi}{3}\right)$ is a *transformation* of $y = \sin(t)$. Use what we know about graph transformations to "transform" the six key points on $y = \sin(t)$ and then connect these points in order to construct a graph of $y = \sin\left(t - \frac{2\pi}{3}\right)$.

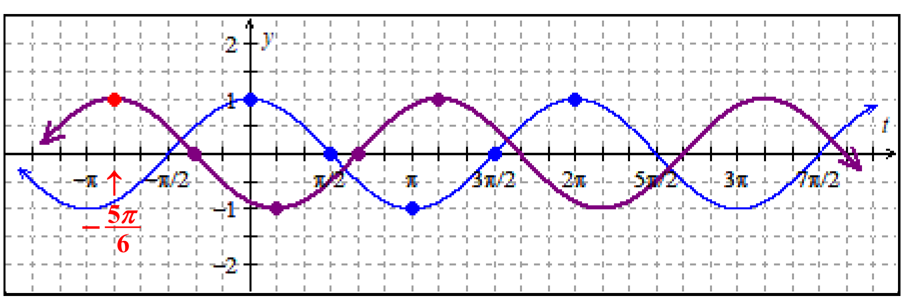
Compared with $y = \sin(t)$, $y = \sin\left(t - \frac{2\pi}{3}\right)$ is shifted right $\frac{2\pi}{3}$ units. Notice that $y = \sin\left(t - \frac{2\pi}{3}\right)$ appears to be a sine wave that "starts" at $t = \frac{2\pi}{3}$.



The graphs of $y = \sin(t)$ and $y = \sin\left(t - \frac{2\pi}{3}\right)$

2. The graph of $y = \cos(t)$ is given below with **five** key points emphasized. As we know, the graph of $y = \cos\left(t + \frac{5\pi}{6}\right)$ is a *transformation* of $y = \cos(t)$. Use what we know about graph transformations to "transform" the **five** key points on $y = \cos(t)$ and then connect these points in order to construct a graph of $y = \cos\left(t + \frac{5\pi}{6}\right)$.

Compared with $y = \cos(t)$, $y = \cos\left(t + \frac{5\pi}{6}\right)$ is shifted left $\frac{5\pi}{6}$ units. Notice that $y = \cos\left(t + \frac{5\pi}{6}\right)$ appears to be a cosine wave that "starts" at $t = -\frac{5\pi}{6}$.

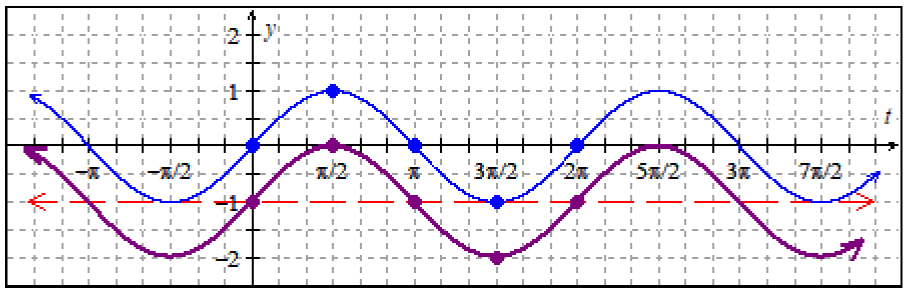


The graphs of $y = \cos(t)$ and $y = \cos\left(t + \frac{5\pi}{6}\right)$.

3. The graph of $y = \sin(t)$ is given below with **five key points** emphasized. As we know, the graph of $y = \sin(t) - 1$ is a *transformation* of $y = \sin(t)$. Use what we know about graph transformations to "transform" the **five** key points on $y = \sin(t)$ and then connect these points in order to construct a graph of $y = \sin(t) - 1$.

Compared with $y = \sin(t)$, $y = \sin(t) - 1$ is shifted down 1 unit.

Notice that the graph of $y = \sin(t) - 1$ has midline $y = -1$ (so the vertical shift produces the midline).

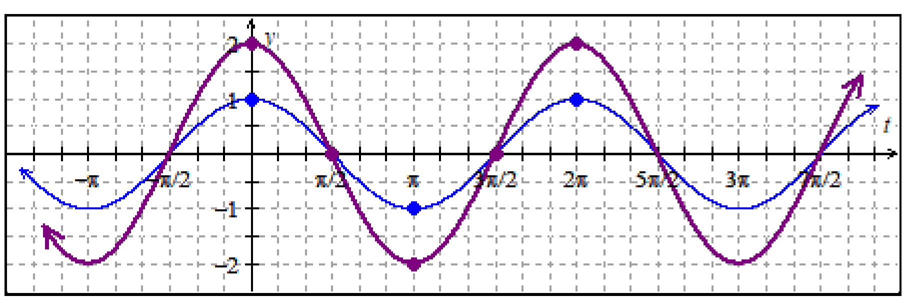


The graphs of $y = \sin(t)$ and $y = \sin(t) - 1$.

4. The graph of $y = \cos(t)$ is given below with **five key points** emphasized. As we know, the graph of $y = 2\cos(t)$ is a *transformation* of $y = \cos(t)$. Use what we know about graph transformations to "transform" the **five key points** on $y = \cos(t)$ and then connect these points in order to construct a graph of $y = 2\cos(t)$.

Compared with $y = \cos(t)$, $y = 2\cos(t)$ is stretched vertically by a factor of 2.

Notice that the graph of $y = 2\cos(t)$ has amplitude 2 units (so the vertical stretch produces the amplitude).



The graphs of $y = \cos(t)$ and $y = 2\cos(t)$.

5. Scale the axes on the given coordinate plane an appropriately for a graph of $y = \sin(2t) + 3$ and then draw a graph of $y = \sin(2t) + 3$ by first plotting the points where the graph will intersect the midline and the points where the graph will reach maximum and minimum values, and then connect these points with an appropriately curved sinusoidal wave.

Compared with $y = \sin(t)$, $y = \sin(2t) + 3$ is compressed horizontally by a factor of $\frac{1}{2}$ and shifted up 3 units.

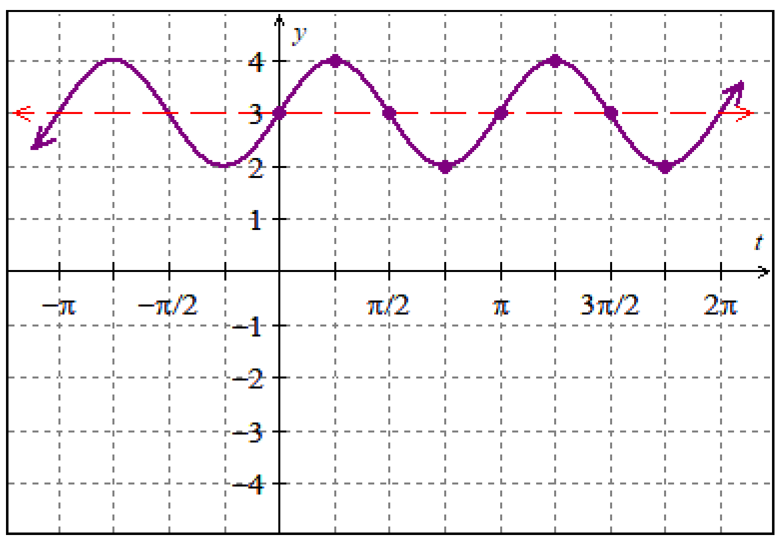
The horizontal compression means that the function has a period of

$$2\pi \cdot \frac{1}{2} = \pi \text{ units.}$$

The vertical shift means that the function has a midline of $y = 3$.

Since the algebraic rule for the function doesn't involve a vertical stretch, its amplitude is 1 unit.

Since the algebraic rule for the function doesn't involve a horizontal shift, its graph will appear like a sine wave "starts" at $t = 0$, i.e., its y-intercept will occur on the midline and the graph will be increasing there.



A graph of $y = \sin(2t) + 3$.

6. Scale the axes on the given coordinate plane an appropriately for a graph of $y = 3\cos(\pi t)$ and then draw a graph of $y = 3\cos(\pi t)$ by first plotting the points where the graph will intersect the midline and the points where the graph will reach maximum and minimum values, and then connect these points with an appropriately curved sinusoidal wave.

Compared with $y = \cos(t)$, $y = 3\cos(\pi t)$ is compressed horizontally by a factor of $\frac{1}{\pi}$ and stretched vertically by a factor of 3.

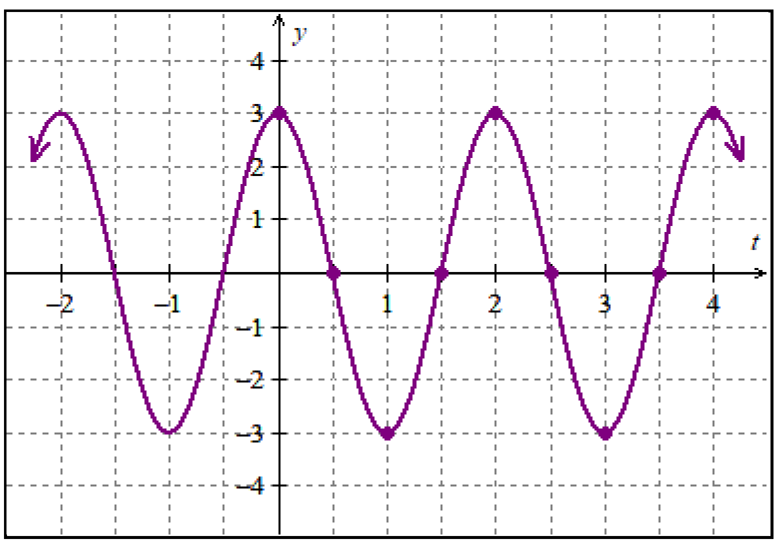
The horizontal compression means that the function has a period of

$$2\pi \cdot \frac{1}{\pi} = 2 \text{ units.}$$

The vertical stretch means that the function's amplitude is 3 units.

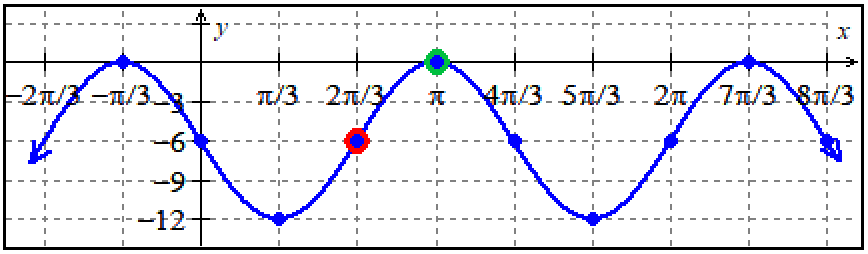
Since the algebraic rule for the function doesn't involve a vertical shift, it must have a midline of $y = 0$.

Since the algebraic rule for the function doesn't involve a horizontal shift, its graph will appear like a cosine wave that "starts" at $t = 0$; i.e., its y-intercept will occur at a maximum value for the function.



A graph of $y = 3\cos(\pi t)$.

7. Find two different algebraic rules for the sinusoidal function $y = p(x)$ graphed below. One of your rules should involve sine and the other should involve cosine.



The graph of $y = p(x)$.

First let's write a rule involving sine, so our rule will have the form $p(x) = A\sin\big(\omega(x-h)\big) + k$ and we need to determine the values of A, ω, h, and k.

- The midline is the line midway between the function's maximum and minimum output values. The function's maximum output value is 0 and its minimum output value is -12. Since -6 is the average of these values, the midline is $y = -6$ so $k = -6$.
- The amplitude is the distance between the function's maximum output value, 0, and its midline $y = -6$, which is 6 units. Therefore, $|A| = 6$.
- The function completes one period between $x = \frac{2\pi}{3}$ and $x = 2\pi$. Thus, the period of the function is $2\pi - \frac{2\pi}{3} = \frac{4\pi}{3}$. To find ω we need to solve $\frac{4\pi}{3} = 2\pi \cdot \frac{1}{\omega}$:

$$\begin{aligned} \frac{4\pi}{3} &= 2\pi \cdot \frac{1}{\omega} \\ \Rightarrow \quad \omega &= 2\pi \cdot \frac{1}{4\pi/3} \\ \Rightarrow \quad \omega &= 2\pi \cdot \frac{3}{4\pi} \\ \Rightarrow \quad \omega &= \frac{3}{2} \end{aligned}$$

- Near the y-axis, the graph of $y = \sin(x)$ is increasing and passes through its midline, so we need to look for a spot in the graph of $y = p(x)$ where it shows this behavior, and one such spot is at $x = \frac{2\pi}{3}$ (this point has been highlighted in **red** in the graph above) so we can so consider this graph a sine wave shifted right $\frac{2\pi}{3}$ units and use $h = \frac{2\pi}{3}$.

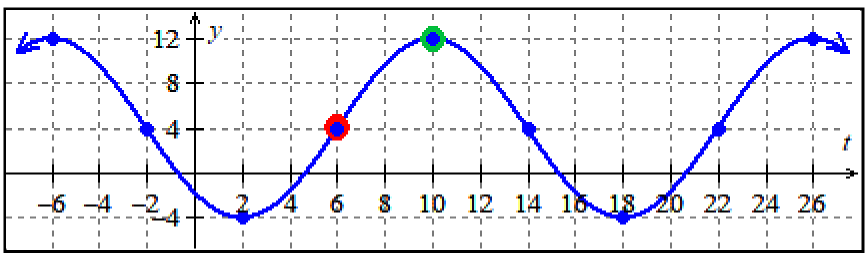
Therefore, an algebraic rule for the graphed function is $p(x) = 6\sin\left(\frac{3}{2}\left(x - \frac{2\pi}{3}\right)\right) - 6$.

Now we'll write a rule involving cosine.

Since we want to use cosine to construct our rule, it will have the form $p(x) = A\cos\big(\omega(x-h)\big)+k$. Since the amplitude, period, and midline aren't dependent on whether we use sine or cosine in our algebraic rule, we can use the same values for A, ω, and k that we used above. So we only need to determine an appropriate horizontal shift, h, that works for cosine. Near the y-axis, the graph of $y=\cos(t)$ is reaches its maximum value, so we need to look for a spot in the graph of $y = p(x)$ where it shows this behavior, and one such spot is at $x=\pi$ (this point has been highlighted in **green** in the graph above) so we can consider this graph a cosine wave shifted right π units and use $h=\pi$.

Therefore, an algebraic rule for the graphed function is $p(x) = 6\cos\left(\frac{3}{2}(x-\pi)\right)-6$.

8. Find two different algebraic rules for the sinusoidal function $y = q(t)$ graphed below. One of your rules should involve sine and the other should involve cosine.



The graph of $y = q(t)$.

First let's write a rule involving sine, so our rule will have the form $q(t) = A\sin\big(\omega(t-h)\big)+k$ and we need to determine the values of A, ω, h, and k.

- The midline is the line midway between the function's maximum and minimum output values. The function's maximum output value is 12 and its minimum output value is -4. Since 4 is the average of these values, the midline is $y=4$ so $k=4$.
- The amplitude is the distance between the function's maximum output value, 12, and its midline $y=4$, which is 8 units. Therefore, $|A|=8$.
- The function completes one period between $t=6$ and $t=22$. Thus, the period of the function is $22-6=16$. To find ω we need to solve $16 = 2\pi\cdot\frac{1}{\omega}$:

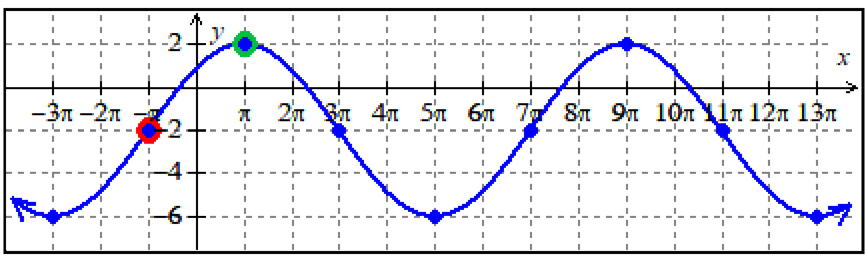
$$16 = 2\pi\cdot\frac{1}{\omega}$$
$$\Rightarrow \quad \omega = 2\pi\cdot\frac{1}{16} = \frac{\pi}{8}$$

- Near the y-axis, the graph of $y=\sin(t)$ is increasing and passes through its midline, so we need to look for a spot in the graph of $y=q(t)$ where it shows this behavior, and one such spot is at $t=6$ (this point has been highlighted in **red** in the graph above) so we can consider this graph a sine wave shifted right 6 units and use $h=6$.

- Therefore, an algebraic rule for the graphed function is $q(t) = 8\sin\left(\frac{\pi}{8}(t-6)\right) + 4$.

Now we'll write a rule involving cosine, so our rule will have the form $q(t) = A\cos\left(\omega(t-h)\right) + k$. Since the amplitude, period, and midline aren't dependent on whether we use sine or cosine in our algebraic rule, we can use the same values for A, ω, and k that we used above. So we only need to determine an appropriate horizontal shift, h, that works for cosine. Near the y-axis, the graph of $y = \cos(t)$ is reaches its maximum value, so we need to look for a spot in the graph of $y = q(t)$ where it shows this behavior, and one such spot is at $t = 10$ (this point has been highlighted in **green** in the graph above) so we can consider this graph a cosine wave shifted right 10 units and use $h = 10$. Therefore, an algebraic rule for the graphed function is $q(t) = 8\cos\left(\frac{\pi}{8}(t-10)\right) + 4$.

9. Find two different algebraic rules for the sinusoidal function $y = m(x)$ graphed below. One of your rules should involve sine and the other should involve cosine.



The graph of $y = m(x)$.

First let's write a rule involving sine, so our rule will have the form $m(x) = A\sin\left(\omega(x-h)\right) + k$ and we need to determine the values of A, ω, h, and k.

- The midline is the line midway between the function's maximum and minimum output values. The function's maximum output value is 2 and its minimum output value is -6. Since -2 is the average of these values, the midline is $y = -2$ so $k = -2$.

- The amplitude is the distance between the function's maximum output value, 2, and its midline $y = -2$, which is 4 units. Therefore, $|A| = 4$.

- The function completes one period between $x = \pi$ and $x = 9\pi$. Thus, the period of the function is $9\pi - \pi = 8\pi$. To find ω we need to solve $8\pi = 2\pi \cdot \frac{1}{\omega}$:

$$8\pi = 2\pi \cdot \frac{1}{\omega}$$
$$\Rightarrow \quad \omega = 2\pi \cdot \frac{1}{8\pi}$$
$$\Rightarrow \quad \omega = \frac{1}{4}$$

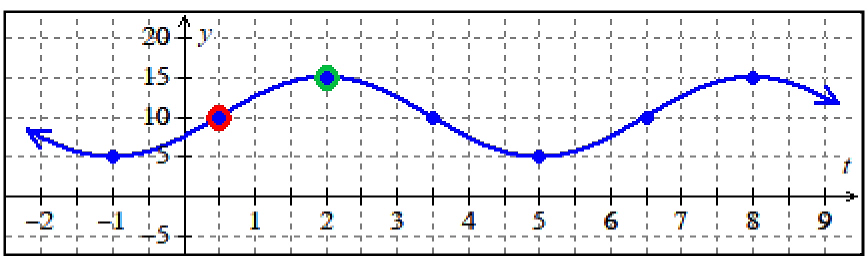
- Near the y-axis, the graph of $y = \sin(x)$ is increasing and passes through its midline, so we need to look for a spot in the graph of $y = m(x)$ where it shows this behavior, and one such spot is at $x = -\pi$ (this point has been highlighted in **red** in the graph above) so we can so consider this graph a sine wave shifted left π units and use $h = -\pi$.

Therefore, an algebraic rule for the graphed function is $m(x) = 4\sin\left(\frac{1}{4}(x - (-\pi))\right) - 2$ which simplifies as $m(x) = 4\sin\left(\frac{1}{4}(x + \pi)\right) - 2$.

Now we'll write a rule involving cosine, so our rule will have the form $m(x) = A\cos(\omega(x - h)) + k$. Since the amplitude, period, and midline aren't dependent on whether we use sine or cosine in our algebraic rule, we can use the same values for A, ω, and k that we used above. So we only need to determine an appropriate horizontal shift, h, that works for cosine. Near the y-axis, the graph of $y = \cos(t)$ is reaches its maximum value, so we need to look for a spot in the graph of $y = m(x)$ where it shows this behavior, and one such spot is at $x = \pi$ (this point has been highlighted in **green** in the graph above) so we can consider this graph a cosine wave shifted right π units and use $h = \pi$.

Therefore, an algebraic rule for the graphed function is $m(x) = 4\cos\left(\frac{1}{4}(x - \pi)\right) - 2$.

10. Find two different algebraic rules for the sinusoidal function $y = n(t)$ graphed below. One of your rules should involve sine and the other should involve cosine.



The graph of $y = n(t)$.

First let's write a rule involving sine, so our rule will have the form $n(t) = A\sin(\omega(t - h)) + k$ and we need to determine the values of A, ω, h, and k.

- The midline is the line midway between the function's maximum and minimum output values. The function's maximum output value is 15 and its minimum output value is 5. Since 10 is the average of these values, the midline is $y = 10$ so $k = 10$.

- The amplitude is the distance between the function's maximum output value, 15, and its midline $y = 10$, which is 5 units. Therefore, $|A| = 5$.

- The function completes one period between $t = 2$ and $t = 8$. Thus, the period of the function is $8 - 2 = 6$. To find ω we need to solve $6 = 2\pi \cdot \frac{1}{\omega}$:

$$6 = 2\pi \cdot \frac{1}{\omega}$$
$$\Rightarrow \quad \omega = 2\pi \cdot \frac{1}{6} = \frac{\pi}{3}$$

- Near the y-axis, the graph of $y = \sin(t)$ is increasing and passes through its midline, so we need to look for a spot in the graph of $y = n(t)$ where it shows this behavior, and one such spot is at $t = \frac{1}{2}$ (this point has been highlighted in **red** in the graph above) so we can consider this graph a sine wave shifted right $\frac{1}{2}$ of a unit and use $h = \frac{1}{2}$.

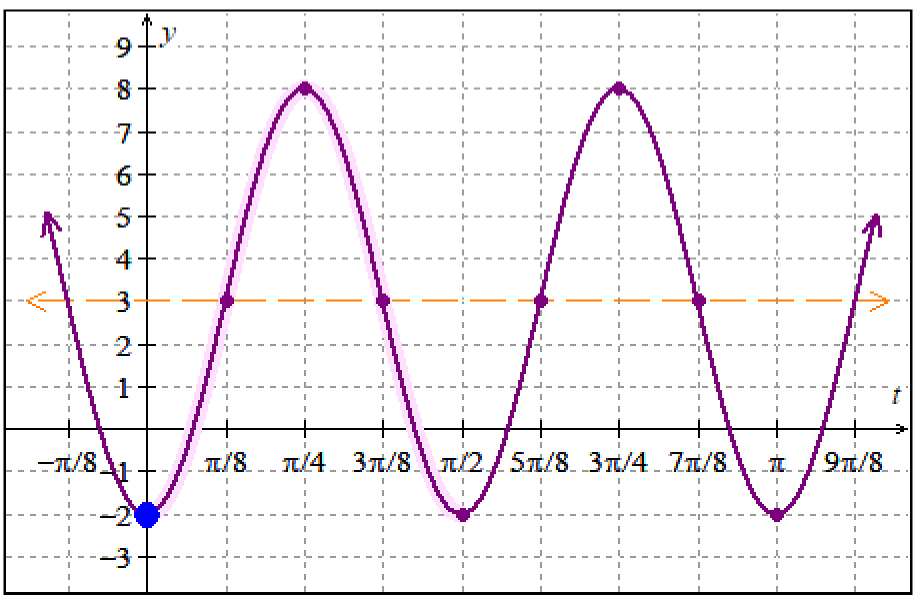
- Therefore, an algebraic rule for the graphed function is $n(t) = 5\sin\left(\frac{\pi}{3}\left(t - \frac{1}{2}\right)\right) + 10$.

Now we'll write a rule involving cosine, so our rule will have the form $n(t) = A\cos\left(\omega(t - h)\right) + k$. Since the amplitude, period, and midline aren't dependent on whether we use sine or cosine in our algebraic rule, we can use the same values for A, ω, and k that we used above. So we only need to determine an appropriate horizontal shift, h, that works for cosine. Near the y-axis, the graph of $y = \cos(t)$ is reaches its maximum value, so we need to look for a spot in the graph of $y = n(t)$ where it shows this behavior, and one such spot is at $t = 2$ (this point has been highlighted in **green** in the graph above) so we can consider this graph a cosine wave shifted right 2 units and use $h = 2$. Therefore, an algebraic rule for the graphed function is $n(t) = 5\cos\left(\frac{\pi}{3}(t - 2)\right) + 10$.

11. Draw a graph of at least two periods of the functions in **a–f** below by first *plotting the points* where the graph will intersect the midline and *plotting the points* where the graph will reach maximum and minimum values, and then *connect these points* with an appropriately curved sinusoidal wave. List the period, midline, and amplitude of each function. (Be sure to label the scale on the axes of your graph.)

a. $f(t) = -5\cos(4t) + 3$

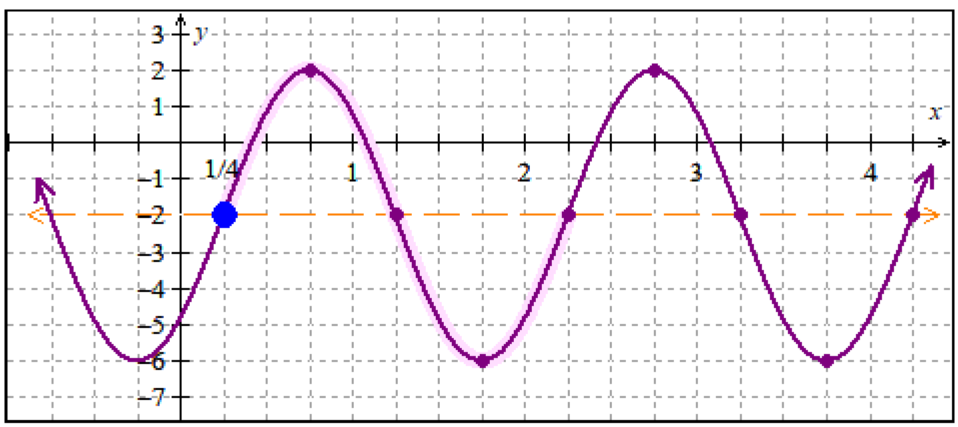
- $|A| = |-5| = 5$ so the **amplitude** is 5 units. Since $A < 0$, we'll need to draw a "reflected cosine wave".
- $k = 3$ so the **midline** is $y = 3$.
- $\omega = 4$ so the **period** is $2\pi \cdot \frac{1}{4} = \frac{\pi}{2}$ units.
- There is no horizontal shift so we'll "start" a reflected cosine wave on the y-axis and make sure it has the appropriate midline, amplitude, and period; we've highlighted the "first" period in pink. (Since this is a "reflected cosine wave", it needs to start at a minimum output value rather than at its maximum output value like $y = \cos(t)$.)



A graph of $f(t) = -5\sin(4t) + 3$.

b. $g(x) = 4\sin\left(\pi\left(x - \frac{1}{4}\right)\right) - 2$

- $|A| = |4| = 4$ so the **amplitude** is 4 units.
- $k = -2$ so the **midline** is $y = -2$.
- $\omega = \pi$ so the **period** is $2\pi \cdot \frac{1}{\pi} = 2$ units.
- $h = \frac{1}{4}$ so the horizontal shift is $\frac{1}{4}$ units to the right, so we'll "start" a sine wave at $x = \frac{1}{4}$ and make sure it has the appropriate midline, amplitude, and period; we've highlighted the "first" period in pink.

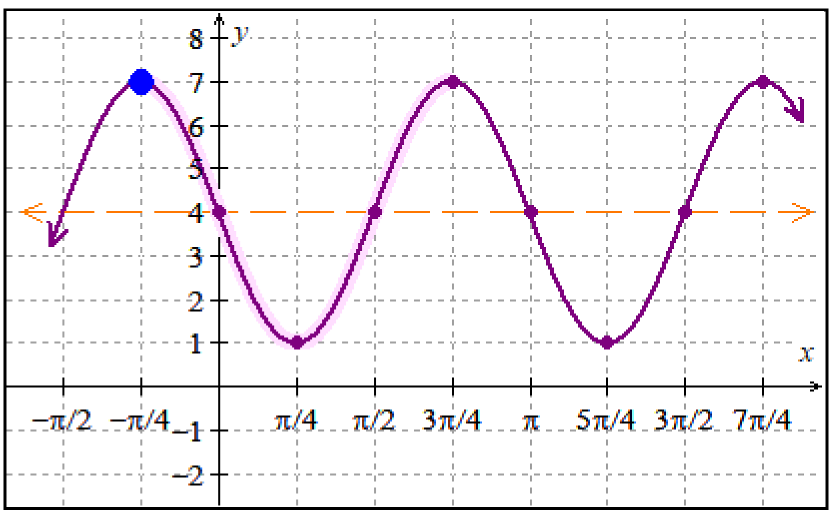


A graph of $g(x) = 4\sin\left(\pi\left(x - \frac{1}{4}\right)\right) - 2$.

c. $G(x) = 3\cos\left(2x + \frac{\pi}{2}\right) + 4$

$= 3\cos\left(2\left(x - \left(-\frac{\pi}{4}\right)\right)\right) + 4$

- $|A| = |3| = 3$ so the **amplitude** is 3 units.
- $k = 4$ so the **midline** is $y = 4$.
- $\omega = 2$ so the **period** is $2\pi \cdot \frac{1}{2} = \pi$ units.
- $h = -\frac{\pi}{4}$ so the horizontal shift is $\frac{\pi}{4}$ units to the left, so we'll "start" a cosine wave at $x = -\frac{\pi}{4}$ and make sure it has the appropriate midline, amplitude, and period; we've highlighted the "first" period in pink.

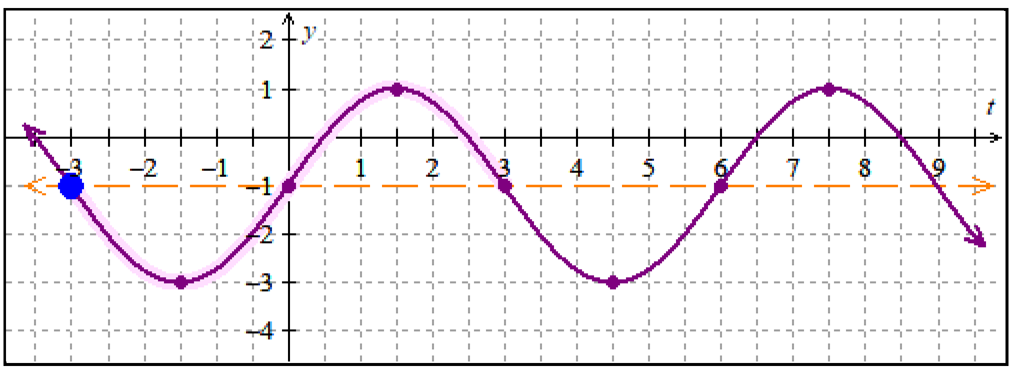


A graph of $G(x) = 3\cos\left(2x + \frac{\pi}{2}\right) + 4$.

d. $F(t) = -2\sin\left(\frac{\pi}{3}t + \pi\right) - 1$

$$= -2\sin\left(\frac{\pi}{3}(t - (-3))\right) - 1$$

- $|A| = |-2| = 2$ so the **amplitude** is 2 units. Since $A < 0$, we'll need to draw a "reflected sine wave".
- $k = -1$ so the **midline** is $y = -1$.
- $\omega = \frac{\pi}{3}$ so the **period** is $2\pi \cdot \frac{1}{\pi/3} = 6$ units.
- $h = -3$ so the horizontal shift is 3 units to the left, so we'll "start" a reflected sine wave at $t = -3$ and make sure it has the appropriate midline, amplitude, and period; we've highlighted the "first" period in pink. (Since this is a "reflected sine wave", it needs to travel *down* from its starting point at its midline.)

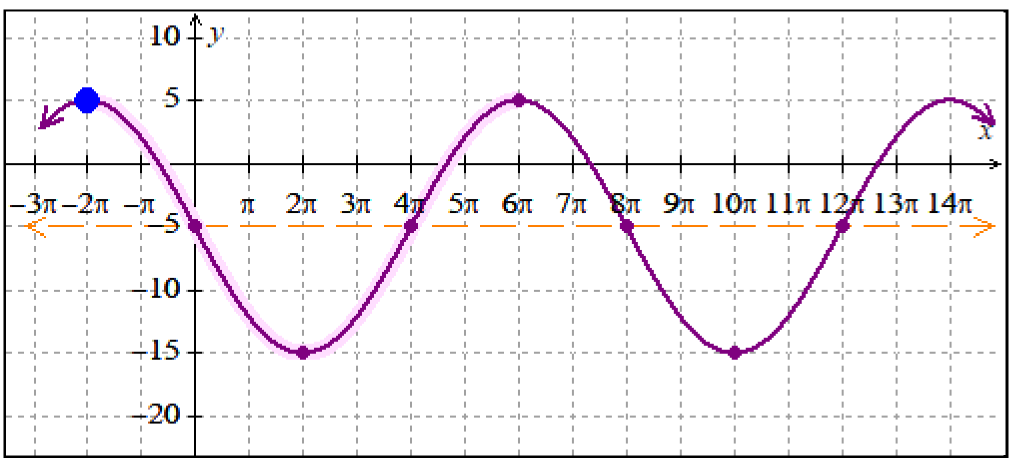


A graph of $F(t) = -2\sin\left(\frac{\pi}{3}t + \pi\right) - 1$.

e. $p(x) = 10\cos\left(\dfrac{x+2\pi}{4}\right) - 5$

$= 10\cos\left(\frac{1}{4}\left(x - (-2\pi)\right)\right) - 5$

- $|A| = |10| = 10$ so the **amplitude** is 10 units.
- $k = -5$ so the **midline** is $y = -5$.
- $\omega = \frac{1}{4}$ so the **period** is $2\pi \cdot \frac{1}{1/4} = 8\pi$ units.
- $h = -2\pi$ so the horizontal shift is 2π units to the left, so we'll "start" a cosine wave at $x = -2\pi$ and make sure it has the appropriate midline, amplitude, and period; we've highlighted the "first" period in pink.

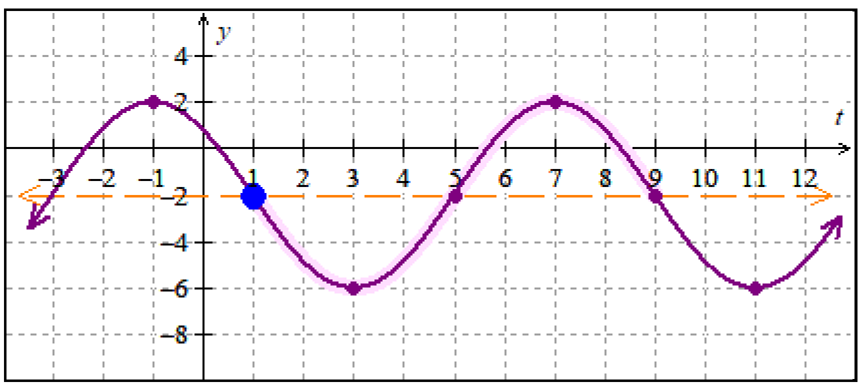


A graph of $p(x) = 10\cos\left(\dfrac{x+2\pi}{4}\right) - 5$.

f. $q(t) = -4\sin\left(\frac{\pi}{4}t - \frac{\pi}{4}\right) - 2$

$= -4\sin\left(\frac{\pi}{4}(t-1)\right) - 2$

- $|A| = |-4| = 4$ so the **amplitude** is 4 units. Since $A < 0$, we'll need to draw a "reflected sine wave".
- $k = -2$ so the **midline** is $y = -2$.
- $\omega = \frac{\pi}{4}$ so the **period** is $2\pi \cdot \frac{1}{\pi/4} = 8$ units.
- $h = 1$ so the horizontal shift is 1 units to the left, so we'll "start" a reflected sine wave at $t = 1$ and make sure it has the appropriate midline, amplitude, and period; we've highlighted the "first" period in pink. (Since this is a "reflected sine wave", it needs to travel *down* from its starting point at its midline.)



A graph of $q(t) = -4\sin\left(\frac{\pi}{4}t - \frac{\pi}{4}\right) - 2$.

12. Determine the domain of $y = \tan(t)$.

Since $\tan(t) = \frac{\sin(t)}{\cos(t)}$, $y = \tan(t)$ is undefined where $\cos(t) = 0$. Recall that $\cos(t) = 0$ when $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$ and all angles coterminal with these two angles. One way to express these values is like this:

$$\left\{t \,\middle|\, t = \tfrac{\pi}{2} + 2k\pi \text{ for all } k \in \mathbb{Z}\right\} \cup \left\{t \,\middle|\, t = \tfrac{3\pi}{2} + 2k\pi \text{ for all } k \in \mathbb{Z}\right\}$$

This set can be simplified and expressed as:

$$\left\{t \,\middle|\, t = \tfrac{\pi}{2} + k\pi \text{ for all } k \in \mathbb{Z}\right\}.$$

(To verify that this simplification is correct, choose a value represented in the unsimplified set and check if it's represented in the simplified set.)

The domain of $y = \tan(t)$ is all real numbers *except* the values listed above where $\cos(t) = 0$. Thus, the domain of $y = \tan(t)$ is:

$$\left\{t \,\middle|\, t \in \mathbb{R} \text{ and } t \neq \tfrac{\pi}{2} + k\pi \text{ for all } k \in \mathbb{Z}\right\}.$$

13. Determine the domain of $y = \sec(t)$.

Since $\sec(t) = \frac{1}{\cos(t)}$, $y = \sec(t)$ is undefined where $\cos(t) = 0$. In #12, we observed that these values can be represented by the following set:

$$\left\{t \,\middle|\, t = \tfrac{\pi}{2} + k\pi \text{ for all } k \in \mathbb{Z}\right\}.$$

The domain of $y = \sec(t)$ is all real numbers *except* the values listed above where $\cos(t) = 0$. Thus, the domain of $y = \sec(t)$ is:

$$\left\{t \,\middle|\, t \in \mathbb{R} \text{ and } t \neq \tfrac{\pi}{2} + k\pi \text{ for all } k \in \mathbb{Z}\right\}.$$

14. Determine the domain of $y = \cot(t)$.

Since $\cot(t) = \frac{\cos(t)}{\sin(t)}$, $y = \cot(t)$ is undefined where $\sin(t) = 0$. Recall that $\sin(t) = 0$ when $t = 0$ and $t = \pi$ and all angles coterminal with these two angles. One way to express these values is like this:

$$\left\{t \;\middle|\; t = 2k\pi \text{ for all } k \in \mathbb{Z}\right\} \cup \left\{t \;\middle|\; t = \pi + 2k\pi \text{ for all } k \in \mathbb{Z}\right\}$$

This set can be simplified and expressed as:

$$\left\{t \;\middle|\; t = k\pi \text{ for all } k \in \mathbb{Z}\right\}.$$

(To verify that this simplification is correct, choose a value represented in the unsimplified set and check if it's represented in the simplified set.)

The domain of $y = \cot(t)$ is all real numbers *except* the values listed above where $\sin(t) = 0$. Thus, the domain of $y = \cot(t)$ is:

$$\left\{t \;\middle|\; t \in \mathbb{R} \text{ and } t \neq k\pi \text{ for all } k \in \mathbb{Z}\right\}.$$

15. Determine the domain of $y = \csc(t)$.

Since $\csc(t) = \frac{1}{\sin(t)}$, $y = \csc(t)$ is undefined where $\sin(t) = 0$. In #14, we observed that these values can be represented by the following set:

$$\left\{t \;\middle|\; t = k\pi \text{ for all } k \in \mathbb{Z}\right\}.$$

The domain of $y = \csc(t)$ is all real numbers *except* the values listed above where $\sin(t) = 0$. Thus, the domain of $y = \csc(t)$ is:

$$\left\{t \;\middle|\; t \in \mathbb{R} \text{ and } t \neq k\pi \text{ for all } k \in \mathbb{Z}\right\}.$$

16. Find the exact value of each of the following expressions; do not use a calculator. Be sure to use proper notation to ***directly communicate*** what the given expressions equal.

a. $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$ (since $\frac{\sqrt{3}}{2} = \sin\left(\frac{\pi}{3}\right)$ and $-\frac{\pi}{2} \le \frac{\pi}{3} \le \frac{\pi}{2}$)

b. $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$

$\cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$ (since $\frac{\sqrt{2}}{2} = \cos\left(\frac{\pi}{4}\right)$ and $0 \le \frac{\pi}{4} \le \pi$)

c. $\sin^{-1}\left(-\frac{1}{2}\right)$

$\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$ (since $-\frac{1}{2} = \sin\left(-\frac{\pi}{6}\right)$ and $-\frac{\pi}{2} \le -\frac{\pi}{6} \le \frac{\pi}{2}$)

d. $\cos^{-1}(0)$

$\cos^{-1}(0) = \frac{\pi}{2}$ (since $0 = \cos\left(\frac{\pi}{2}\right)$ and $0 \le \frac{\pi}{2} \le \pi$)

e. $\tan^{-1}\left(-\sqrt{3}\right)$

$\tan^{-1}\left(-\sqrt{3}\right) = -\frac{\pi}{3}$ (since $-\sqrt{3} = \tan\left(-\frac{\pi}{3}\right)$ and $-\frac{\pi}{2} < -\frac{\pi}{3} < \frac{\pi}{2}$)

f. $\tan^{-1}(-1)$

$\tan^{-1}(-1) = -\frac{\pi}{4}$ (since $-1 = \tan\left(-\frac{\pi}{4}\right)$ and $-\frac{\pi}{2} < -\frac{\pi}{4} < \frac{\pi}{2}$)

17. Find the exact value of each of the following expressions; do not use a calculator. Be sure to use proper notation to ***directly communicate*** what the given expressions equal.

a. $\sin\left(\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)$

$$\sin\left(\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right) = \sin\left(\frac{\pi}{3}\right) \qquad \text{(since } \frac{\sqrt{3}}{2} = \sin\left(\frac{\pi}{3}\right) \text{ and } -\frac{\pi}{2} \le \frac{\pi}{3} \le \frac{\pi}{2}\text{)}$$
$$= \frac{\sqrt{3}}{2} \qquad \text{(since } \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}\text{)}$$

b. $\cos\left(\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)\right)$

$$\cos\left(\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)\right) = \cos\left(\frac{3\pi}{4}\right) \qquad \text{(since } -\frac{\sqrt{2}}{2} = \cos\left(\frac{3\pi}{4}\right) \text{ and } 0 \le \frac{3\pi}{4} \le \pi\text{)}$$
$$= -\frac{\sqrt{2}}{2} \qquad \text{(since } \cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}\text{)}$$

c. $\cos^{-1}\left(\cos\left(\frac{5\pi}{3}\right)\right)$

$$\cos^{-1}\left(\cos\left(\frac{5\pi}{3}\right)\right) = \cos^{-1}\left(\frac{1}{2}\right) \qquad \text{(since } \cos\left(\frac{5\pi}{3}\right) = \frac{1}{2}\text{)}$$
$$= \frac{\pi}{3} \qquad \text{(since } \cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \text{ and } 0 \le \frac{\pi}{3} \le \pi\text{)}$$

d. $\sin^{-1}\left(\sin\left(\frac{4\pi}{3}\right)\right)$

$$\sin^{-1}\left(\sin\left(\frac{4\pi}{3}\right)\right) = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) \qquad \text{(since } \sin\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{3}}{2}\text{)}$$
$$= -\frac{\pi}{3} \qquad \text{(since } \sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2} \text{ and } -\frac{\pi}{2} \le -\frac{\pi}{3} \le \frac{\pi}{2}\text{)}$$

e. $\sin\left(\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$

$$\sin\left(\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right) = \sin\left(\frac{5\pi}{6}\right) \qquad \text{(since } -\frac{\sqrt{3}}{2} = \cos\left(\frac{5\pi}{6}\right) \text{ and } 0 \le \frac{5\pi}{6} \le \pi\text{)}$$
$$= \frac{1}{2} \qquad \text{(since } \sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}\text{)}$$

18. Find the exact value of each of the following expressions; do not use a calculator. Be sure to use proper notation to ***directly communicate*** what the given expressions equal.

a. $\sin^{-1}\left(\cos\left(-\frac{\pi}{6}\right)\right)$

$$\sin^{-1}\left(\cos\left(-\frac{\pi}{6}\right)\right) = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) \quad \text{(since } \cos\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}\text{)}$$

$$= \frac{\pi}{3} \quad \text{(since } \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \text{ and } -\frac{\pi}{2} \le \frac{\pi}{3} \le \frac{\pi}{2}\text{)}$$

b. $\tan^{-1}\left(\tan\left(\frac{2\pi}{3}\right)\right)$

$$\tan^{-1}\left(\tan\left(\frac{2\pi}{3}\right)\right) = \tan^{-1}\left(-\sqrt{3}\right) \quad \text{(since } \tan\left(\frac{2\pi}{3}\right) = -\sqrt{3}\text{)}$$

$$= -\frac{\pi}{3} \quad \text{(since } \tan\left(-\frac{\pi}{3}\right) = -\sqrt{3} \text{ and } -\frac{\pi}{2} < -\frac{\pi}{3} < \frac{\pi}{2}\text{)}$$

c. $\cos^{-1}\left(\tan\left(\frac{3\pi}{4}\right)\right)$

$$\cos^{-1}\left(\tan\left(\frac{3\pi}{4}\right)\right) = \cos^{-1}(-1) \quad \text{(since } \tan\left(\frac{3\pi}{4}\right) = -1\text{)}$$

$$= \pi \quad \text{(since } \cos(\pi) = -1 \text{ and } 0 \le \pi \le \pi\text{)}$$

d. $\tan^{-1}\left(\sin\left(\frac{\pi}{2}\right)\right)$

$$\tan^{-1}\left(\sin\left(\frac{\pi}{2}\right)\right) = \tan^{-1}(1) \quad \text{(since } \sin\left(\frac{\pi}{2}\right) = 1\text{)}$$

$$= \frac{\pi}{4} \quad \text{(since } \tan\left(\frac{\pi}{4}\right) = 1 \text{ and } -\frac{\pi}{2} \le \frac{\pi}{4} \le \frac{\pi}{2}\text{)}$$

e. $\sin\left(\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right)$

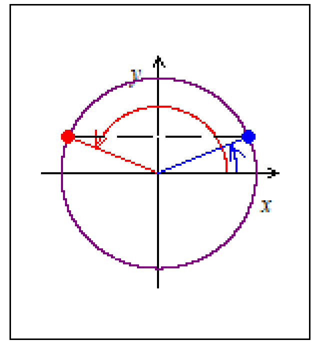
$$\sin\left(\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right) = \sin\left(\frac{\pi}{6}\right) \quad \text{(since } \frac{1}{\sqrt{3}} = \tan\left(\frac{\pi}{6}\right) \text{ and } -\frac{\pi}{2} < \frac{\pi}{6} < \frac{\pi}{2}\text{)}$$

$$= \frac{1}{2} \quad \text{(since } \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}\text{)}$$

19. Find the exact value of each of the following expressions; do not use a calculator. Be sure to use proper notation to ***directly communicate*** what the given expressions equal.

a. $\sin^{-1}\left(\sin\left(\frac{7\pi}{8}\right)\right)$

$\frac{7\pi}{8}$ isn't a "friendly angle" so we aren't familiar with its sine so we need to rely on our conceptual understanding of the sine function along with the symmetry of a circle:

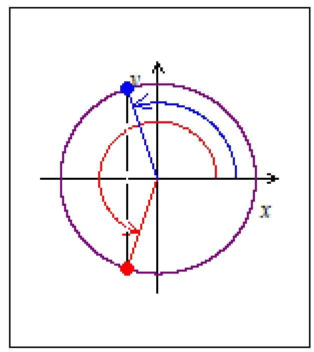


$\sin^{-1}\left(\sin\left(\frac{7\pi}{8}\right)\right) = \sin^{-1}\left(\sin\left(\frac{\pi}{8}\right)\right)$ (using the symmetry of a circle)

$= \frac{\pi}{8}$ (since $-\frac{\pi}{2} \le \frac{\pi}{8} \le \frac{\pi}{2}$)

b. $\cos^{-1}\left(\cos\left(\frac{7\pi}{5}\right)\right)$

As in part (**a**) above, $\frac{7\pi}{5}$ isn't a "friendly angle" so we need to rely on our conceptual understanding of the cosine function along with the symmetry of a circle:

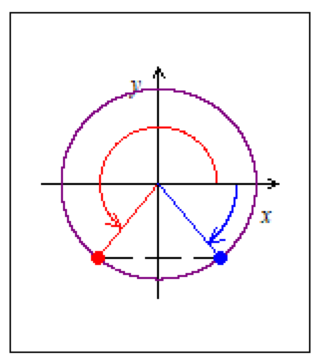


$\cos^{-1}\left(\cos\left(\frac{7\pi}{5}\right)\right) = \cos^{-1}\left(\cos\left(\frac{3\pi}{5}\right)\right)$ (using the symmetry of a circle)

$= \frac{3\pi}{5}$ (since $0 \le \frac{3\pi}{5} \le \pi$)

c. $\sin^{-1}\left(\sin\left(\frac{9\pi}{7}\right)\right)$

As in parts (**a**) and (**b**) above, $\frac{9\pi}{7}$ isn't a "friendly angle" so we need to rely on our conceptual understanding of the sine function along with the symmetry of a circle:



$\sin^{-1}\left(\sin\left(\frac{9\pi}{7}\right)\right) = \sin^{-1}\left(\sin\left(-\frac{2\pi}{7}\right)\right)$ (using the symmetry of a circle)

$= -\frac{2\pi}{7}$ (since $-\frac{\pi}{2} \le -\frac{2\pi}{7} \le \frac{\pi}{2}$)

20. Find the exact value of each of the following expressions; do not use a calculator. Be sure to use proper notation to ***directly communicate*** what the given expressions equal.

a. $\sin(t) = -\frac{\sqrt{2}}{2}$

$$\sin(t) = -\frac{\sqrt{2}}{2}$$

$$\Rightarrow \quad t = \sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) + 2k\pi \quad \text{or} \quad t = \pi - \sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) + 2k\pi, \quad k \in \mathbb{Z}$$

$$\Rightarrow \quad t = -\frac{\pi}{4} + 2k\pi \quad \text{or} \quad t = \pi - \left(-\frac{\pi}{4}\right) + 2k\pi, \quad k \in \mathbb{Z}$$

$$\Rightarrow \quad t = -\frac{\pi}{4} + 2k\pi \quad \text{or} \quad t = \frac{5\pi}{4} + 2k\pi, \quad k \in \mathbb{Z}$$

(In the step indicated by a red arrow ($\Rightarrow$) we've used the identity $\sin(t) = \sin(\pi - t)$ in order to generate the second family of solutions Instead of using the inverse trig function in this step, you might choose to obtain the step indicated by a pink arrow ($\Rightarrow$) by utilizing your awareness of these two facts: $\sin\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$ and $\sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}$ although using "awareness" might inspire you to use "$\frac{7\pi}{4}$" instead of "$-\frac{\pi}{4}$" for the first family of solutions, which will to lead you to $t = \frac{7\pi}{4} + 2k\pi$" instead of $t = -\frac{\pi}{4} + 2k\pi$ for the second family of solutions)

b. $2\cos(x) - \sqrt{3} = 0$

$$2\cos(x) - \sqrt{3} = 0$$

$$\Rightarrow \quad 2\cos(x) = \sqrt{3}$$

$$\Rightarrow \quad \cos(x) = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad x = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) + 2k\pi \quad \text{or} \quad x = -\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) + 2k\pi, \quad k \in \mathbb{Z}$$

$$\Rightarrow \quad x = \tfrac{\pi}{6} + 2k\pi \quad \text{or} \quad x = -\tfrac{\pi}{6} + 2k\pi, \quad k \in \mathbb{Z}$$

$$\Rightarrow \quad x = \tfrac{\pi}{6} + 2k\pi \quad \text{or} \quad x = -\tfrac{\pi}{6} + 2k\pi, \quad k \in \mathbb{Z}$$

(In the step indicated by a red arrow ($\Rightarrow$) we've used the identity $\cos(x) = -\cos(x)$ in order to generate the second family of solutions. Instead of using the inverse trig function in this step, you might choose to obtain the step indicated by a pink arrow ($\Rightarrow$) by utilizing your awareness of these two facts: $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ and $\cos\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$, although using "awareness" might inspire you to use "$\frac{11\pi}{6}$" instead of "$-\frac{\pi}{6}$" for the second family of solutions, which will to lead you to $x = \frac{11\pi}{6} + 2k\pi$ instead of $x = -\frac{\pi}{6} + 2k\pi$.)

21. Find the solutions on the interval $[0, 2\pi)$ for the equations below; provide *exact* solutions.

a. $\cos(t) = -\frac{1}{2}$

$$\cos(t) = -\frac{1}{2}$$

$$\Rightarrow \quad t = \cos^{-1}\left(-\frac{1}{2}\right) + 2k\pi \quad \text{or} \quad x = -\cos^{-1}\left(-\frac{1}{2}\right) + 2k\pi, \ \ k \in \mathbb{Z}$$

$$\Rightarrow \quad t = \frac{2\pi}{3} + 2k\pi \quad \text{or} \quad x = -\frac{2\pi}{3} + 2k\pi, \ \ k \in \mathbb{Z}$$

Now we need to substitute particular values of k in order to determine which solutions fall on the interval $[0, 2\pi)$:

$k = -1$: $x = \frac{2\pi}{3} + 2(-1)\pi = \frac{2\pi}{3} - \frac{6\pi}{3} = -\frac{4\pi}{3}$ or $x = -\frac{2\pi}{3} + 2(-1)\pi = -\frac{2\pi}{3} - \frac{6\pi}{3} = -\frac{8\pi}{3}$

Since both of these values are negative, they aren't in the interval $[0, 2\pi)$. There's no reason to try smaller values of k since they'll produce yet smaller solutions which will certainly be outside the interval.

$k = 0$: $x = \frac{2\pi}{3} + 2(0)\pi = \frac{2\pi}{3}$ or $x = -\frac{2\pi}{3} + 2(0)\pi = -\frac{2\pi}{3}$

Only $\frac{2\pi}{3}$ is in the given interval.

$k = 1$: $x = \frac{2\pi}{3} + 2(1)\pi = \frac{2\pi}{3} + 2\pi > 2\pi$ or $x = -\frac{2\pi}{3} + 2(1)\pi = -\frac{2\pi}{3} + \frac{6\pi}{3} > 2\pi = \frac{4\pi}{3}$

Only $\frac{2\pi}{3}$ is in the given interval. There's no reason to try larger values of k since they'll produce yet greater solutions which will certainly be outside the given interval.

Therefore, the solution set for $\cos(t) = -\frac{1}{2}$ on the interval $[0, 2\pi)$ is $\left\{\frac{2\pi}{3}, \frac{4\pi}{3}\right\}$.

b. $\frac{\sin(x)}{2} - \frac{\sqrt{3}}{4} = 0$

$$\frac{\sin(x)}{2} - \frac{\sqrt{3}}{4} = 0$$

$$2 \cdot \frac{\sin(x)}{2} = \frac{\sqrt{3}}{4} \cdot 2$$

$$\sin(x) = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad t = \sin^{-1}\left(\tfrac{\sqrt{3}}{2}\right) + 2k\pi \quad \text{or} \quad t = \pi - \sin^{-1}\left(\tfrac{\sqrt{3}}{2}\right) + 2k\pi, \quad k \in \mathbb{Z}$$

$$\Rightarrow \quad t = \tfrac{\pi}{3} + 2k\pi \quad \text{or} \quad t = \pi - \left(\tfrac{\pi}{3}\right) + 2k\pi, \quad k \in \mathbb{Z}$$

$$\Rightarrow \quad t = \tfrac{\pi}{3} + 2k\pi \quad \text{or} \quad t = \tfrac{2\pi}{3} + 2k\pi, \quad k \in \mathbb{Z}$$

Now we need to substitute particular values of k in order to determine which solutions fall on the interval $[0, 2\pi)$:

$k = -1$: $x = \tfrac{\pi}{3} + 2(-1)\pi = \tfrac{\pi}{3} - \tfrac{6\pi}{3} = -\tfrac{5\pi}{3}$ or $x = \tfrac{2\pi}{3} + 2(-1)\pi = \tfrac{2\pi}{3} - \tfrac{6\pi}{3} = -\tfrac{4\pi}{3}$

Since both of these values are negative, they aren't in the interval $[0, 2\pi)$. There's no reason to try smaller values of k since they'll produce yet smaller solutions which will certainly be outside the interval.

$k = 0$: $x = \tfrac{\pi}{3} + 2(0)\pi = \tfrac{\pi}{3}$ or $x = \tfrac{2\pi}{3} + 2(0)\pi = \tfrac{2\pi}{3}$

Both $\tfrac{\pi}{3}$ and $\tfrac{2\pi}{3}$ are in the given interval.

$k = 1$: $x = \tfrac{\pi}{3} + 2(1)\pi = \tfrac{\pi}{3} + 2\pi > 2\pi$ or $x = \tfrac{2\pi}{3} + 2(1)\pi = \tfrac{2\pi}{3} + 2\pi > 2\pi$

Both of these solutions are outside the given interval. There's no reason to try larger values of k since they'll produce yet greater solutions which will certainly be outside the given interval.

Therefore, the solution set for $\frac{\sin(x)}{2} - \frac{\sqrt{3}}{4} = 0$ on the interval $[0, 2\pi)$ is $\left\{\tfrac{\pi}{3}, \tfrac{2\pi}{3}\right\}$.

22. Find *all* of the solutions to the equations below; provide *exact* solutions.

a. $\sin(6t) = -\frac{\sqrt{3}}{2}$

$$\sin(6t) = -\frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad 6t = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) + 2k\pi \quad \text{or} \quad 6t = \pi - \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) + 2k\pi, \quad k \in \mathbb{Z}$$

$$\Rightarrow \quad 6t = -\frac{\pi}{3} + 2k\pi \quad \text{or} \quad 6t = \pi - \left(-\frac{\pi}{3}\right) + 2k\pi, \quad k \in \mathbb{Z}$$

$$\Rightarrow \quad \frac{6t}{6} = \frac{-\frac{\pi}{3}}{6} + \frac{2k\pi}{6} \quad \text{or} \quad \frac{6t}{6} = \frac{\frac{4\pi}{3}}{6} + \frac{2k\pi}{6}, \quad k \in \mathbb{Z}$$

$$\Rightarrow \quad t = -\frac{\pi}{18} + \frac{k\pi}{3} \quad \text{or} \quad t = \frac{2\pi}{9} + \frac{k\pi}{3}, \quad k \in \mathbb{Z}$$

b. $5 + 4\cos(2\theta) = 1$

$$5 + 4\cos(2\theta) = 1$$

$$\Rightarrow \quad 4\cos(2\theta) = -4$$

$$\Rightarrow \quad \cos(2\theta) = -1$$

$$\Rightarrow \quad 2\theta = \cos^{-1}(-1) + 2k\pi, \quad k \in \mathbb{Z}$$

(there's only one "family" of solutions since the cosine function achieves the output -1 only once in each period)

$$\Rightarrow \quad 2\theta = \pi + 2k\pi, \quad k \in \mathbb{Z}$$

$$\Rightarrow \quad \frac{2\theta}{2} = \frac{\pi}{2} + \frac{2k\pi}{2}, \quad k \in \mathbb{Z}$$

$$\Rightarrow \quad \theta = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}$$

c. $16\cos(4x) + 11 = 3$

$$16\cos(4x) + 11 = 3$$

$$\Rightarrow \quad 16\cos(4x) = -8$$

$$\Rightarrow \quad \cos(4x) = \tfrac{-8}{16} = -\tfrac{1}{2}$$

$$\Rightarrow \quad 4x = \cos^{-1}\left(-\tfrac{1}{2}\right) + 2k\pi \quad \text{or} \quad 4x = -\cos^{-1}\left(-\tfrac{1}{2}\right) + 2k\pi, \quad k \in \mathbb{Z}$$

$$\Rightarrow \quad 4x = \tfrac{2\pi}{3} + 2k\pi \quad \text{or} \quad 4x = -\tfrac{2\pi}{3} + 2k\pi, \quad k \in \mathbb{Z}$$

$$\Rightarrow \quad \frac{4x}{4} = \frac{\frac{2\pi}{3}}{4} + \frac{2k\pi}{4} \quad \text{or} \quad \frac{4x}{4} = \frac{-\frac{2\pi}{3}}{4} + \frac{2k\pi}{4}, \quad k \in \mathbb{Z}$$

$$\Rightarrow \quad x = \tfrac{2\pi}{3\cdot 4} + \tfrac{2k\pi}{4} \quad \text{or} \quad x = -\tfrac{2\pi}{3\cdot 4} + \tfrac{k\pi}{2}, \quad k \in \mathbb{Z}$$

$$\Rightarrow \quad x = \tfrac{\pi}{6} + \tfrac{k\pi}{2} \quad \text{or} \quad x = -\tfrac{\pi}{6} + \tfrac{k\pi}{2}, \quad k \in \mathbb{Z}$$

d. $16 - 24\sin(8t) = 4$

$$16 - 24\sin(8t) = 4$$

$$\Rightarrow \quad -24\sin(8t) = -12$$

$$\Rightarrow \quad \sin(8t) = \tfrac{-12}{-24} = \tfrac{1}{2}$$

$$\Rightarrow \quad 8t = \sin^{-1}\left(\tfrac{1}{2}\right) + 2k\pi \quad \text{or} \quad 8t = \pi - \sin^{-1}\left(\tfrac{1}{2}\right) + 2k\pi, \quad k \in \mathbb{Z}$$

$$\Rightarrow \quad 8t = \tfrac{\pi}{6} + 2k\pi \quad \text{or} \quad 8t = \pi - \tfrac{\pi}{6} + 2k\pi, \quad k \in \mathbb{Z}$$

$$\Rightarrow \quad \frac{8t}{8} = \frac{\frac{\pi}{6}}{8} + \frac{2k\pi}{8} \quad \text{or} \quad \frac{8t}{8} = \frac{\frac{5\pi}{6}}{8} + \frac{2k\pi}{8}, \quad k \in \mathbb{Z}$$

$$\Rightarrow \quad t = \tfrac{\pi}{48} + \tfrac{k\pi}{4} \quad \text{or} \quad t = \tfrac{5\pi}{48} + \tfrac{k\pi}{4}, \quad k \in \mathbb{Z}$$

23. Find the solutions on the interval $[0, 2\pi)$ to following equations.

a. $5 + 4\cos(2\theta) = 1$

$$5 + 4\cos(2\theta) = 1$$

$$\Rightarrow \quad 4\cos(2\theta) = -4$$

$$\Rightarrow \quad \cos(2\theta) = -1$$

$$\Rightarrow \quad 2\theta = \cos^{-1}(-1) + 2k\pi, \quad k \in \mathbb{Z}$$

(there's only one "family" of solutions since the cosine function achieves the output -1 only once in each period)

$$\Rightarrow \quad 2\theta = \pi + 2k\pi, \quad k \in \mathbb{Z}$$

$$\Rightarrow \quad \frac{2\theta}{2} = \frac{\pi}{2} + \frac{2k\pi}{2}, \quad k \in \mathbb{Z}$$

$$\Rightarrow \quad \theta = \tfrac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}$$

Now we need to substitute particular values of k in order to determine which solutions fall on the interval $[0, 2\pi)$:

$k = -1$: $\theta = \tfrac{\pi}{2} + (-1)\cdot\pi = -\tfrac{\pi}{2} < 0$ so $-\tfrac{\pi}{2} \notin [0, 2\pi)$.

We could try smaller values of k but it should be clear that since $k = -1$ produced a value of θ that's too small, smaller values of k will produces even smaller values of θ so they won't produce solutions in the given interval so there's no need in trying smaller values of k.

$k = 0$: $\theta = \tfrac{\pi}{2} + 0\cdot\pi = \tfrac{\pi}{2} \in [0, 2\pi)$, so $\tfrac{\pi}{2}$ is a solution in the given interval.

$k = 1$: $\theta = \tfrac{\pi}{2} + 1\cdot\pi = \tfrac{3\pi}{2} \in [0, 2\pi)$, so $\tfrac{3\pi}{2}$ is a solution in the given interval.

$k = 2$: $\theta = \tfrac{\pi}{2} + 2\cdot\pi = \tfrac{5\pi}{2} > 2\pi$ so $\tfrac{5\pi}{2} \notin [0, 2\pi)$.

We could try larger values of k but it should be clear that since $k = 2$ produced a value θ that's too large, larger values of k will produces even larger values of θ, so they won't produce solutions in the given interval so there's no need in trying larger values of k.

Therefore, the solution set to $5 + 4\cos(2\theta) = 1$ on the interval $[0, 2\pi)$ is $\left\{\tfrac{\pi}{2}, \tfrac{3\pi}{2}\right\}$.

b. $4 - 6\sin(2x) = 7$

$$4 - 6\sin(2x) = 7$$

$$\Rightarrow \quad -6\sin(2x) = 3$$

$$\Rightarrow \quad \sin(2x) = \tfrac{3}{-6} = -\tfrac{1}{2}$$

$$\Rightarrow 2x = \sin^{-1}\left(-\tfrac{1}{2}\right) + 2k\pi \text{ or } 2x = \pi - \sin^{-1}\left(-\tfrac{1}{2}\right) + 2k\pi, \quad k \in \mathbb{Z}$$

$$\Rightarrow 2x = -\tfrac{\pi}{6} + 2k\pi \quad \text{or} \quad 2x = \pi - \left(-\tfrac{\pi}{6}\right) + 2k\pi, \quad k \in \mathbb{Z}$$

$$\Rightarrow \frac{2x}{2} = \frac{-\frac{\pi}{6}}{2} + \frac{2k\pi}{2} \quad \text{or} \quad \frac{2x}{2} = \frac{\frac{7\pi}{6}}{2} + \frac{2k\pi}{2}, \quad k \in \mathbb{Z}$$

$$\Rightarrow \quad x = -\tfrac{\pi}{12} + k\pi \quad \text{or} \quad x = \tfrac{7\pi}{12} + k\pi, \quad k \in \mathbb{Z}$$

Now we need to substitute particular values of k into these equations to determine which solutions fall on the interval $[0, 2\pi)$:

$k = -1$: $x = -\tfrac{\pi}{12} + (-1)\pi = -\tfrac{\pi}{12} - \tfrac{12\pi}{12} = -\tfrac{13\pi}{12} < 0$ or $x = \tfrac{7\pi}{12} + (-1)\pi = \tfrac{7\pi}{12} - \tfrac{12\pi}{12} = -\tfrac{5\pi}{12} < 0$

Since both of these values are negative, they aren't in the interval $[0, 2\pi)$. We could try smaller values of k but it should be clear that since $k = -1$ resulted in a values of α that are too small, smaller values of k will produces even smaller values of α so they won't produce solutions in the given interval so there's no need in trying smaller values of k.

$k = 0$: $x = -\tfrac{\pi}{12} + (0)\pi = -\tfrac{\pi}{12} < 0$ or $x = \tfrac{7\pi}{12} + (0)\pi = \tfrac{7\pi}{12}$

Only $\tfrac{7\pi}{12}$ is in the given interval.

$k = 1$: $x = -\tfrac{\pi}{12} + (1)\pi = -\tfrac{\pi}{12} + \tfrac{12\pi}{12} = \tfrac{11\pi}{12}$ or $x = \tfrac{7\pi}{12} + (1)\pi = \tfrac{7\pi}{12} + \tfrac{12\pi}{12} = \tfrac{19\pi}{12}$

Both $\tfrac{11\pi}{12}$ and $\tfrac{19\pi}{12}$ are in the given interval.

$k = 2$: $x = -\frac{\pi}{12} + (2)\pi$ or $x = \frac{7\pi}{12} + (2)\pi$

$= -\frac{\pi}{12} + \frac{24\pi}{12}$ $= \frac{7\pi}{12} + \frac{24\pi}{12}$

$= \frac{23\pi}{12}$ $= \frac{31\pi}{12} > 0$

Only $\frac{23\pi}{12}$ is in the given interval.

$k = 3$: $x = -\frac{\pi}{12} + (3)\pi$ or $x = \frac{7\pi}{12} + (3)\pi$

$= -\frac{\pi}{12} + \frac{36\pi}{12}$ $= \frac{7\pi}{12} + \frac{36\pi}{12}$

$= \frac{35\pi}{12} > 2\pi$ $= \frac{43\pi}{12} > 2\pi$

Since both of these values are greater than 2π, they aren't in the interval $[0, 2\pi)$. We could try larger values of k but it should be clear that since $k = 3$ resulted in a values of x that are too larger, larger values of k will produces even larger values of x so they won't produce solutions in the given interval so there's no need in trying larger values of k.

Thus, the solution set to $4 - 6\sin(2x) = 7$ on the interval $[0, 2\pi)$ is $\left\{\frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}\right\}$.

c. $6\sqrt{2}\cos(3\alpha) + 10 = 4$

$$
\begin{aligned}
& 6\sqrt{2}\cos(3\alpha) + 10 = 4 \\
\Rightarrow\quad & 6\sqrt{2}\cos(3\alpha) = -6 \\
\Rightarrow\quad & \cos(3\alpha) = \tfrac{-6}{6\sqrt{2}} = -\tfrac{1}{\sqrt{2}} \\
\Rightarrow\quad & 3\alpha = \cos^{-1}\left(-\tfrac{1}{\sqrt{2}}\right) + 2k\pi \quad \text{or} \quad 3\alpha = -\cos^{-1}\left(-\tfrac{1}{\sqrt{2}}\right) + 2k\pi,\ \ k \in \mathbb{Z} \\
\Rightarrow\quad & 3\alpha = \tfrac{3\pi}{4} + 2k\pi \quad \text{or} \quad 3\alpha = -\tfrac{3\pi}{4} + 2k\pi,\ \ k \in \mathbb{Z} \\
\Rightarrow\quad & \frac{3\alpha}{3} = \frac{\frac{3\pi}{4}}{3} + \frac{2k\pi}{3} \quad \text{or} \quad \frac{3\alpha}{3} = \frac{-\frac{3\pi}{4}}{3} + \frac{2k\pi}{3},\ \ k \in \mathbb{Z} \\
\Rightarrow\quad & \alpha = \tfrac{3\pi}{4\cdot 3} + \tfrac{2k\pi}{3} \quad \text{or} \quad \alpha = -\tfrac{3\pi}{4\cdot 3} + \tfrac{2k\pi}{3},\ \ k \in \mathbb{Z} \\
\Rightarrow\quad & \alpha = \tfrac{\pi}{4} + \tfrac{2k\pi}{3} \quad \text{or} \quad \alpha = -\tfrac{\pi}{4} + \tfrac{2k\pi}{3},\ \ k \in \mathbb{Z}
\end{aligned}
$$

Now we need to substitute particular values of k into these equations to determine which solutions fall on the interval $[0, 2\pi)$:

$k = -1$: $\alpha = \frac{\pi}{4} + \frac{2(-1)\pi}{3} = \frac{3\pi}{12} - \frac{8\pi}{12} = -\frac{5\pi}{12}$ or $\alpha = -\frac{\pi}{4} + \frac{2(-1)\pi}{3} = -\frac{3\pi}{12} - \frac{8\pi}{12} = -\frac{11\pi}{12}$

Since both of these values are negative, they aren't in the interval $[0, 2\pi)$. We could try smaller values of k but it should be clear that since $k = -1$ resulted in a values of α that are too small, smaller values of k will produces even smaller values of α so they won't produce solutions in the given interval so there's no need in trying smaller values of k.

$k = 0$: $\alpha = \frac{\pi}{4} + \frac{2\cdot 0\cdot\pi}{3} = \frac{\pi}{4}$ or $\alpha = -\frac{\pi}{4} + \frac{2\cdot 0\cdot\pi}{3} = -\frac{\pi}{4}$

Only $\frac{\pi}{4}$ is in the given interval.

$k = 1$: $\alpha = \frac{\pi}{4} + \frac{2\cdot 1\cdot\pi}{3} = \frac{3\pi}{14} + \frac{8\pi}{12} = \frac{11\pi}{12}$ or $\alpha = -\frac{\pi}{4} + \frac{2\cdot 1\cdot\pi}{3} = -\frac{3\pi}{12} + \frac{8\pi}{12} = \frac{5\pi}{12}$

Both $\frac{11\pi}{12}$ and $\frac{5\pi}{12}$ are in the given interval.

$k = 2$: $\alpha = \frac{\pi}{4} + \frac{2 \cdot 2 \cdot \pi}{3}$ or $\alpha = -\frac{\pi}{4} + \frac{2 \cdot 2 \cdot \pi}{3}$

$= \frac{3\pi}{12} + \frac{16\pi}{12}$ $= -\frac{3\pi}{12} + \frac{16\pi}{12}$

$= \frac{19\pi}{12}$ $= \frac{13\pi}{12}$

Both $\frac{19\pi}{12}$ and $\frac{13\pi}{12}$ are in the given interval.

$k = 3$: $\alpha = \frac{\pi}{4} + \frac{2 \cdot 3 \cdot \pi}{3}$ or $\alpha = -\frac{\pi}{4} + \frac{2 \cdot 3 \cdot \pi}{3}$

$= \frac{\pi}{4} + \frac{8\pi}{4}$ $= -\frac{\pi}{4} + \frac{8\pi}{4}$

$= \frac{9\pi}{4} > 2\pi$ $= \frac{7\pi}{4}$

Only $\frac{7\pi}{4}$ is in the given interval.

$k = 4$: $\alpha = \frac{\pi}{4} + \frac{2 \cdot 4 \cdot \pi}{3}$ or $\alpha = -\frac{\pi}{4} + \frac{2 \cdot 4 \cdot \pi}{3}$

$= \frac{3\pi}{12} + \frac{32\pi}{12}$ $= -\frac{3\pi}{12} + \frac{32\pi}{12}$

$= \frac{35\pi}{12} > 2\pi$ $= \frac{29\pi}{12} > 2\pi$

Since both of these values are greater than 2π, they aren't in the interval $[0, 2\pi)$. We could try larger values of k but it should be clear that since $k = 4$ resulted in a values of α that are too larger, larger values of k will produces even larger values of α so they won't produce solutions in the given interval so there's no need in trying larger values of k.

Thus, the solution set to $6\sqrt{2}\cos(3\alpha) + 10 = 4$ on the interval $[0, 2\pi)$ is $\left\{\frac{\pi}{4}, \frac{5\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{19\pi}{12}, \frac{7\pi}{4}\right\}$.