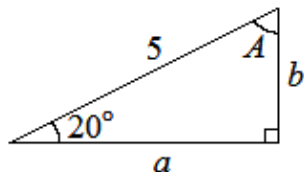


SOLUTIONS: Practice Worksheet**Some Additional Practice for the Final Exam**

1. Find the missing side(s) and missing angle(s) for the triangle given below. (The triangles may not be drawn to scale.)

a.



We can find A using the fact that the sum of the angles in a triangle is 180° :

$$\begin{aligned} A + 20^\circ + 90^\circ &= 180^\circ \\ \Rightarrow A &= 180^\circ - 20^\circ - 90^\circ \\ \Rightarrow A &= 70^\circ \end{aligned}$$

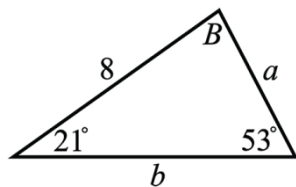
We can find b using sine:

$$\begin{aligned} \sin(20^\circ) &= \frac{b}{5} \\ \Rightarrow b &= 5 \sin(20^\circ) \\ &\approx 1.71 \end{aligned}$$

And we can use cosine to find a :

$$\begin{aligned} \cos(20^\circ) &= \frac{a}{5} \\ \Rightarrow a &= 5 \cos(20^\circ) \\ &\approx 4.698 \end{aligned}$$

b.



We can find B using the fact that the sum of the angles in a triangle is 180° :

$$\begin{aligned}21^\circ + B + 53^\circ &= 180^\circ \\ \Rightarrow B &= 180^\circ - 21^\circ - 53^\circ \\ \Rightarrow B &= 106^\circ\end{aligned}$$

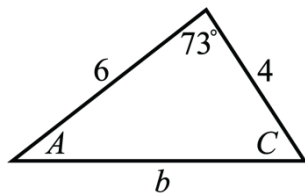
Now we can use the Law of Sines to find b :

$$\begin{aligned}\frac{8}{\sin(53^\circ)} &= \frac{b}{\sin(106^\circ)} \\ \Rightarrow b &= \frac{8\sin(106^\circ)}{\sin(53^\circ)} \approx 9.62\end{aligned}$$

And we can use the Law of Sines again to find a :

$$\begin{aligned}\frac{8}{\sin(53^\circ)} &= \frac{a}{\sin(21^\circ)} \\ \Rightarrow a &= \frac{8\sin(21^\circ)}{\sin(53^\circ)} \approx 3.59\end{aligned}$$

c.



First let's use the Law of Cosines to find b :

$$\begin{aligned} b^2 &= 4^2 + 6^2 - 2 \cdot 4 \cdot 6 \cos(73^\circ) \\ \Rightarrow b &= \sqrt{16 + 36 - 48 \cos(73^\circ)} \\ \Rightarrow b &= \sqrt{52 - 48 \cos(73^\circ)} \\ \Rightarrow b &\approx 6.16 \end{aligned}$$

Now we can use the Law of Sines to find A . (We choose to find A first instead of C since A must be smaller than C since A is across a shorter side than C .)

$$\begin{aligned} \frac{\sin(A)}{4} &= \frac{\sin(73^\circ)}{b} \\ \Rightarrow \sin(A) &= \frac{4 \sin(73^\circ)}{b} \\ \Rightarrow A &= \sin^{-1}\left(\frac{4 \sin(73^\circ)}{b}\right) \\ \Rightarrow A &\approx \sin^{-1}\left(\frac{4 \sin(73^\circ)}{6.16}\right) \\ &\approx 38.39^\circ \end{aligned}$$

Finally, we can find C using the fact that the sum of the angles in a triangle is 180° :

$$\begin{aligned} A + 73^\circ + C &= 180^\circ \\ \Rightarrow C &= 180^\circ - 73^\circ - A \\ \Rightarrow C &\approx 180^\circ - 73^\circ - 38.39^\circ \\ \Rightarrow C &\approx 68.61^\circ \end{aligned}$$

2. Suppose that $\sin(\alpha) = \frac{5}{13}$ and $\cos(\beta) = \frac{3}{5}$, and where $0 < \alpha < \frac{\pi}{2}$ and $\frac{3\pi}{2} < \beta < 2\pi$.

a. Find the exact value of $\sin(\alpha + \beta)$.

To find $\sin(\alpha + \beta)$ we'll need to use the sine-of-a-sum identity:

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)$$

We'll need $\cos(\alpha)$ and $\sin(\beta)$ to use this formula. Let's use the Pythagorean identity, and start by finding $\cos(\alpha)$:

$$\begin{aligned} \sin^2(\alpha) + \cos^2(\alpha) &= 1 \\ \Rightarrow \left(\frac{5}{13}\right)^2 + \cos^2(\alpha) &= 1 \\ \Rightarrow \frac{25}{169} + \cos^2(\alpha) &= 1 \\ \Rightarrow \cos^2(\alpha) &= 1 - \frac{25}{169} \\ \Rightarrow \cos^2(\alpha) &= \frac{144}{169} \\ \Rightarrow \cos(\alpha) &= \frac{12}{13} \end{aligned}$$

We take the positive square root of $\frac{144}{169}$ since $0 < \alpha < \frac{\pi}{2}$, i.e., α is in the quadrant 1.

Now let's use the Pythagorean identity to find $\sin(\beta)$:

$$\begin{aligned} \sin^2(\beta) + \cos^2(\beta) &= 1 \\ \Rightarrow \sin^2(\beta) + \left(\frac{3}{5}\right)^2 &= 1 \\ \Rightarrow \sin^2(\beta) + \frac{9}{25} &= 1 \\ \Rightarrow \sin^2(\beta) &= 1 - \frac{9}{25} \\ \Rightarrow \sin^2(\beta) &= \frac{16}{25} \\ \Rightarrow \sin(\beta) &= -\frac{4}{5} \end{aligned}$$

We take the negative square root of $\frac{16}{25}$ since $\frac{3\pi}{2} < \beta < 2\pi$, i.e., β is in the quadrant 4.

Therefore,

$$\begin{aligned}
 \sin(\alpha + \beta) &= \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha) \\
 &= \frac{5}{13} \cdot \frac{3}{5} + \left(-\frac{4}{5}\right) \cdot \frac{12}{13} \\
 &= \frac{15}{65} - \frac{48}{65} \\
 &= -\frac{33}{65}
 \end{aligned}$$

- b. Find the exact value of $\cos(\alpha - \beta)$.

To find $\cos(\alpha - \beta)$ we'll need to use the cosine-of-a-difference identity:

$$\cos(\alpha - \beta) = \sin(\alpha)\sin(\beta) + \cos(\alpha)\cos(\beta)$$

We can use the values we found in part (a):

$$\begin{aligned}
 \cos(\alpha - \beta) &= \sin(\alpha)\sin(\beta) + \cos(\alpha)\cos(\beta) \\
 &= \frac{5}{13} \cdot \left(-\frac{4}{5}\right) + \frac{12}{13} \cdot \frac{3}{5} \\
 &= \frac{16}{65}
 \end{aligned}$$

- c. Find the exact value of $\sin(2\beta)$.

We can use the double-angle identity:

$$\sin(2\beta) = 2\sin(\beta)\cos(\beta).$$

(Note that we can use the value of $\cos(\beta)$ that we found in part (a).)

$$\begin{aligned}
 \sin(2\beta) &= 2\sin(\beta)\cos(\beta) \\
 &= 2 \cdot \left(-\frac{4}{5}\right) \cdot \frac{3}{5} \\
 &= -\frac{24}{25}
 \end{aligned}$$

- d. Find the exact value of $\cos(2\beta)$.

We can use the double-angle identity $\cos(2\beta) = 1 - 2\sin^2(\beta)$:

$$\begin{aligned}
 \cos(2\beta) &= 1 - 2\sin^2(\beta) \\
 &= 1 - 2 \cdot \left(-\frac{4}{5}\right)^2 \\
 &= 1 - \frac{32}{25} \\
 &= -\frac{7}{25}
 \end{aligned}$$

e. Find the exact value of $\sin\left(\frac{\beta}{2}\right)$.

Recall that $\frac{3\pi}{2} < \beta < 2\pi$. Thus,

$$\begin{aligned}\frac{\frac{3\pi}{2}}{2} &< \frac{\beta}{2} < \frac{2\pi}{2} \\ \Rightarrow \frac{3\pi}{4} &< \frac{\beta}{2} < \pi.\end{aligned}$$

Since $\frac{3\pi}{4} < \frac{\beta}{2} < \pi$, we know that $\frac{\beta}{2}$ is in quadrant 2 so $\sin\left(\frac{\beta}{2}\right) > 0$; therefore, we need to choose the *positive* square root in the half-angle identity:

$$\begin{aligned}\sin\left(\frac{\beta}{2}\right) &= +\sqrt{\frac{1 - \cos(\beta)}{2}} \\ &= \sqrt{\frac{1 - \frac{3}{5}}{2}} \\ &= \sqrt{\frac{2}{5} \cdot \frac{1}{2}} \\ &= \sqrt{\frac{1}{5}} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}\end{aligned}$$

f. Find the exact value of $\cos\left(\frac{\beta}{2}\right)$.

In (e) we discovered that $\frac{\beta}{2}$ is in quadrant 2 so $\cos\left(\frac{\beta}{2}\right) < 0$; therefore, we need to choose the *negative* square root in the half-angle identity:

$$\begin{aligned}\cos\left(\frac{\beta}{2}\right) &= -\sqrt{\frac{1 + \cos(\beta)}{2}} \\ &= -\sqrt{\frac{1 + \frac{3}{5}}{2}} \\ &= -\sqrt{\frac{8}{5} \cdot \frac{1}{2}} \\ &= -\sqrt{\frac{4}{5}} \\ &= -\frac{2}{\sqrt{5}} = -\frac{2\sqrt{5}}{5}\end{aligned}$$

3. Prove the following identities.

a. $\tan(x) + \cot(x) = \sec(x)\csc(x)$

$$\begin{aligned}\tan(x) + \cot(x) &= \frac{\sin(x)}{\cos(x)} + \frac{\cos(x)}{\sin(x)} \\ &= \frac{\sin(x)}{\cos(x)} \cdot \frac{\sin(x)}{\sin(x)} + \frac{\cos(x)}{\sin(x)} \cdot \frac{\cos(x)}{\cos(x)} \\ &= \frac{\sin^2(x)}{\cos(x)\sin(x)} + \frac{\cos^2(x)}{\sin(x)\cos(x)} \\ &= \frac{\sin^2(x) + \cos^2(x)}{\cos(x)\sin(x)} \\ &= \frac{1}{\cos(x)\sin(x)} \\ &= \frac{1}{\cos(x)} \cdot \frac{1}{\sin(x)} \\ &= \sec(x)\csc(x)\end{aligned}$$

b. $\tan^2(x) - \sin^2(x) = \tan^2(x)\sin^2(x)$

$$\begin{aligned}\tan^2(x) - \sin^2(x) &= \frac{\sin^2(x)}{\cos^2(x)} - \sin^2(x) \cdot \frac{\cos^2(x)}{\cos^2(x)} \\ &= \frac{\sin^2(x) - \sin^2(x)\cos^2(x)}{\cos^2(x)} \\ &= \frac{\sin^2(x) \cdot (1 - \cos^2(x))}{\cos^2(x)} \\ &= \frac{\sin^2(x) \cdot \sin^2(x)}{\cos^2(x)} \\ &= \frac{\sin^2(x)}{\cos^2(x)} \cdot \sin^2(x) \\ &= \tan^2(x)\sin^2(x)\end{aligned}$$

c. $\cos(2x) = \cos^4(x) - \sin^4(x)$

$$\begin{aligned}\cos^4(x) - \sin^4(x) &= (\cos^2(x) - \sin^2(x))(\cos^2(x) + \sin^2(x)) \\ &= (\cos^2(x) - \sin^2(x)) \cdot 1 \\ &= \cos^2(x) - \sin^2(x) \\ &= \cos(2x)\end{aligned}$$

4. Convert the following polar ordered pairs into Cartesian (i.e., rectangular) coordinates.

a. $\left(3, \frac{\pi}{2}\right)$

The polar coordinates (r, θ) correspond to the Cartesian coordinates $(x, y) = (r \cos(\theta), r \sin(\theta))$. So

$$\begin{aligned}x &= 3 \cos\left(\frac{\pi}{2}\right) & \text{and} & & y &= 3 \sin\left(\frac{\pi}{2}\right) \\ &= 0 & & & &= 3\end{aligned}$$

Therefore, the polar ordered pair $\left(3, \frac{\pi}{2}\right)$ corresponds to the Cartesian ordered pair $(0, 3)$.

b. $\left(\pi, \frac{5\pi}{3}\right)$

The polar coordinates (r, θ) correspond to the Cartesian coords. $(x, y) = (r \cos(\theta), r \sin(\theta))$. So

$$\begin{aligned}x &= \pi \cos\left(\frac{5\pi}{3}\right) & & & y &= \pi \sin\left(\frac{5\pi}{3}\right) \\ &= \pi \cdot \frac{1}{2} & \text{and} & & &= \pi \cdot -\frac{\sqrt{3}}{2} \\ &= \frac{\pi}{2} & & & &= -\frac{\pi\sqrt{3}}{2}\end{aligned}$$

Therefore, the polar ordered pair $\left(\pi, \frac{5\pi}{3}\right)$ corresponds to the Cartesian ordered pair $\left(\frac{\pi}{2}, -\frac{\pi\sqrt{3}}{2}\right)$.

c. $(10, -10^\circ)$

The polar coordinates (r, θ) correspond to the Cartesian coords. $(x, y) = (r \cos(\theta), r \sin(\theta))$. So

$$\begin{aligned} x &= 10 \cos(-10^\circ) & \text{and} & & y &= 10 \sin(-10^\circ) \\ &\approx 9.85 & & & &\approx -1.74 \end{aligned}$$

Therefore, the polar ordered pair $(10, -10^\circ)$ corresponds to the Cartesian ordered pair $(10 \cos(10^\circ), 10 \sin(10^\circ)) \approx (9.85, -1.74)$.

5. Convert the following Cartesian (i.e., rectangular) ordered pairs into polar coordinates.

a. $(10, -10)$

We need to convert Cartesian coordinates (x, y) into polar coordinates (r, θ) . The distance from the origin to the point, r , is given by $r = \sqrt{x^2 + y^2}$:

$$\begin{aligned} r &= \sqrt{10^2 + (-10)^2} \\ &= 10\sqrt{2} \end{aligned}$$

To find θ (which should be in the 4th quadrant since that's where the given point sits), we can use the formula $\theta = \tan^{-1}\left(\frac{y}{x}\right)$:

$$\begin{aligned} \theta &= \tan^{-1}\left(\frac{10}{-10}\right) \\ &= \tan^{-1}(-1) \\ &= -\frac{\pi}{4} \quad \left(\text{since } \tan\left(-\frac{\pi}{4}\right) = -1\right) \end{aligned}$$

Therefore, the Cartesian ordered pair $(10, -10)$ corresponds to the polar ordered pair $(10\sqrt{2}, -\frac{\pi}{4})$.

b. $(-3, 0)$

We need to convert Cartesian coordinates (x, y) into polar coordinates (r, θ) . Clearly $r = 3$ and $\theta = \pi$ (since the point lies on the negative x -axis). Therefore, the Cartesian ordered pair $(-3, 0)$ corresponds to the polar ordered pair $(3, \pi)$.

c. $(-8, -8\sqrt{3})$

We need to convert Cartesian coordinates (x, y) into polar coordinates (r, θ) . The distance from the origin to the point, r , is given by $r = \sqrt{x^2 + y^2}$:

$$\begin{aligned} r &= \sqrt{(-8)^2 + (-8\sqrt{3})^2} \\ &= \sqrt{256} \\ &= 16 \end{aligned}$$

To find θ (which should be in the 3rd quadrant since that's where the given point sits), we can use the formula $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ – but we'll need to add π to the result since the range of arctangent is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, i.e., arctangent can only give us angles in quadrants 1 or 4 but our point is in quadrant 3, so we need to add a half-revolution to rotate the angle into the correct quadrant:

$$\begin{aligned} \theta &= \tan^{-1}\left(\frac{-8\sqrt{3}}{-8}\right) + \pi \\ &= \tan^{-1}(\sqrt{3}) + \pi \\ &= \frac{\pi}{3} + \pi \quad (\text{since } \tan\left(\frac{\pi}{3}\right) = \sqrt{3}) \\ &= \frac{4\pi}{3} \end{aligned}$$

So the Cartesian ordered pair $(-8, -8\sqrt{3})$ corresponds to the polar ordered pair $\left(16, \frac{4\pi}{3}\right)$.

6. Translate the complex number $z = -3 + 3\sqrt{3} \cdot i$ into its polar form $z = re^{i\theta}$.

The number z has form $z = a + bi$ where $a = -3$ and $b = 3\sqrt{3}$. So

$$\begin{aligned}r &= \sqrt{a^2 + b^2} \\&= \sqrt{(-3)^2 + (3\sqrt{3})^2} \\&= \sqrt{9 + 27} \\&= 6\end{aligned}$$

If we plot z in the Cartesian plane, we see that it lies in the second quadrant, so the angle θ is in the second quadrant, so we'll need to add π to the arctangent value:

$$\begin{aligned}\theta &= \tan^{-1}\left(\frac{b}{a}\right) + \pi \\&= \tan^{-1}\left(\frac{3\sqrt{3}}{-3}\right) + \pi \\&= \tan^{-1}\left(-\sqrt{3}\right) + \pi \\&= \frac{2\pi}{3}\end{aligned}$$

Therefore, $z = 6e^{i \cdot \frac{2\pi}{3}}$.

7. Translate the polar form of the complex number $z = 4e^{i \cdot \frac{5\pi}{6}}$ into its rectangular form $z = a + bi$.

$$\begin{aligned}z &= 4e^{i \cdot \frac{5\pi}{6}} \\&= 4\left(\cos\left(\frac{5\pi}{6}\right) + i \sin\left(\frac{5\pi}{6}\right)\right) \\&= 4\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) \\&= -2\sqrt{3} + 2i\end{aligned}$$

8. Determine the magnitude and direction (with respect to the positive x -axis) of the vector $\vec{v} = \langle -3, -7 \rangle$.

Magnitude:

$$\begin{aligned}\|\vec{v}\| &= \sqrt{(-3)^2 + (-7)^2} \\ &= \sqrt{9 + 49} \\ &= \sqrt{58}\end{aligned}$$

Direction:

$$\begin{aligned}\theta &= \tan^{-1}\left(\frac{-7}{-3}\right) \\ &\approx 66.8^\circ + 180^\circ \quad (\text{we add } 180^\circ \text{ since the vector} \\ &\quad \text{points towards the 3rd quadrant)} \\ &\approx 246.8^\circ\end{aligned}$$

Therefore the vector $\vec{v} = \langle -3, -7 \rangle$ has magnitude $\sqrt{58}$ and direction approximately 246.8° with respect to the positive x -axis.

9. a. Find the horizontal and vertical components of the vector \vec{v} that starts at the point $P = (5, 6)$ and ends at the point $Q = (2, 2)$.

Since the vector \vec{v} starts at x -coordinate 5 and ends at x -coordinate 2, we see that the horizontal component of the vector is $2 - 5 = -3$.

Since the vector \vec{v} starts at y -coordinate 6 and ends at y -coordinate 2, we see that the vertical component of the vector is $2 - 6 = -4$.

Therefore the vector is $\vec{v} = \langle -3, -4 \rangle$.

- b. Find the magnitude, $\|\vec{v}\|$, and the direction (with respect to the positive x -axis) of the vector \vec{v} that you found in part a?

Magnitude:

$$\begin{aligned}\|\vec{v}\| &= \sqrt{(-3)^2 + (-4)^2} \\ &= 5\end{aligned}$$

Direction:

$$\begin{aligned}\theta &= \tan^{-1}\left(\frac{-4}{-3}\right) \\ &\approx 53.13^\circ + 180^\circ \quad (\text{we add } 180^\circ \text{ since the vector} \\ &\quad \text{points towards the 3rd quadrant)} \\ &\approx 233.13^\circ\end{aligned}$$

Therefore the vector $\vec{v} = \langle -3, -4 \rangle$ has magnitude 5 and direction approximately 233.13° with respect to the positive x -axis.

10. Suppose $\vec{v} = \langle -4, 1 \rangle$ and $\vec{u} = \langle 3, -6 \rangle$.

- a. Find $\vec{w} = \vec{v} - 2\vec{u}$.

$$\begin{aligned}\vec{w} &= \vec{v} - 2\vec{u} \\ &= \langle -4, 1 \rangle - 2 \cdot \langle 3, -6 \rangle \\ &= \langle -4, 1 \rangle - \langle 6, -12 \rangle \\ &= \langle -4 - 6, 1 - (-12) \rangle \\ &= \langle -10, 13 \rangle\end{aligned}$$

- b. Use the *dot product* to find the angle between $\vec{v} = \langle -4, 1 \rangle$ and $\vec{u} = \langle 3, -6 \rangle$?

We can use the fact that $\vec{v} \cdot \vec{u} = \|\vec{v}\| \cdot \|\vec{u}\| \cos(\theta)$, where θ is the angle between vectors \vec{v} and \vec{u} . First, let's find $\|\vec{v}\|$, $\|\vec{u}\|$, and $\vec{v} \cdot \vec{u}$:

$$\begin{aligned}\|\vec{v}\| &= \sqrt{(-4)^2 + (1)^2} & \|\vec{u}\| &= \sqrt{(3)^2 + (-6)^2} \\ &= \sqrt{16 + 1} & \text{and} & & = \sqrt{9 + 36} \\ &= \sqrt{17} & & & = \sqrt{45}\end{aligned}$$

and

$$\begin{aligned}\vec{v} \cdot \vec{u} &= (-4) \cdot 3 + 1 \cdot (-6) \\ &= -12 - 6 \\ &= -18.\end{aligned}$$

Thus,

$$\begin{aligned}\vec{v} \cdot \vec{u} &= \|\vec{v}\| \cdot \|\vec{u}\| \cos(\theta) \\ \Rightarrow -18 &= \sqrt{17} \cdot \sqrt{45} \cos(\theta) \\ \Rightarrow \cos(\theta) &= \frac{-18}{\sqrt{17} \cdot \sqrt{45}} \\ \Rightarrow \theta &= \cos^{-1}\left(\frac{-18}{\sqrt{17} \cdot \sqrt{45}}\right) \approx 130.6^\circ\end{aligned}$$

So the angle between vectors \vec{v} and \vec{u} is about 130.6° .