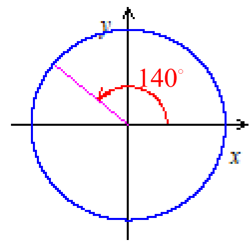
SOLUTIONS: Week 1 Practice Worksheet

Angles, Arc-length, and Periodic Functions

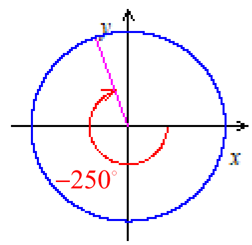
1. a. Draw 140° in standard position on the provided coordinate plane.

Since $90^\circ < 140^\circ < 180^\circ$, 140° falls in Quad 2.



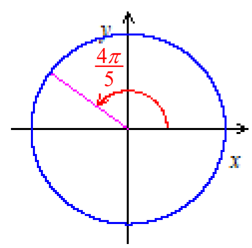
b. Draw -250° in standard position on the provided coordinate plane.

Since $-180^\circ < -250^\circ < -270^\circ$, -250° falls in Quad 2. (Since $-250^\circ < 0^\circ$, the rotation is clockwise.)



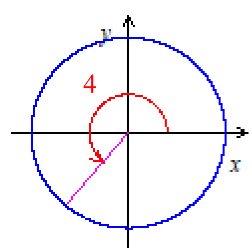
c. Draw $\frac{4\pi}{5}$ radians in standard position on the provided coordinate plane.

Since $\frac{4\pi}{5} > \frac{2.5\pi}{5} = \frac{\pi}{2}$, we know that $\frac{4\pi}{5}$ rotates beyond the border between Quads 1 and 2; and since $\frac{4\pi}{5} < \frac{5\pi}{5} = \pi$ so we know that $\frac{4\pi}{5}$ radians doesn't rotate beyond Quad 2, so it's in Quad 2.



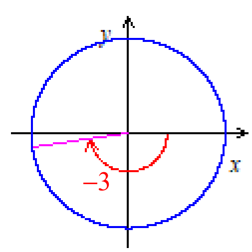
d. Draw 4 radians in standard position on the provided coordinate plane.

Certainly $4 > \pi$ so we know that 4 radians is at least in Quad 3: the only question is whether it's large enough to get into Quad 4. The border between Quads 3 and 4 is $\frac{3\pi}{2} \approx \frac{9}{2}$ and $4 < \frac{9}{2}$ so we know that 4 radians is in Quad 3.



d. Draw -3 radians in standard position on the provided coordinate plane.

-3 is *almost* $-\pi$ but a little bit greater than $-\pi$ so -3 radians rotates *almost* to $-\pi$ but falls a little short, so it lands in Quad 3. (Since $-3 < 0$, the rotation is clockwise.)



2. The Greek letters θ and ϕ are often used as variables in mathematics. They are ***different*** symbols so it's important to distinguish between them. Which of these symbols is pronounced "phi"? And which is pronounced "theta"?

θ is "theta."

ϕ is "phi."

3. a. Convert $\frac{\pi}{10}$ radians into degrees.

Let's try this using two different methods:

(1) One option is to use the fact that $\frac{180^\circ}{\pi \text{ rad}} = 1$ (since $180^\circ = \pi$ rad):

$$\begin{aligned}\tfrac{\pi}{10}\text{rad}\cdot 1 &= \tfrac{\pi}{10}\,\cancel{\text{rad}}\cdot\frac{180^\circ}{\pi\ \cancel{\text{rad}}}\\ &= \frac{\cancel{\pi}\cdot 180^\circ}{10\cdot\cancel{\pi}}\\ &= \frac{180^\circ}{10}\\ &= 18^\circ\end{aligned}$$

Thus, $\frac{\pi}{10} = 18^\circ$.

(2) Another option is to recognize that, since π rad $= 180^\circ$,

$$\frac{\pi}{10} = \frac{180^\circ}{10} = 18^\circ.$$

Therefore, $\frac{\pi}{10} = 18^\circ$.

b. Convert 4 radians into degrees.

$$\begin{aligned}4\,\text{rad}\cdot 1 &= 4\,\cancel{\text{rad}}\cdot\frac{180^\circ}{\pi\ \cancel{\text{rad}}}\\ &= \frac{720^\circ}{\pi}\\ &\approx 229.18^\circ\end{aligned}$$

Therefore, 4 radians is exactly equal to $\frac{720^\circ}{\pi}$ and approximately equal to 229.18°.

c. Convert 10° into radians.

$$10^\circ \cdot 1 = 10^{\not\circ} \cdot \frac{\pi \text{ rad}}{180^{\not\circ}}$$
$$= \frac{10\pi}{180} \text{ rad}$$
$$= \frac{\pi}{18} \text{ rad}$$

d. Convert 140° into radians.

$$140^\circ \cdot 1 = 140^{\not\circ} \cdot \frac{\pi \text{ rad}}{180^{\not\circ}}$$
$$= \frac{140\pi}{180} \text{ rad}$$
$$= \frac{14\pi}{18} \text{ rad}$$
$$= \frac{7\pi}{9} \text{ rad}$$

e. Convert 1200° into radians.

$$1200^\circ \cdot 1 = 1200^{\not\circ} \cdot \frac{\pi \text{ rad}}{180^{\not\circ}}$$
$$= \frac{1200\pi}{180} \text{ rad}$$
$$= \frac{120\pi}{18} \text{ rad}$$
$$= \frac{60\pi}{9} \text{ rad}$$
$$= \frac{20\pi}{3} \text{ rad}$$

4. a. Find both a positive and a negative angle that are coterminal with 140°. (Answer in degrees.)

To find a positive angle coterminal with 140° we can add 360°:

$$140^\circ + 360^\circ = 500^\circ$$

So 500° is coterminal with 140°.

To find a negative angle coterminal with 140° we can subtract 360°:

$$140^\circ - 360^\circ = -220^\circ$$

So -220° is coterminal with 140°.

b. Find both a positive and a negative angle that are coterminal with $-\frac{5\pi}{8}$. (Answer in radians.)

To find a positive angle coterminal with $-\frac{5\pi}{8}$ we can add 2π:

$$\begin{aligned} -\frac{5\pi}{8} + 2\pi &= -\frac{5\pi}{8} + \frac{16\pi}{8} \\ &= \frac{11\pi}{8} \end{aligned}$$

So $\frac{11\pi}{8}$ is coterminal with $-\frac{5\pi}{8}$.

To find a negative angle coterminal with $-\frac{5\pi}{8}$ we can subtract 2π:

$$\begin{aligned} -\frac{5\pi}{8} - 2\pi &= -\frac{5\pi}{8} - \frac{16\pi}{8} \\ &= -\frac{21\pi}{8} \end{aligned}$$

So $-\frac{21\pi}{8}$ is coterminal with $-\frac{5\pi}{8}$.

c. Represent infinitely many different angles that are coterminal with 70°.

To find coterminal angles, we know that we need to add (or subtract) any number of full revolutions. For example,

$$70^\circ + 360^\circ \cdot 1 = 430^\circ$$
$$70^\circ + 360^\circ \cdot 2 = 790^\circ$$
$$70^\circ + 360^\circ \cdot 3 = 1150^\circ$$
etc.

are all coterminal with 70°. To represent all of the infinitely many different possibilities, we can let the symbol k represent any integer (i.e., any whole number) using the notation "$k \in \mathbb{Z}$" (the symbol $\mathbb{Z}$ represents the set of integers and the symbol $\in$ means "is an element of" so this means, "k is an element of the set of integers") and multiply 360° by k to represent all of the different possible quantities of full revolutions. Therefore the expression

$$70^\circ + 360^\circ \cdot k \text{ for all } k \in \mathbb{Z}$$

represents infinitely many different angles that are coterminal with 70°. (Note that $\mathbb{Z}$ includes the negative whole numbers so this also represents coterminal angles resulting from subtracting full revolutions from 70°.)

d. Represent infinitely many different angles that are coterminal with $\frac{3\pi}{5}$.

To find coterminal angles, we know that we need to add (or subtract) any number of full revolutions. For example,

$$\frac{3\pi}{5} + 2\pi \cdot 1 = \frac{3\pi}{5} + \frac{10\pi}{5} = \frac{13\pi}{5}$$
$$\frac{3\pi}{5} + 2\pi \cdot 2 = \frac{3\pi}{5} + \frac{20\pi}{5} = \frac{23\pi}{5}$$
$$\frac{3\pi}{5} + 2\pi \cdot 3 = \frac{3\pi}{5} + \frac{30\pi}{5} = \frac{33\pi}{5}$$
etc.

are all coterminal with $\frac{3\pi}{5}$. To represent all of the infinitely many different possibilities, we can define k to be an integer and multiply 2π by k to represent all of the different possible quantities of full revolutions. Therefore the expression

$$\frac{3\pi}{5} + 2\pi \cdot k \text{ for all } k \in \mathbb{Z}$$

represents infinitely many different angles that are coterminal with $\frac{3\pi}{5}$. (Note that we often represent "$2\pi \cdot k$" as "$2k\pi$".)

5. Complete the table below.

θ (degrees)	0°	30°	45°	60°	90°	180°	270°	360°	3600°
θ (radians)	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	20π

6. What is the length of the arc spanned by the angle 4 radians on a circle of radius 20 yards?

We can use the formula $s = r|\theta|$ since the angle 4 is a radian measure:

$$\begin{aligned} s &= r|\theta| \\ &= (20 \text{ yards}) \cdot 4 \\ &= 80 \text{ yards} \end{aligned}$$

Therefore, the length of the arc spanned by the angle 4 on a circle of radius 20 feet is 80 yards.

7. What is the length of the arc spanned by the angle 25° on a circle of radius 30 inches?

Before we can use the formula $s = r|\theta|$, we need to convert the angle 25° into radians:

$$\begin{aligned} 25^{\not\circ} \cdot \frac{2\pi \text{ rad}}{360^{\not\circ}} &= \frac{50\pi}{360} \text{ rad} \\ &= \frac{5\pi}{36} \text{ rad} \end{aligned}$$

Now we can find the desired arc-length:

$$\begin{aligned} s &= r|\theta| \\ &= (30 \text{ inches}) \cdot \frac{5\pi}{36} \\ &= \frac{25\pi}{6} \text{ inches} \end{aligned}$$

Thus, the length of the arc spanned by the angle 25° on a circle of radius 30 feet is $\frac{25\pi}{6}$ inches.

8. What is the length of the arc spanned by the angle $\frac{4\pi}{5}$ radians on a circle of radius 15 feet?

We can use the formula $s = r|\theta|$ since the angle $\frac{4\pi}{5}$ is a radian measure:

$$\begin{aligned} s &= r|\theta| \\ &= (15 \text{ feet}) \cdot \tfrac{4\pi}{5} \\ &= 12\pi \text{ feet} \end{aligned}$$

Therefore, the length of the arc spanned by the angle $\frac{4\pi}{5}$ on a circle of radius 15 feet is 12π feet.

9. What is the length of the arc spanned by the angle 135° on a circle of radius 8 meters?

Before we can use the formula $s = r|\theta|$, we need to convert the angle 135° into radians:

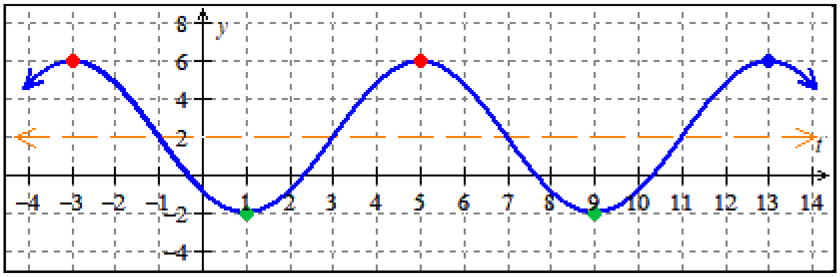
$$\begin{aligned} 135^{\circ} \cdot \frac{\pi \text{ rad}}{180^{\circ}} &= \frac{135\pi}{180} \text{ rad} \\ &= \frac{45 \cdot 3\pi}{45 \cdot 4} \text{ rad} \\ &= \frac{3\pi}{4} \text{ rad} \end{aligned}$$

Now we can find the desired arc-length:

$$\begin{aligned} s &= r|\theta| \\ &= (8 \text{ meters}) \cdot \tfrac{3\pi}{4} \\ &= 6\pi \text{ meters} \end{aligned}$$

Therefore, the length of the arc spanned by the angle 25° on a circle of radius 8 meters is 6π meters.

10. Determine the period, midline and amplitude of the function $y = f(t)$ graphed below. Note that the following points are on the graph: $(-3, 6)$, $(1, -2)$, $(5, 6)$, $(9, -2)$, and $(13, 6)$.



The graph of $y = f(t)$.

The <u>period</u> of the function graphed above is 8 units. There are a few ways to compute this. For example, we can compare the x-values for the ordered pairs $(\mathbf{-3}, 6)$ and $(\mathbf{5}, 6)$ (colored red in the graph above) or the x-values for the ordered pairs $(\mathbf{1}, -2)$ and $(\mathbf{9}, -2)$ (colored green in the graph above):

$$5 - (-3) = 8 \quad \text{and} \quad 9 - 1 = 8 .$$

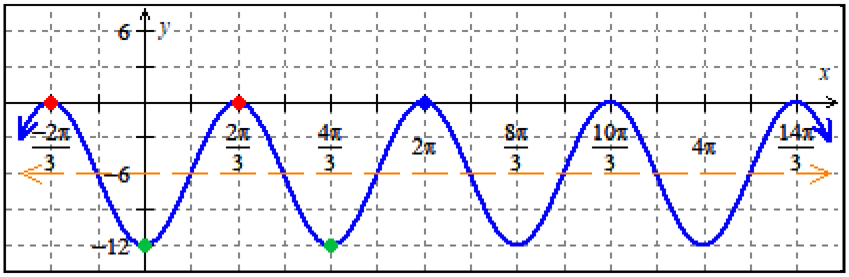
The <u>midline</u> of the graphed function is $y = 2$. There are a few ways to compute this. For example, we can find the average of the y-values for the ordered pairs $(-3, \mathbf{6})$ and $(1, \mathbf{-2})$ (colored red and green in the graph above) since the function reaches, respectively, maximum and minimum values at these points:

$$y = \frac{6 + (-2)}{2} = 2 .$$

The <u>amplitude</u> of the graphed function is 4 units. There are a few ways to compute this. For example, we can determine the distance between the midline, $y = \mathbf{2}$, and the y-value of the ordered pair $(-3, \mathbf{6})$ where the function reaches its maximum:

$$6 - 2 = 4 .$$

11. Determine the period, midline and amplitude of the function $y = g(t)$ graphed below. Note that the following points are on the graph: $\left(-\frac{2\pi}{3}, 0\right)$, $(0, -12)$, $\left(\frac{2\pi}{3}, 0\right)$, $\left(\frac{4\pi}{3}, -12\right)$, and $(2\pi, 0)$.



The graph of $y = g(t)$.

The period of the function graphed above is $\frac{4\pi}{3}$ units. There are a few ways to compute this. For example, we can compare the x-values for the ordered pairs $\left(-\frac{2\pi}{3}, 0\right)$ and $\left(\frac{2\pi}{3}, 0\right)$ (colored red in the graph above) or the x-values for the ordered pairs $(0, -12)$ and $\left(\frac{4\pi}{3}, -12\right)$ (colored green in the graph above):

$$\frac{2\pi}{3} - \left(-\frac{2\pi}{3}\right) = \frac{4\pi}{3} \quad \text{and} \quad \frac{4\pi}{3} - 0 = \frac{4\pi}{3}.$$

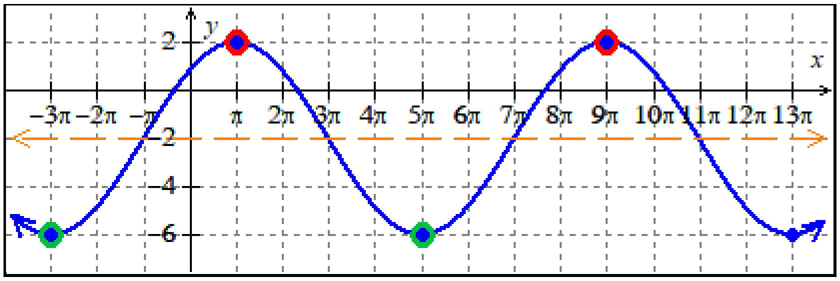
The midline of the graphed function is $y = -6$. There are a few ways to compute this. For example, we can find the average of the y-values for the ordered pairs $\left(-\frac{2\pi}{3}, 0\right)$ and $(0, -12)$ (colored red and green in the graph above) since the function reaches, respectively, maximum and minimum values at these points:

$$y = \frac{0 + (-12)}{2} = -6.$$

The amplitude of the graphed function is 6 units. There are a few ways to compute this. For example, we can determine the distance between the midline, $y = -6$, and the y-value of the ordered pair $\left(-\frac{2\pi}{3}, 0\right)$ where the function reaches its maximum:

$$0 - (-6) = 6.$$

12. Determine the period, midline and amplitude of the function $y = p(x)$ graphed below. Note that the following points are on the graph: $(-3\pi, -6)$, $(\pi, 2)$, $(5\pi, -6)$, and $(9\pi, 2)$.



The graph of $y = p(x)$.

The <u>period</u> of the function graphed above is 8π units. There are a few ways to compute this. For example, we can compare the x-values for the ordered pairs $(\pi, 2)$ and $(9\pi, 2)$ (colored red in the graph above) or the x-values for the ordered pairs $(-3\pi, -6)$ and $(5\pi, -6)$ (colored green in the graph above):

$$9\pi - \pi = 8\pi \quad \text{and} \quad 5\pi - (-3\pi) = 8\pi .$$

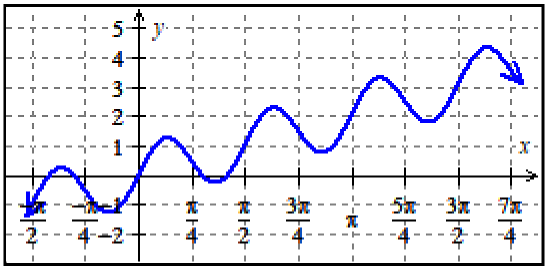
The <u>midline</u> of the graphed function is $y = -2$. There are a few ways to compute this. For example, we can find the average of the y-values for the ordered pairs $(\pi, 2)$ and $(5\pi, -6)$ (colored red and green in the graph above) since the function reaches, respectively, maximum and minimum values at these points:

$$y = \frac{2 + (-6)}{2} = -2 .$$

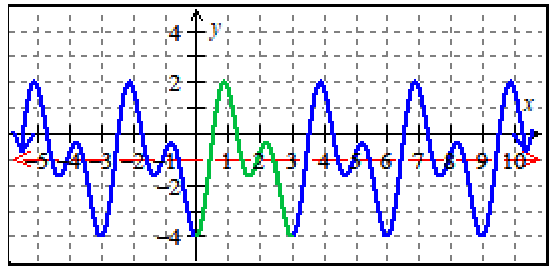
The <u>amplitude</u> of the graphed function is 4 units. There are a few ways to compute this. For example, we can determine the distance between the midline, $y = -2$, and the y-value of the ordered pair $(\pi, 2)$ where the function reaches its maximum:

$$2 - (-2) = 4 .$$

13. Determine which of the functions graphed below are periodic functions and find the period, midline, and amplitude of the periodic functions.



$y = A(x)$ is **not** periodic.

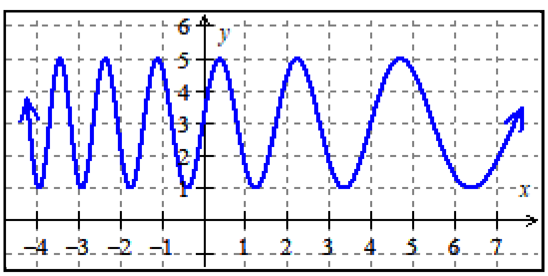


$y = B(x)$ is periodic.

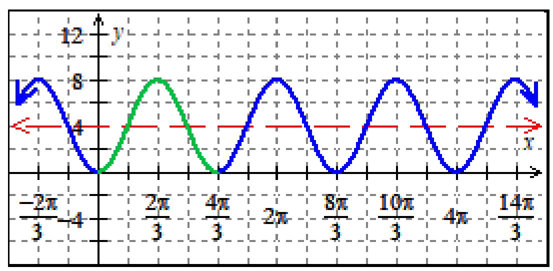
The **period** of $y = B(x)$ is 3 units. In the graph above, we've colored one period green.

The **midline** of $y = B(x)$ is $y = -1$. In the graph above, we've colored the midline red.

The **amplitude** of $y = B(x)$ is 3 units. This is half of the distance between the maximum y-value ($y = 2$) and the minimum y-value ($y = -4$).



$y = C(x)$ is **not** periodic.

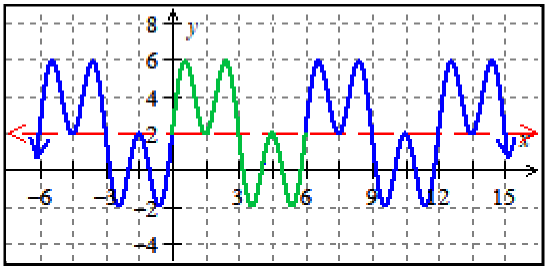


$y = D(x)$ is periodic.

The **period** of $y = D(x)$ is $\frac{4\pi}{3}$ units. In the graph above, we've colored one period green.

The **midline** of $y = D(x)$ is $y = 4$. In the graph above, we've colored the midline red.

The **amplitude** of $y = D(x)$ is 4 units. This is half of the distance between the maximum y-value ($y = 8$) and the minimum y-value ($y = 0$).

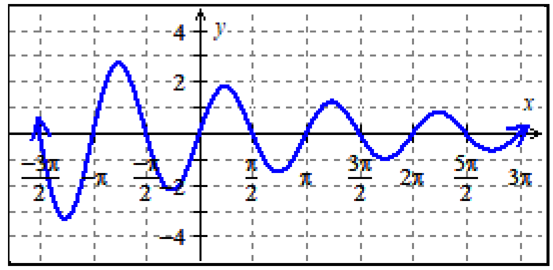


$y = E(x)$ is periodic.

The **period** of $y = E(x)$ is 6 units. In the graph above, we've colored one period green.

The **midline** of $y = E(x)$ is $y = 2$. In the graph above, we've colored the midline red.

The **amplitude** of $y = E(x)$ is 4 units. This is half of the distance between the maximum y-value ($y = 6$) and the minimum y-value ($y = -2$).



$y = F(x)$ is **not** periodic.

14. There's a Ferris wheel with a diameter of 80 feet. The wheel rotates at a constant rate, and completes a full rotation every 20 minutes. The wheel is lifted 10 feet above the ground level, and passengers load into carriages at the lowest point in the wheel's travel (so passengers start their trip 10 feet above the ground). Determine the period, midline and amplitude of the function that associates amount of time a passenger spends in a carriage travelling around the wheel with the height (in feet) above the ground of such a passenger.

The period of the function described above is 20 minutes since that's how long it takes the Ferris wheel to complete a full rotation.

The midline of the function is $y = 50$. We can compute this by finding the vertical-center of the Ferris wheel and adding 10 feet because it's lifted up 10 feet:

$$y = \frac{80}{2} + 10 = 50.$$

The amplitude of the function is 40 feet since that is half of the wheel's diameter.

15. The following verbally-described functions are approximately-periodic: assume that they are truly periodic.

a. Determine the period, midline, and amplitude of the function that associates calendar date (so input-values for this function are dates like 01/01/2023, 01/02/2023, … , 12/31/2023, … and continue for a few years) with the high temperature (in degrees Fahrenheit) at PDX on that date.

The period of the function described above is 1 year since Earth's weather cycle determined by the Earth's rotation about the sun.

The midline of the function is about $y = 65$ degrees since that's my guess at the average annual high temperature at PDX.

The amplitude of the function is about 40 degrees since my guess is that the high daily temperature varies from about 105 degrees in August and 15 degrees in January.

b. Determine the period, midline, and amplitude of the function that associates calendar date (same input values as in part (a)) with the number of hours of daylight (i.e., time between sunrise and sunset) at PDX on that date.

The period of the function described above is 1 year since daylight-length is determined by Earth's rotation about the sun.

The midline of the function is about $y = 12$ hours since that's my guess at the average number of hours of daylight at PDX.

The amplitude of the function is about 3.5 hours since my guess is that the number of hours of daylight varies from about 15.5 hours in mid-summer and 8.5 hours in winter.

c. Suppose that the water depth in a harbor oscillates between 40 feet at high tide and 10 feet at low tide, and then back to 40 feet at the next high tide 12 hours after the previous high tide. Determine the period, midline, and amplitude of the function that associates time-of-day (e.g., 12am, 1am, …, 10pm, 11pm, … and continue for a few days) and the water depth in the harbor at that time-of-day.

The period of the function described above is 12 hours since that's the amount of time between high tides.

The midline of the function is about $y = 25$ feet. .We can compute this by finding the average water depth in the harbor:

$$y = \frac{40 + 10}{2} = 25.$$

The amplitude of the function is about 15 feet since that's the difference between high tide (40 feet) and the midline (25 feet).