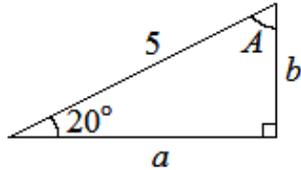## SOLUTIONS: **Week __ Practice Worksheet**

Some Additional Practice for the Final Exam

1. Find the missing side(s) and missing angle(s) for the triangle given below. (The triangles may not be drawn to scale.)

a.

We can find A using the fact that the sum of the angles in a triangle is $180°$:

$$A + 20° + 90° = 180°$$
$$\Rightarrow \qquad A = 180° - 20° - 90°$$
$$\Rightarrow \qquad A = 70°$$

We can find b using sine:

$$\sin(20°) = \frac{b}{5}$$
$$\Rightarrow \qquad b = 5\sin(20°)$$
$$\approx 1.71$$

And we can use cosine to find a:

$$\cos(20°) = \frac{a}{5}$$
$$\Rightarrow \qquad a = 5\cos(20°)$$
$$\approx 4.698$$

b.
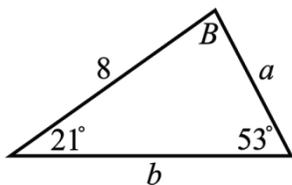
We can find B using the fact that the sum of the angles in a triangle is $180°$:

$$21° + B + 53° = 180°$$

$$\Rightarrow \qquad B = 180° - 21° - 53°$$

$$\Rightarrow \qquad B = 106°$$

Now we can use the Law of Sines to find b :

$$\frac{8}{\sin(53°)} = \frac{b}{\sin(106°)}$$

$$\Rightarrow \qquad b = \frac{8\sin(106°)}{\sin(53°)} \approx 9.62$$

And we can use the Law of Sines again to find a :

$$\frac{8}{\sin(53°)} = \frac{a}{\sin(21°)}$$

$$\Rightarrow \qquad a = \frac{8\sin(21°)}{\sin(53°)} \approx 3.59$$

c.
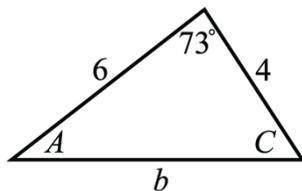
First let's use the Law of Cosines to find b :

$$b^2 = 4^2 + 6^2 - 2 \cdot 4 \cdot 6 \cos(73°)$$
$$\Rightarrow b = \sqrt{16 + 36 - 48\cos(73°)}$$
$$\Rightarrow b = \sqrt{52 - 48\cos(73°)}$$
$$\Rightarrow b \approx 6.16$$

Now we can use the Law of Sines to find A. (We choose to find A first instead of C since A must be smaller than C since A is across a shorter side than C.)

$$\frac{\sin(A)}{4} = \frac{\sin(73°)}{b}$$
$$\Rightarrow \sin(A) = \frac{4\sin(73°)}{b}$$
$$\Rightarrow A = \sin^{-1}\left(\frac{4\sin(73°)}{b}\right)$$
$$\Rightarrow A \approx \sin^{-1}\left(\frac{4\sin(73°)}{6.16}\right)$$
$$\approx 38.39°$$

Finally, we can find C using the fact that the sum of the angles in a triangle is $180°$:

$$A + 73° + C = 180°$$
$$\Rightarrow C = 180° - 73° - A$$
$$\Rightarrow C \approx 180° - 73° - 38.39°$$
$$\Rightarrow C \approx 68.61°$$

2. Suppose that $\sin(\alpha) = \frac{5}{13}$ and $\cos(\beta) = \frac{3}{5}$, and where $0 < \alpha < \frac{\pi}{2}$ and $\frac{3\pi}{2} < \beta < 2\pi$.

a. Find the exact value of $\sin(\alpha + \beta)$.

To find $\sin(\alpha + \beta)$ we'll need to use the sine-of-a-sum identity:

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)$$

We'll need $\cos(\alpha)$ and $\sin(\beta)$ to use this formula. Let's use the Pythagorean identity, and start by finding $\cos(\alpha)$:

$$\sin^2(\alpha) + \cos^2(\alpha) = 1$$

$$\Rightarrow \left(\frac{5}{13}\right)^2 + \cos^2(\alpha) = 1$$

$$\Rightarrow \quad \frac{25}{169} + \cos^2(\alpha) = 1$$

$$\Rightarrow \qquad \cos^2(\alpha) = 1 - \frac{25}{169}$$

$$\Rightarrow \qquad \cos^2(\alpha) = \frac{144}{169}$$

$$\Rightarrow \qquad \cos(\alpha) = \frac{12}{13} \qquad \text{We take the positive square root of } \frac{144}{169} \text{ since}$$

$$0 < \alpha < \frac{\pi}{2}, \text{ i.e., } \alpha \text{ is in the quadrant 1.}$$

Now let's use the Pythagorean identity to find $\sin(\beta)$:

$$\sin^2(\beta) + \cos^2(\beta) = 1$$

$$\Rightarrow \sin^2(\beta) + \left(\frac{3}{5}\right)^2 = 1$$

$$\Rightarrow \sin^2(\beta) + \frac{9}{25} = 1$$

$$\Rightarrow \qquad \sin^2(\beta) = 1 - \frac{9}{25}$$

$$\Rightarrow \qquad \sin^2(\beta) = \frac{16}{25}$$

$$\Rightarrow \qquad \sin(\beta) = -\frac{4}{5} \qquad \text{We take the negative square root of } \frac{16}{25} \text{ since}$$

$$\frac{3\pi}{2} < \beta < 2\pi, \text{ i.e., } \beta \text{ is in the quadrant 4.}$$

Therefore,

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)$$
$$= \frac{5}{13} \cdot \frac{3}{5} + \left(-\frac{4}{5}\right) \cdot \frac{12}{13}$$
$$= \frac{15}{65} - \frac{48}{65}$$
$$= -\frac{33}{65}$$

b. Find the exact value of $\cos(\alpha - \beta)$.

To find $\cos(\alpha - \beta)$ we'll need to use the cosine-of-a-difference identity:

$$\cos(\alpha - \beta) = \sin(\alpha)\sin(\beta) + \cos(\alpha)\cos(\beta)$$

We can use the values we found in part (a):

$$\cos(\alpha - \beta) = \sin(\alpha)\sin(\beta) + \cos(\alpha)\cos(\beta)$$
$$= \frac{5}{13} \cdot \left(-\frac{4}{5}\right) + \frac{12}{13} \cdot \frac{3}{5}$$
$$= \frac{16}{65}$$

c. Find the exact value of $\sin(2\beta)$.

We can use the double-angle identity:

$$\sin(2\beta) = 2\sin(\beta)\cos(\beta).$$

(Note that we can use the value of $\cos(\beta)$ that we found in part (a).)

$$\sin(2\beta) = 2\sin(\beta)\cos(\beta)$$
$$= 2 \cdot \left(-\frac{4}{5}\right) \cdot \frac{3}{5}$$
$$= -\frac{24}{25}$$

d. Find the exact value of $\cos(2\beta)$.

We can use the double-angle identity $\cos(2\beta) = 1 - 2\sin^2(\beta)$:

$$\cos(2\beta) = 1 - 2\sin^2(\beta)$$
$$= 1 - 2 \cdot \left(-\frac{4}{5}\right)^2$$
$$= 1 - \frac{32}{25}$$
$$= -\frac{7}{25}$$

e. Find the exact value of $\sin\left(\frac{\beta}{2}\right)$.

Recall that $\frac{3\pi}{2} < \beta < 2\pi$. Thus,

$$\frac{\frac{3\pi}{2}}{2} < \frac{\beta}{2} < \frac{2\pi}{2}$$

$$\Rightarrow \quad \frac{3\pi}{4} < \frac{\beta}{2} < \pi.$$

Since $\frac{3\pi}{4} < \frac{\beta}{2} < \pi$, we know that $\frac{\beta}{2}$ is in quadrant 2 so $\sin\left(\frac{\beta}{2}\right) > 0$; therefore, we need to choose the *positive* square root in the half-angle identity:

$$\sin\left(\tfrac{\beta}{2}\right) = +\sqrt{\frac{1 - \cos(\beta)}{2}}$$

$$= \sqrt{\frac{1 - \frac{3}{5}}{2}}$$

$$= \sqrt{\tfrac{2}{5} \cdot \tfrac{1}{2}}$$

$$= \sqrt{\tfrac{1}{5}} = \tfrac{1}{\sqrt{5}} = \tfrac{\sqrt{5}}{5}$$

f. Find the exact value of $\cos\left(\frac{\beta}{2}\right)$.

In (e) we discovered that $\frac{\beta}{2}$ is in quadrant 2 so $\cos\left(\frac{\beta}{2}\right) < 0$; therefore, we need to choose the *negative* square root in the half-angle identity:

$$\cos\left(\tfrac{\beta}{2}\right) = -\sqrt{\frac{1 + \cos(\beta)}{2}}$$

$$= -\sqrt{\frac{1 + \frac{3}{5}}{2}}$$

$$= -\sqrt{\tfrac{8}{5} \cdot \tfrac{1}{2}}$$

$$= -\sqrt{\tfrac{4}{5}}$$

$$= -\tfrac{2}{\sqrt{5}} = -\tfrac{2\sqrt{5}}{5}$$

3. Prove the following identities.

a. $\tan(x) + \cot(x) = \sec(x)\csc(x)$

$$\tan(x) + \cot(x) = \frac{\sin(x)}{\cos(x)} + \frac{\cos(x)}{\sin(x)}$$

$$= \frac{\sin(x)}{\cos(x)} \cdot \frac{\sin(x)}{\sin(x)} + \frac{\cos(x)}{\sin(x)} \cdot \frac{\cos(x)}{\cos(x)}$$

$$= \frac{\sin^2(x)}{\cos(x)\sin(x)} + \frac{\cos^2(x)}{\sin(x)\cos(x)}$$

$$= \frac{\sin^2(x) + \cos^2(x)}{\cos(x)\sin(x)}$$

$$= \frac{1}{\cos(x)\sin(x)}$$

$$= \frac{1}{\cos(x)} \cdot \frac{1}{\sin(x)}$$

$$= \sec(x)\csc(x)$$

b. $\tan^2(x) - \sin^2(x) = \tan^2(x)\sin^2(x)$

$$\tan^2(x) - \sin^2(x) = \frac{\sin^2(x)}{\cos^2(x)} - \sin^2(x) \cdot \frac{\cos^2(x)}{\cos^2(x)}$$

$$= \frac{\sin^2(x) - \sin^2(x)\cos^2(x)}{\cos^2(x)}$$

$$= \frac{\sin^2(x) \cdot \left(1 - \cos^2(x)\right)}{\cos^2(x)}$$

$$= \frac{\sin^2(x) \cdot \sin^2(x)}{\cos^2(x)}$$

$$= \frac{\sin^2(x)}{\cos^2(x)} \cdot \sin^2(x)$$

$$= \tan^2(x)\sin^2(x)$$

c. $\cos(2x) = \cos^4(x) - \sin^4(x)$

$$\cos^4(x) - \sin^4(x) = \left(\cos^2(x) - \sin^2(x)\right)\left(\cos^2(x) + \sin^2(x)\right)$$
$$= \left(\cos^2(x) - \sin^2(x)\right) \cdot 1$$
$$= \cos^2(x) - \sin^2(x)$$
$$= \cos(2x)$$

4. Convert the following polar ordered pairs into Cartesian (i.e., rectangular) coordinates.

a. $\left(3, \frac{\pi}{2}\right)$

The polar coordinates (r, θ) correspond to the Cartesian coordinates $(x, y) = (r\cos(\theta), r\sin(\theta))$. So

$$x = 3\cos\left(\frac{\pi}{2}\right) \qquad \text{and} \qquad y = 3\sin\left(\frac{\pi}{2}\right)$$
$$= 0 \qquad\qquad\qquad = 3$$

Therefore, the polar ordered pair $\left(3, \frac{\pi}{2}\right)$ corresponds to the Cartesian ordered pair $(0, 3)$.

b. $\left(\pi, \frac{5\pi}{3}\right)$

The polar coordinates (r, θ) correspond to the Cartesian coords. $(x, y) = (r\cos(\theta), r\sin(\theta))$. So

$$x = \pi\cos\left(\frac{5\pi}{3}\right) \qquad\qquad y = \pi\sin\left(\frac{5\pi}{3}\right)$$
$$= \pi \cdot \frac{1}{2} \qquad \text{and} \qquad = \pi \cdot -\frac{\sqrt{3}}{2}$$
$$= \frac{\pi}{2} \qquad\qquad\qquad = -\frac{\pi\sqrt{3}}{2}$$

Therefore, the polar ordered pair $\left(\pi, \frac{5\pi}{3}\right)$ corresponds to the Cartesian ordered pair $\left(\frac{\pi}{2}, -\frac{\pi\sqrt{3}}{2}\right)$.

c. $(10, -10°)$

The   polar   coordinates (r, θ) correspond   to   the   Cartesian   coords. $(x, y) = (r\cos(\theta),\ r\sin(\theta))$.   So

$$x = 10\cos(-10°) \qquad \text{and} \qquad y = 10\sin(-10°)$$
$$\approx 9.85 \qquad\qquad\qquad\qquad \approx -1.74$$

Therefore, the polar ordered pair $(10, -10°)$ corresponds to the Cartesian ordered pair $\left(10\cos(10°),\ 10\sin(10°)\right) \approx (9.85, -1.74)$.

5. Convert the following Cartesian (i.e., rectangular) ordered pairs into polar coordinates.

a. $(10, -10)$

We need to convert Cartesian coordinates (x, y) into polar coordinates (r, θ). The distance from the origin to the point, r, is given by $r = \sqrt{x^2 + y^2}$:

$$r = \sqrt{10^2 + (-10)^2}$$
$$= 10\sqrt{2}$$

To find θ (which should be in the 4th quadrant since that's where the given point sits), we can use the formula $\theta = \tan^{-1}\left(\frac{y}{x}\right)$:

$$\theta = \tan^{-1}\left(\frac{10}{-10}\right)$$
$$= \tan^{-1}(-1)$$
$$= -\frac{\pi}{4} \qquad \text{(since } \tan\left(-\frac{\pi}{4}\right) = -1\text{)}$$

Therefore, the Cartesian ordered pair $(10, -10)$ corresponds to the polar ordered pair $\left(10\sqrt{2},\ -\frac{\pi}{4}\right)$.

b. $(-3, 0)$

We need to convert Cartesian coordinates (x, y) into polar coordinates (r, θ). Clearly $r = 3$ and $\theta = \pi$ (since the point lies on the negative x-axis). Therefore, the Cartesian ordered pair $(-3, 0)$ corresponds to the polar ordered pair $(3, \pi)$.

c. $\left(-8, -8\sqrt{3}\right)$

We need to convert Cartesian coordinates (x, y) into polar coordinates (r, θ). The distance from the origin to the point, r, is given by $r = \sqrt{x^2 + y^2}$:

$$r = \sqrt{(-8)^2 + (-8\sqrt{3})^2}$$
$$= \sqrt{256}$$
$$= 16$$

To find θ (which should be in the 3rd quadrant since that's where the given point sits), we can use the formula $\theta = \tan^{-1}\left(\dfrac{y}{x}\right)$ – but we'll need to add π to the result since the range of arctangent is $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$, i.e., arctangent can only give us angles in quadrants 1 or 4 but our point is in quadrant 3, so we need to add a half-revolution to rotate the angle into the correct quadrant:

$$\theta = \tan^{-1}\left(\frac{-8\sqrt{3}}{-8}\right) + \pi$$
$$= \tan^{-1}(\sqrt{3}) + \pi$$
$$= \frac{\pi}{3} + \pi \qquad \text{(since } \tan\left(\frac{\pi}{3}\right) = \sqrt{3})$$
$$= \frac{4\pi}{3}$$

So the Cartesian ordered pair $\left(8, -8\sqrt{3}\right)$ corresponds to the polar ordered pair $\left(16, \dfrac{4\pi}{3}\right)$.

6. Translate the complex number $z = -3 + 3\sqrt{3} \cdot i$ into its polar form $z = re^{i\theta}$.

The number z has form $z = a + bi$ where $a = -3$ and $b = 3\sqrt{3}$. So

$$r = \sqrt{a^2 + b^2}$$
$$= \sqrt{(-3)^2 + \left(3\sqrt{3}\right)^2}$$
$$= \sqrt{9 + 27}$$
$$= 6$$

If we plot z in the Cartesian plane, we see that it lies in the second quadrant, so the angle θ is in the second quadrant, so we'll need to add π to the arctangent value:

$$\theta = \tan^{-1}\left(\frac{b}{a}\right) + \pi$$
$$= \tan^{-1}\left(\frac{3\sqrt{3}}{-3}\right) + \pi$$
$$= \tan^{-1}\left(-\sqrt{3}\right) + \pi$$
$$= \frac{2\pi}{3}$$

Therefore, $z = 6e^{i \cdot \frac{2\pi}{3}}$.

7. Translate the polar form of the complex number $z = 4e^{i \cdot \frac{5\pi}{6}}$ into its rectangular form $z = a + bi$.

$$z = 4e^{i \cdot \frac{5\pi}{6}}$$
$$= 4\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right)$$
$$= 4\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$
$$= -2\sqrt{3} + 2i$$

8. Determine the magnitude and direction (with respect to the positive x-axis) of the vector $\vec{v} = \langle -3, -7 \rangle$.

Magnitude:

$$\|\vec{v}\| = \sqrt{(-3)^2 + (-7)^2}$$
$$= \sqrt{9 + 49}$$
$$= \sqrt{58}$$

Direction:

$$\theta = \tan^{-1}\left(\tfrac{-7}{-3}\right)$$

$$\approx 66.8° + 180° \quad \text{(we add } 180° \text{ since the vector}$$
$$\text{points towards the 3rd quadrant)}$$

$$\approx 246.8°$$

Therefore the vector $\vec{v} = \langle -3, -7 \rangle$ has magnitude $\sqrt{58}$ and direction approximately $246.8°$ with respect to the positive x-axis.

9. a. Find the horizontal and vertical components of the vector $\vec{v}$ that starts at the point $P = (5, 6)$ and ends at the point $Q = (2, 2)$.

Since the vector $\vec{v}$ starts at x-coordinate 5 and ends at x-coordinate 2, we see that the horizontal component of the vector is $2 - 5 = -3$.

Since the vector $\vec{v}$ starts at y-coordinate 6 and ends at x-coordinate 2, we see that the vertical component of the vector is $2 - 6 = -4$.

Therefore the vector is $\vec{v} = \langle -3, -4 \rangle$.

b. Find the magnitude, $\|\vec{v}\|$, and the direction (with respect to the positive x-axis) of the vector $\vec{v}$ that you found in part **a**?

Magnitude:

$$\|\vec{v}\| = \sqrt{(-3)^2 + (-4)^2}$$

$$= 5$$

Direction:

$$\theta = \tan^{-1}\left(\frac{-4}{-3}\right)$$

$$\approx 53.13° + 180° \quad \text{(we add } 180° \text{ since the vector points towards the 3rd quadrant)}$$

$$\approx 233.13°$$

Therefore the vector $\vec{v} = \langle -3, -4 \rangle$ has magnitude 5 and direction approximately $233.13°$ with respect to the positive x-axis.

10. Suppose $\vec{v} = \langle -4, 1 \rangle$ and $\vec{u} = \langle 3, -6 \rangle$.

a. Find $\vec{w} = \vec{v} - 2\vec{u}$.

$$\vec{w} = \vec{v} - 2\vec{u}$$
$$= \langle -4, 1 \rangle - 2 \cdot \langle 3, -6 \rangle$$
$$= \langle -4, 1 \rangle - \langle 6, -12 \rangle$$
$$= \langle -4 - 6, \ 1 - (-12) \rangle$$
$$= \langle -10, 13 \rangle$$

b. Use the *dot product* to find the angle between $\vec{v} = \langle -4, 1 \rangle$ and $\vec{u} = \langle 3, -6 \rangle$?

We can use the fact that $\vec{v} \cdot \vec{u} = \|\vec{v}\| \cdot \|\vec{u}\| \cos(\theta)$, where θ is the angle between vectors $\vec{v}$ and $\vec{u}$. First, let's find $\|\vec{v}\|$, $\|\vec{u}\|$, and $\vec{v} \cdot \vec{u}$:

$$\|\vec{v}\| = \sqrt{(-4)^2 + (1)^2} \qquad\qquad \|\vec{u}\| = \sqrt{(3)^2 + (-6)^2}$$
$$= \sqrt{16 + 1} \qquad \text{and} \qquad = \sqrt{9 + 36}$$
$$= \sqrt{17} \qquad\qquad\qquad\qquad = \sqrt{45}$$

and

$$\vec{v} \cdot \vec{u} = (-4) \cdot 3 + 1 \cdot (-6)$$
$$= -12 - 6$$
$$= -18.$$

Thus,

$$\vec{v} \cdot \vec{u} = \|\vec{v}\| \cdot \|\vec{u}\| \cos(\theta)$$
$$\Rightarrow \quad -18 = \sqrt{17} \cdot \sqrt{45} \, \cos(\theta)$$
$$\Rightarrow \cos(\theta) = \frac{-18}{\sqrt{17} \cdot \sqrt{45}}$$
$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{-18}{\sqrt{17} \cdot \sqrt{45}}\right) \approx 130.6°$$

So the angle between vectors $\vec{v}$ and $\vec{u}$ is about $130.6°$.