SOLUTIONS: Week __ Practice Worksheet

Polar Coordinates

1. Find a polar ordered pair (r, θ) that's plotted at the same location as the given rectangular ordered pair (x, y). Answers should be ordered pairs involving exact values, with θ in radians.

a. $(x, y) = (-4, -4)$

We can use the Pythagorean Theorem to find r:

$$r^2 = (-4)^2 + (-4)^2$$
$$\Rightarrow r^2 = 16 + 16$$
$$\Rightarrow r = \sqrt{32} = 4\sqrt{2}$$

We can use tangent to find θ: $\tan(\theta) = \frac{-4}{-4} = 1$. We're familiar with the fact $\tan\left(\frac{\pi}{4}\right) = 1$ so we know that the angle should be a multiple of $\frac{\pi}{4}$. Since the given point is in the third quadrant, we can conclude that $\theta = \frac{5\pi}{4}$.

Therefore, the rectangular coordinates $(-4, -4)$ are plotted in the same location as the polar coordinates $\left(4\sqrt{2}, \frac{5\pi}{4}\right)$.

b. $(x, y) = \left(6, -6\sqrt{3}\right)$

We can use the Pythagorean Theorem to find r:

$$r^2 = (6)^2 + \left(-6\sqrt{3}\right)^2$$
$$\Rightarrow r^2 = 36 + 36 \cdot 3$$
$$\Rightarrow r = \sqrt{144} = 12$$

We can use tangent to find θ: $\tan(\theta) = \frac{-6\sqrt{3}}{6} = -\sqrt{3}$

We're familiar with the fact $\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$ so we know that the angle is a multiple of $\frac{\pi}{3}$. Since the given point is in the fourth quadrant, we can conclude that $\theta = \frac{5\pi}{3}$. (Note that we could also use $\theta = -\frac{\pi}{3}$.)

So the rectangular coordinates $\left(-6, -6\sqrt{3}\right)$ are plotted in the same location as the polar coordinates $\left(12, \frac{5\pi}{3}\right)$.

2. Find a polar ordered pair (r, θ) that's plotted at the same location as the given rectangular ordered pair (x, y). Answers should be ordered pairs involving exact values, with θ in radians.

a. $(x, y) = (10, -2)$

We can use the Pythagorean Theorem to find r:

$$r^2 = (10)^2 + (-2)^2$$
$$\Rightarrow \quad r = \sqrt{104} = 2\sqrt{26}$$

We can use tangent to find θ: $\tan(\theta) = \frac{-2}{10}$. To solve this equation for θ we'll use arctangent. Since the given point is in the fourth quadrant and since the range of arctangent is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we know that the output of arctangent will be in the correct quadrant so we won't need to adjust it:

$$\theta = \tan^{-1}\left(-\tfrac{2}{10}\right) \approx -0.197$$

So the rectangular coordinates $(10, -2)$ are plotted in approximately the same location as the polar coordinates $\left(2\sqrt{26}, -0.197\right)$.

b. $(x, y) = (-3, 7)$

We can use the Pythagorean Theorem to find r:

$$r^2 = (-3)^2 + (7)^2$$
$$\Rightarrow \quad r = \sqrt{58}$$

We can use tangent to find θ: $\tan(\theta) = \frac{7}{-3}$. To solve this equation for θ we'll use arctangent. Since the given point is in the second quadrant but the range of arctangent is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we know we'll have to add π to the output of arctangent in order to put the angle into the correct quadrant:

$$\theta = \tan^{-1}\left(-\tfrac{7}{3}\right) + \pi \approx 1.98$$

Therefore, the rectangular coordinates $(-3, 7)$ are plotted in approximately the same location as the polar coordinates $\left(\sqrt{58}, 1.98\right)$.

3. Find a rectangular ordered pair (x, y) that's plotted at the same location as the given polar ordered pair (r, θ). Answers should be ordered pairs involving exact values.

a. $(r, \theta) = \left(5, \frac{2\pi}{3}\right)$

$$\begin{aligned} x &= r \cdot \cos(\theta) \\ &= 5 \cdot \cos\left(\tfrac{2\pi}{3}\right) \\ &= 5 \cdot \left(-\tfrac{1}{2}\right) \\ &= -\tfrac{5}{2} \end{aligned}$$

and

$$\begin{aligned} y &= r \cdot \sin(\theta) \\ &= 5 \cdot \sin\left(\tfrac{2\pi}{3}\right) \\ &= 5 \cdot \left(\tfrac{\sqrt{3}}{2}\right) \\ &= \tfrac{5\sqrt{3}}{2} \end{aligned}$$

So the rectangular coordinates $\left(-\frac{5}{2}, \frac{5\sqrt{3}}{2}\right)$ are plotted in the same location as the polar coordinates $\left(5, \frac{2\pi}{3}\right)$.

b. $(r, \theta) = \left(16, 210^\circ\right)$

$$\begin{aligned} x &= r \cdot \cos(\theta) \\ &= 16 \cdot \cos\left(210^\circ\right) \\ &= 16 \cdot \left(-\tfrac{\sqrt{3}}{2}\right) \\ &= -8\sqrt{3} \end{aligned}$$

and

$$\begin{aligned} y &= r \cdot \sin(\theta) \\ &= 16 \cdot \sin\left(210^\circ\right) \\ &= 16 \cdot \left(-\tfrac{1}{2}\right) \\ &= -8 \end{aligned}$$

Therefore, the rectangular coordinates $\left(-8\sqrt{3}, -8\right)$ are plotted in the same location as the polar coordinates $\left(16, 210^\circ\right)$.

4. Find a rectangular ordered pair $(x,\ y)$ that's plotted at the same location as the given polar ordered pair $(r,\ \theta)$. Answers should be ordered pairs involving approximate values.

a. $(r,\ \theta) = \left(3,\ 80^\circ\right)$

Note that here we need to use a calculator to compute an approximation of the sine and cosine values since 80° isn't "friendly."

$$\begin{aligned} x &= r\cdot\cos(\theta) \\ &= 3\cdot\cos\left(80^\circ\right) \\ &\approx 0.52 \end{aligned} \qquad \text{and} \qquad \begin{aligned} y &= r\cdot\sin(\theta) \\ &= 3\cdot\sin\left(80^\circ\right) \\ &\approx 2.95 \end{aligned}$$

Therefore, the rectangular coordinates $(0.52,\ 2.95)$ are plotted in approximately the same location as the polar coordinates $\left(3,\ 80^\circ\right)$.

b. $(r,\ \theta) = \left(7,\ -\frac{\pi}{10}\right)$

Note that here we need to use a calculator to compute an approximation of the sine and cosine values since $-\frac{\pi}{10}$ isn't "friendly."

$$\begin{aligned} x &= r\cdot\cos(\theta) \\ &= 7\cdot\cos\left(-\tfrac{\pi}{10}\right) \\ &\approx 6.66 \end{aligned} \qquad \text{and} \qquad \begin{aligned} y &= r\cdot\sin(\theta) \\ &= 7\cdot\sin\left(-\tfrac{\pi}{10}\right) \\ &\approx -2.16 \end{aligned}$$

So the rectangular coordinates $(6.66,\ -2.16)$ are plotted in approximately the same location as the polar coordinates $\left(7,\ -\frac{\pi}{10}\right)$.

5. The polar ordered pair $\left(5, \frac{\pi}{3}\right)$ can be plotted as a "dot" on the polar coordinate plane. List four other (different) polar ordered pairs that are plotted on the same "dot." Use "-5" for "r" in at least one of your ordered pairs.

One way that we can create different polar ordered pairs that are plotted on the same "dot" is to use conterminal angles.

- Since $\frac{\pi}{3} + 2\pi = \frac{7\pi}{3}$ is coterminal with $\frac{\pi}{3}$, the polar ordered pair $\left(5, \frac{7\pi}{3}\right)$ is plotted in the same location as $\left(5, \frac{\pi}{3}\right)$.
- Since $\frac{\pi}{3} - 2\pi = -\frac{5\pi}{3}$ is coterminal with $\frac{\pi}{3}$, the polar ordered pair $\left(5, -\frac{5\pi}{3}\right)$ is plotted in the same location as $\left(5, \frac{\pi}{3}\right)$.

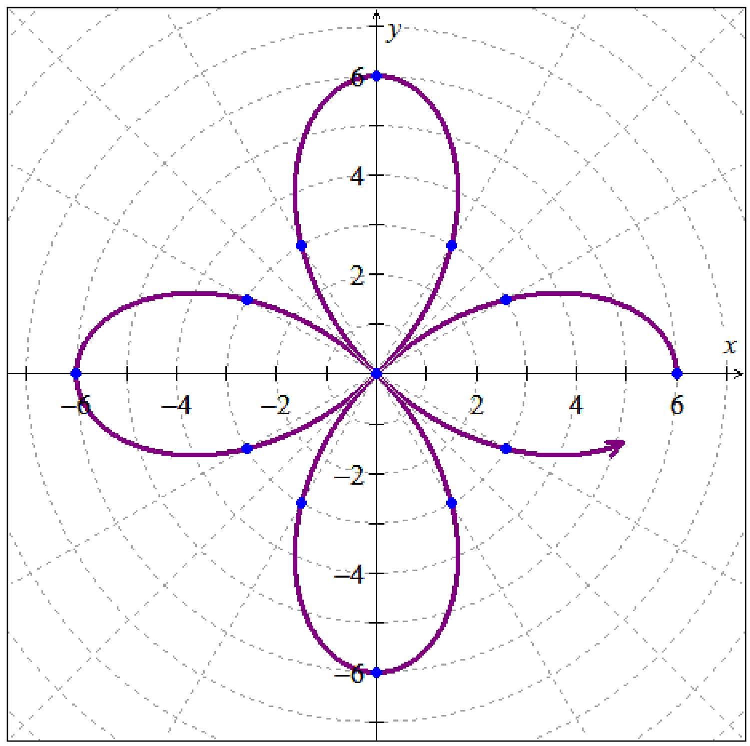
Another way that we can create different polar ordered pairs that are plotted on the same "dot" is to use a negative value for r: when r is negative, it's equivalent of adding (or subtracting) a half-revolution.

- Since $\frac{\pi}{3} + \pi = \frac{4\pi}{3}$ is a half-revolution greater than $\frac{\pi}{3}$, the polar ordered pair $\left(-5, \frac{4\pi}{3}\right)$ is plotted in the same location as $\left(5, \frac{\pi}{3}\right)$.
- Since $\frac{\pi}{3} - \pi = -\frac{2\pi}{3}$ is a half-revolution less than $\frac{\pi}{3}$, the polar ordered pair $\left(-5, -\frac{2\pi}{3}\right)$ is plotted in the same location as $\left(5, \frac{\pi}{3}\right)$.

Consider opening Desmos.com and plotting these points to see that they all land in the same location.

6. Complete the 1st column of the table below with appropriate multiples of $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$; then determine the corresponding values of r if $r = 6\cos(2\theta)$ in order to complete the 2nd column of the table; then plot the points implied by each of the 16 rows of the table on the polar plane below and connect those points in order to draw a graph of $r = 6\cos(2\theta)$.
(HINT: Graph the function in Desmos so that you can predict its shape before you draw it.)

	θ	r
	0	6
Quad. 1 angles	$\frac{\pi}{6}$	3
	$\frac{\pi}{4}$	0
	$\frac{\pi}{3}$	-3
	$\pi/2$	-6
Quad. 2 angles	$\frac{2\pi}{3}$	-3
	$\frac{3\pi}{4}$	0
	$\frac{5\pi}{6}$	3
	π	6
Quad. 3 angles	$\frac{7\pi}{6}$	3
	$\frac{5\pi}{4}$	0
	$\frac{4\pi}{3}$	-3
	$3\pi/2$	-6
Quad. 4 angles	$\frac{5\pi}{3}$	-3
	$\frac{7\pi}{4}$	0
	$\frac{11\pi}{6}$	3



A graph of $r = 6\cos(2\theta)$.

7. Express the complex number $z = 10e^{i\cdot\frac{11\pi}{6}}$ in the form $z = a + bi$.

$$\begin{aligned} z &= 10e^{i\cdot\frac{11\pi}{6}} \\ &= 10\cos\left(\tfrac{11\pi}{6}\right) + 10\sin\left(\tfrac{11\pi}{6}\right)\cdot i \\ &= 10\cdot\left(\tfrac{\sqrt{3}}{2}\right) + 10\cdot\left(-\tfrac{1}{2}\right)\cdot i \\ &= 5\sqrt{3} - 5i \end{aligned}$$

8. Express the complex number $z = 8e^{i\cdot\frac{2\pi}{3}}$ in the form $z = a + bi$.

$$\begin{aligned} z &= 8e^{i\cdot\frac{2\pi}{3}} \\ &= 8\cos\left(\tfrac{2\pi}{3}\right) + 8\sin\left(\tfrac{2\pi}{3}\right)\cdot i \\ &= 8\cdot\left(-\tfrac{1}{2}\right) + 8\cdot\left(\tfrac{\sqrt{3}}{2}\right)\cdot i \\ &= -4 + 4\sqrt{3}\,i \end{aligned}$$

9. Express the complex number $z = -7e^{i\cdot\frac{5\pi}{4}}$ in the form $z = a + bi$.

$$\begin{aligned} z &= -7e^{i\cdot\frac{5\pi}{4}} \\ &= -7\cos\left(\tfrac{5\pi}{4}\right) + (-7)\sin\left(\tfrac{5\pi}{4}\right)\cdot i \\ &= -7\cdot\left(-\tfrac{\sqrt{2}}{2}\right) - 7\cdot\left(-\tfrac{\sqrt{2}}{2}\right)\cdot i \\ &= \tfrac{7\sqrt{2}}{2} + \tfrac{7\sqrt{2}}{2}i \end{aligned}$$

10. Find a polar form, $z = re^{i\theta}$, of the complex number $z = -3 - 3\sqrt{3}\,i$.

We can associate the complex number $z = -3 - 3\sqrt{3}\,i$ with the rectangular ordered pair $(-3, -3\sqrt{3})$, and then translate this ordered pair into polar coordinates (r, θ), and finally use this polar ordered pair to obtain the polar form $z = re^{i\theta}$. First, let's find r:

$$\begin{aligned} r &= \sqrt{(-3)^2 + (-3\sqrt{3})^2} \\ &= \sqrt{9 + 9\cdot 3} \\ &= 6. \end{aligned}$$

Now, let's find θ:

$$\tan(\theta) = \frac{-3\sqrt{3}}{-3} = \sqrt{3}$$

$$\Rightarrow \qquad \theta = \tan^{-1}\left(\sqrt{3}\right) + \pi$$

(we add π since the given point is in quadrant 3 but the range of arctangent is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$)

$$\Rightarrow \qquad \theta = \frac{\pi}{3} + \pi = \frac{4\pi}{3}$$

Therefore, $z = 6e^{i\cdot\frac{4\pi}{3}}$ is a polar form of the complex number $z = -3 - 3\sqrt{3}\,i$.

11. Find a polar form, $z = re^{i\theta}$, of the complex number $z = 2 - 2i$.

We can associate the complex number $z = 2 - 2i$ with the rectangular ordered pair $(2, -2)$, and then translate this ordered pair into polar coordinates (r, θ), and finally use this polar ordered pair to obtain the polar form $z = re^{i\theta}$. First, let's find r:

$$\begin{aligned} r &= \sqrt{(2)^2 + (-2)^2} \\ &= \sqrt{4 + 4} \\ &= 2\sqrt{2}. \end{aligned}$$

Now, let's find θ:

$$\tan(\theta) = \tfrac{-2}{2}$$
$$\Rightarrow \qquad \theta = \tan^{-1}(-1)$$
$$\Rightarrow \qquad \theta = -\tfrac{\pi}{4}$$

Therefore, $z = 2\sqrt{2}\,e^{i\cdot\left(-\frac{\pi}{4}\right)}$ is a polar form of the complex number $z = 2 - 2i$.

12. Find a polar form, $z = re^{i\theta}$, of the complex number $z = -4\sqrt{3} + 12i$.

We can associate the complex number $z = -4\sqrt{3} + 12i$ with the rectangular ordered pair $(-4\sqrt{3}, 12)$, and then translate this ordered pair into polar coordinates (r, θ) , and finally use this polar ordered pair to obtain the polar form $z = re^{i\theta}$. First, let's find r :

$$\begin{aligned} r &= \sqrt{\left(-4\sqrt{3}\right)^2 + (12)^2} \\ &= \sqrt{16\cdot 3 + 144} \\ &= \sqrt{192} \\ &= 8\sqrt{3}. \end{aligned}$$

Now, let's find θ :

$$\tan(\theta) = \frac{12}{-4\sqrt{3}} = -\frac{3}{\sqrt{3}} = -\sqrt{3}$$
$$\Rightarrow \qquad \theta = \tan^{-1}\left(-\sqrt{3}\right) + \pi$$

(we add π since the given point is in quadrant 2 but the range of arctangent is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$)

$$\Rightarrow \qquad \theta = -\tfrac{\pi}{3} + \pi = \tfrac{2\pi}{3}$$

Therefore, $z = 8\sqrt{3}\,e^{i\cdot\left(\frac{2\pi}{3}\right)}$ is a polar form of the complex number $z = -4\sqrt{3} + 12i$.

13. Find three different a polar forms, $z = re^{i\theta}$, of the complex number $z = 4i$. (HINT: $i = 0 + 4i$ can be associated with the point $(0, 4)$ so find three different angles that can be used to represent the "direction of this point" and use each angle to create a polar form.)

As suggested in the HINT, we can associate the complex number $z = 4i = 0 + 4i$ with the rectangular ordered pair $(0, 4)$. We're going to translate this ordered pair into polar coordinates (r, θ), so we need and then use this polar ordered pair to obtain the polar form $z = re^{i\theta}$.

First, let's find r:

$$\begin{aligned} r &= \sqrt{(0)^2 + (4)^2} \\ &= \sqrt{16} \\ &= 4. \end{aligned}$$

Now, let's find angles that we can use for θ to align with the point $(0, 4)$. This point is on the "positive y-axis": so one angle that will work is $\frac{\pi}{2}$, and we can use two other angles that are coterminal with $\frac{\pi}{2}$ for two additional polar representations: let's use $-\frac{3\pi}{2}$ and $\frac{5\pi}{2}$.

Based on the discussion above we can conclude that the following are three different polar forms of the complex number $z = 4i$:

$$z = 4e^{i\cdot\frac{\pi}{2}}, \quad z = 4e^{i\cdot\left(-\frac{3\pi}{2}\right)}, \text{ and } \quad z = 4e^{i\cdot\frac{5\pi}{2}}$$