SOLUTIONS: Practice Worksheet: Complex Numbers

1. Express the complex number $z = 10e^{i\cdot\frac{11\pi}{6}}$ in the form $z = a + bi$.

$$\begin{aligned} z &= 10e^{i\cdot\frac{11\pi}{6}} \\ &= 10\cos\left(\tfrac{11\pi}{6}\right) + 10\sin\left(\tfrac{11\pi}{6}\right)\cdot i \\ &= 10\cdot\left(\tfrac{\sqrt{3}}{2}\right) + 10\cdot\left(-\tfrac{1}{2}\right)\cdot i \\ &= 5\sqrt{3} - 5i \end{aligned}$$

2. Express the complex number $z = 8e^{i\cdot\frac{2\pi}{3}}$ in the form $z = a + bi$.

$$\begin{aligned} z &= 8e^{i\cdot\frac{2\pi}{3}} \\ &= 8\cos\left(\tfrac{2\pi}{3}\right) + 8\sin\left(\tfrac{2\pi}{3}\right)\cdot i \\ &= 8\cdot\left(-\tfrac{1}{2}\right) + 8\cdot\left(\tfrac{\sqrt{3}}{2}\right)\cdot i \\ &= -4 + 4\sqrt{3}\,i \end{aligned}$$

3. Express the complex number $z = -7e^{i\cdot\frac{5\pi}{4}}$ in the form $z = a + bi$.

$$\begin{aligned} z &= -7e^{i\cdot\frac{5\pi}{4}} \\ &= -7\cos\left(\tfrac{5\pi}{4}\right) + (-7)\sin\left(\tfrac{5\pi}{4}\right)\cdot i \\ &= -7\cdot\left(-\tfrac{\sqrt{2}}{2}\right) - 7\cdot\left(-\tfrac{\sqrt{2}}{2}\right)\cdot i \\ &= \tfrac{7\sqrt{2}}{2} + \tfrac{7\sqrt{2}}{2}i \end{aligned}$$

4. Express the complex number $z = 20e^{i\cdot\pi}$ in the form $z = a + bi$.

$$\begin{aligned} z &= 20e^{i\cdot\pi} \\ &= 20\cos(\pi) + 20\sin(\pi)\cdot i \\ &= 20\cdot(-1) + 20\cdot(0)\cdot i \\ &= -20 + 0 \\ &= -20 \end{aligned}$$

5. Find a polar form, $z = re^{i\theta}$, of the complex number $z = -3 - 3\sqrt{3}\,i$.

We can associate the complex number $z = -3 - 3\sqrt{3}\,i$ with the rectangular ordered pair $(-3, -3\sqrt{3})$, and then translate this ordered pair into polar coordinates (r, θ), and finally use this polar ordered pair to obtain the polar form $z = re^{i\theta}$. First, let's find r:

$$\begin{aligned} r &= \sqrt{(-3)^2 + (-3\sqrt{3})^2} \\ &= \sqrt{9 + 9\cdot 3} \\ &= 6. \end{aligned}$$

Now, let's find θ:

$$\tan(\theta) = \frac{-3\sqrt{3}}{-3} = \sqrt{3}$$

$$\Rightarrow \qquad \theta = \tan^{-1}\left(\sqrt{3}\right) + \pi$$

(we add π since the given point is in quadrant 3 but the range of arctangent is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$)

$$\Rightarrow \qquad \theta = \frac{\pi}{3} + \pi = \frac{4\pi}{3}$$

Therefore, $z = 6e^{i\cdot\frac{4\pi}{3}}$ is a polar form of the complex number $z = -3 - 3\sqrt{3}\,i$.

6. Find a polar form, $z = re^{i\theta}$, of the complex number $z = 2 - 2i$.

We can associate the complex number $z = 2 - 2i$ with the rectangular ordered pair $(2, -2)$, and then translate this ordered pair into polar coordinates (r, θ), and finally use this polar ordered pair to obtain the polar form $z = re^{i\theta}$. First, let's find r:

$$
\begin{aligned}
r &= \sqrt{(2)^2 + (-2)^2} \\
&= \sqrt{4 + 4} \\
&= 2\sqrt{2}.
\end{aligned}
$$

Now, let's find θ:

$$
\begin{aligned}
&\tan(\theta) = \tfrac{-2}{2} \\
\Rightarrow \quad &\theta = \tan^{-1}(-1) \\
\Rightarrow \quad &\theta = -\tfrac{\pi}{4}
\end{aligned}
$$

Therefore, $z = 2\sqrt{2}\, e^{i \cdot \left(-\frac{\pi}{4}\right)}$ is a polar form of the complex number $z = 2 - 2i$.

7. Find a polar form, $z = re^{i\theta}$, of the complex number $z = -4\sqrt{3} + 12i$.

We can associate the complex number $z = -4\sqrt{3} + 12i$ with the rectangular ordered pair $(-4\sqrt{3}, 12)$, and then translate this ordered pair into polar coordinates (r, θ), and finally use this polar ordered pair to obtain the polar form $z = re^{i\theta}$. First, let's find r:

$$\begin{aligned} r &= \sqrt{\left(-4\sqrt{3}\right)^2 + (12)^2} \\ &= \sqrt{16 \cdot 3 + 144} \\ &= \sqrt{192} \\ &= 8\sqrt{3}. \end{aligned}$$

Now, let's find θ:

$$\tan(\theta) = \frac{12}{-4\sqrt{3}} = -\frac{3}{\sqrt{3}} = -\sqrt{3}$$

$$\Rightarrow \quad \theta = \tan^{-1}\left(-\sqrt{3}\right) + \pi$$

(we add π since the given point is in quadrant 2 but the range of arctangent is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$)

$$\Rightarrow \quad \theta = -\frac{\pi}{3} + \pi = \frac{2\pi}{3}$$

Therefore, $z = 8\sqrt{3}\, e^{i \cdot \left(\frac{2\pi}{3}\right)}$ is a polar form of the complex number $z = -4\sqrt{3} + 12i$.

8. Find a polar form, $z = re^{i\theta}$, of the complex number $z = -100$.

As a complex number $z = -100$ has no imaginary part, so we can write $z = -100 + 0i$ and associate this number with the rectangular ordered pair $(-100, 0)$. We're going to translate this ordered pair into polar coordinates (r, θ), and then use this polar ordered pair to obtain the polar form $z = re^{i\theta}$.

First, let's find r:

$$\begin{aligned} r &= \sqrt{(-100)^2 + (0)^2} \\ &= \sqrt{(-100)^2} \\ &= 100. \end{aligned}$$

Now, let's find θ: using math/formulas, we could find $\tan(\theta) = \frac{0}{-100} = 0$ and use this to find θ but we know that complex numbers with no imaginary part are real numbers that we plot on the x-axis, and we know that negative numbers are plotted on the negative x-axis, and we know that the angle π aligns with that direction. So we know that π is an appropriate angle for the point $(-100, 0)$. (We can confirm that $\tan(\pi) = 0$, so this agrees with the math/formula we considered using to find the angle.)

Therefore, $z = 100e^{i \cdot \pi}$ is a polar form of the complex number $z = -100$.

9. Find three different a polar forms, $z = re^{i\theta}$, of the complex number $z = 4i$. (HINT: $i = 0 + 4i$ can be associated with the point $(0, 4)$ so find three different angles that can be used to represent the "direction of this point" and use each angle to create a polar form.)

As suggested in the HINT, we can associate the complex number $z = 4i = 0 + 4i$ with the rectangular ordered pair $(0, 4)$. We're going to translate this ordered pair into polar coordinates (r, θ), and then use this polar ordered pair to obtain the polar form $z = re^{i\theta}$.

First, let's find r:

$$\begin{aligned} r &= \sqrt{(0)^2 + (4)^2} \\ &= \sqrt{16} \\ &= 4. \end{aligned}$$

Now, let's find angles that we can use for θ to align with the point $(0, 4)$. This point is on the "positive y-axis": so one angle that will work is $\frac{\pi}{2}$, and we can use two other angles that are coterminal with $\frac{\pi}{2}$ for two additional polar representations: let's use $-\frac{3\pi}{2}$ and $\frac{5\pi}{2}$.

Based on the discussion above we can conclude that the following are three different polar forms of the complex number $z = 4i$:

$$z = 4e^{i \cdot \frac{\pi}{2}}, \quad z = 4e^{i \cdot \left(-\frac{3\pi}{2}\right)}, \text{ and } \quad z = 4e^{i \cdot \frac{5\pi}{2}}$$