SOLUTIONS: Practice Worksheet: Some Important Identities

1. Use a **sum-of-angles** or **difference-of-angles identity** to calculate the *exact value* of each of the following. (These identities are included on the Identities and Formulas Reference Sheet that will be provided to you during the Final Exam so you don't need to memorize them.)

a. $\sin(165^\circ)$

$$\begin{aligned}\sin(165^\circ) &= \sin(120^\circ + 45^\circ)\\ &= \sin(120^\circ)\cos(45^\circ) + \sin(45^\circ)\cos(120^\circ)\\ &= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\cdot\left(-\frac{1}{2}\right)\\ &= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

b. $\cos\left(\frac{13\pi}{12}\right)$

$$\begin{aligned}\cos\left(\frac{13\pi}{12}\right) &= \cos\left(\frac{3\pi}{12} + \frac{10\pi}{12}\right)\\ &= \cos\left(\frac{\pi}{4} + \frac{5\pi}{6}\right)\\ &= \cos\left(\frac{\pi}{4}\right)\cos\left(\frac{5\pi}{6}\right) - \sin\left(\frac{\pi}{4}\right)\sin\left(\frac{5\pi}{6}\right)\\ &= \frac{\sqrt{2}}{2}\cdot\left(-\frac{\sqrt{3}}{2}\right) - \frac{\sqrt{2}}{2}\cdot\frac{1}{2}\\ &= -\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}\\ &= \frac{-\sqrt{6}-\sqrt{2}}{4} = -\frac{\sqrt{6}+\sqrt{2}}{4}\end{aligned}$$

c. $\tan\left(\frac{17\pi}{12}\right)$

We could use the sum/difference formula for tangent but, assuming that we were trying to memorize the identities rather than copy them off of our Identities and Formulas Reference Sheet, it's not worth trying to memorize the relatively obscure identities for tangent since we can use what we know about sine and cosine along with the fact that $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$:

$$\begin{aligned}
\tan\left(\frac{17\pi}{12}\right) &= \frac{\sin\left(\frac{17\pi}{12}\right)}{\cos\left(\frac{17\pi}{12}\right)} \\
&= \frac{\sin\left(\frac{20\pi}{12} - \frac{3\pi}{12}\right)}{\cos\left(\frac{20\pi}{12} - \frac{3\pi}{12}\right)} \\
&= \frac{\sin\left(\frac{5\pi}{3} - \frac{\pi}{4}\right)}{\cos\left(\frac{5\pi}{3} - \frac{\pi}{4}\right)} \\
&= \frac{\sin\left(\frac{5\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{5\pi}{3}\right)}{\cos\left(\frac{5\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{5\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)} \\
&= \frac{-\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2}}{\frac{1}{2}\cdot\frac{\sqrt{2}}{2} + \left(-\frac{\sqrt{3}}{2}\right)\cdot\frac{\sqrt{2}}{2}} \\
&= \frac{\frac{-\sqrt{6} - \sqrt{2}}{4}}{\frac{\sqrt{2} - \sqrt{6}}{4}} \\
&= -\frac{\sqrt{2} + \sqrt{6}}{\sqrt{2} - \sqrt{6}} \cdot \frac{\sqrt{2} + \sqrt{6}}{\sqrt{2} + \sqrt{6}}
\end{aligned}$$

(we'll use the conjugate of the denominator to rationalize the denominator; this isn'tt required)

$$\begin{aligned}
&= -\frac{2 + 2\sqrt{12} + 6}{2 - 6} \\
&= -\frac{8 + 4\sqrt{3}}{-4} \\
&= 2 + \sqrt{3}
\end{aligned}$$

2. In order to get familiar with the **sum-of-angles**, **difference-of-angles**, **double-angle** and **half-angle identities**, we'll use these identities to calculate some "friendly" sine and cosine values – so we already know these sine and cosine values and we'll verify that the identities lead us to these values.

a. Find $\sin\left(\frac{\pi}{2}\right)$ using the fact that $\frac{\pi}{2} = \frac{\pi}{6} + \frac{\pi}{3}$.

$$\begin{aligned}\sin\left(\frac{\pi}{2}\right) &= \sin\left(\frac{\pi}{6} + \frac{\pi}{3}\right)\\ &= \sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{3}\right)\\ &= \frac{1}{2}\cdot\frac{1}{2} + \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}\\ &= \frac{1}{4} + \frac{3}{4}\\ &= 1\end{aligned}$$

We know that $\sin\left(\frac{\pi}{2}\right) = 1$ so the identity gave us the correct value.

b. Find $\cos(\pi)$ using the fact that $\pi = \frac{5\pi}{6} + \frac{\pi}{6}$.

$$\begin{aligned}\cos(\pi) &= \cos\left(\frac{5\pi}{6} + \frac{\pi}{6}\right)\\ &= \cos\left(\frac{5\pi}{6}\right)\cos\left(\frac{\pi}{6}\right) - \sin\left(\frac{5\pi}{6}\right)\sin\left(\frac{\pi}{6}\right)\\ &= -\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{1}{2}\cdot\frac{1}{2}\\ &= -\frac{3}{4} - \frac{1}{4}\\ &= -1\end{aligned}$$

We know that $\cos(\pi) = -1$ so the identity gave us the correct value.

c. Find $\sin\left(\frac{2\pi}{3}\right)$ using the fact that $\frac{2\pi}{3} = 2\cdot\frac{\pi}{3}$. (hint: use a double-angle identity)

$$\begin{aligned}\sin\left(\frac{2\pi}{3}\right) &= \sin\left(2\cdot\frac{\pi}{3}\right)\\ &= 2\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}\right)\\ &= 2\cdot\frac{\sqrt{3}}{2}\cdot\frac{1}{2}\\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

We know that $\sin\left(\frac{\pi}{2}\right) = 1$ so the identity gave us the correct value.

d. Find $\cos\left(\frac{\pi}{3}\right)$ using the fact that $\frac{\pi}{3} = 2\cdot\frac{\pi}{6}$. (hint: use a double-angle identity)

$$\begin{aligned}\cos\left(\tfrac{\pi}{3}\right) &= \cos\left(2\cdot\tfrac{\pi}{6}\right)\\ &= 1 - 2\sin^2\left(\tfrac{\pi}{6}\right)\\ &= 1 - 2\cdot\left(\tfrac{1}{2}\right)^2\\ &= 1 - 2\cdot\tfrac{1}{4}\\ &= \tfrac{1}{2}\end{aligned}$$

We know that $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$ so the identity gave us the correct value.

e. Find $\sin\left(\frac{\pi}{4}\right)$ using the fact that $\frac{\pi}{4} = \frac{\pi/2}{2}$. (hint: use a half-angle identity)

$$\begin{aligned}\sin\left(\tfrac{\pi}{4}\right) &= \sin\left(\tfrac{\pi/2}{2}\right)\\ &= +\sqrt{\frac{1-\cos\left(\frac{\pi}{2}\right)}{2}}\\ &= \sqrt{\frac{1-0}{2}}\\ &= \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}\end{aligned}$$

We know that $\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ so the identity gave us the correct value.

f. Find $\cos(30^\circ)$ using the fact that $30^\circ = \frac{60^\circ}{2}$. (hint: use a half-angle identity)

$$\begin{aligned}\cos\left(30^\circ\right) &= \cos\left(\tfrac{60^\circ}{2}\right)\\ &= +\sqrt{\frac{1+\cos(60^\circ)}{2}}\\ &= \sqrt{\frac{1+1/2}{2}}\\ &= \sqrt{\frac{3}{4}}\\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

We know that $\cos(30^\circ) = \frac{\sqrt{3}}{2}$ so the identity gave us the correct value.

3. Suppose that $\sin(\alpha) = -\frac{\sqrt{65}}{9}$ and $\pi < \alpha < \frac{3\pi}{2}$. Calculate the *exact value* of each of the following using an appropriate **double-angle** or **half-angle identity**. (These identities are included on the Identities and Formulas Reference Sheet.)

a. $\sin(2\alpha)$. [Hint: first find $\cos(\alpha)$]

Since the double-angle identity for sine involves $\cos(\alpha)$ let's first find this using the Pythagorean identity:

$$\sin^2(\alpha) + \cos^2(\alpha) = 1$$

$$\Rightarrow \left(-\frac{\sqrt{65}}{9}\right)^2 + \cos^2(\alpha) = 1 \quad \text{(since } \sin(\alpha) = -\tfrac{\sqrt{65}}{9}\text{)}$$

$$\Rightarrow \cos^2(\alpha) = 1 - \frac{65}{81}$$

$$\Rightarrow \cos^2(\alpha) = \frac{16}{81}$$

$$\Rightarrow \cos(\alpha) = -\frac{4}{9}$$

(note that we take the negative square root since cosine is negative in the 3rd quadrant)

Thus,

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$$

$$= 2\left(-\frac{\sqrt{65}}{9}\right)\left(-\frac{4}{9}\right) \quad \text{(since } \sin(\alpha) = -\tfrac{\sqrt{65}}{9} \text{ and } \cos(\alpha) = -\tfrac{4}{9}\text{)}$$

$$= \frac{8\sqrt{65}}{81}$$

b. $\cos(2\alpha)$.

We could use any one of the three double-angle identities for cosine but we'll choose the identity that only involves sine since we are given the sine value in the question. (If we use given info rather than info that we've discovered in part **a.**, we avoid the possibility of using incorrect info if we made mistakes in part **a.**)

$$\cos(2\alpha) = 1 - 2\sin^2(\alpha)$$

$$= 1 - 2\cdot\left(-\frac{\sqrt{65}}{9}\right)^2 \quad \text{(since } \sin(\alpha) = -\tfrac{\sqrt{65}}{9}\text{)}$$

$$= 1 - 2\cdot\frac{65}{81}$$

$$= \frac{81}{81} - \frac{130}{81}$$

$$= -\frac{49}{81}$$

c. $\sin\left(\frac{\alpha}{2}\right)$.

The half-angle identity for sine is $\sin\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos(\alpha)}{2}}$, and the "$\pm$" in the identity suggests that we need to determine which sign is correct in this case. Since

$$\begin{aligned} &\pi < \alpha < \frac{3\pi}{2} \\ \Rightarrow\ &\frac{\pi}{2} < \frac{\alpha}{2} < \frac{3\pi}{2\cdot 2} \\ \Rightarrow\ &\frac{\pi}{2} < \frac{\alpha}{2} < \frac{3\pi}{4}, \end{aligned}$$

we see that $\frac{\alpha}{2}$ is in the 2nd quadrant so $\sin\left(\frac{\alpha}{2}\right) > 0$, so we need the positive value:

$$\begin{aligned} \sin\left(\frac{\alpha}{2}\right) &= +\sqrt{\frac{1-\cos(\alpha)}{2}} \\ &= \sqrt{\frac{1-\left(-\frac{4}{9}\right)}{2}} \qquad \text{(since } \cos(\alpha) = -\tfrac{4}{9}\text{)} \\ &= \sqrt{\frac{13}{9}\cdot\frac{1}{2}} \\ &= \sqrt{\frac{13}{18}} \\ &= \frac{\sqrt{13}}{\sqrt{9\cdot 2}} = \frac{\sqrt{13}}{3\sqrt{2}} = \frac{\sqrt{26}}{6} \end{aligned}$$

4. Suppose that $\cos(\theta) = \frac{7}{10}$ and $\frac{3\pi}{2} < \theta < 2\pi$. Calculate the *exact value* of each of the following using an appropriate **double-angle** or **half-angle identity**. (These identities are included on the Identities and Formulas Reference Sheet.)

a. $\sin(2\theta)$. [Hint: first find $\sin(\theta)$]

Since the double-angle identity for sine involves $\cos(\theta)$ let's first find this using the Pythagorean identity:

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

$$\Rightarrow \sin^2(\theta) + \left(\frac{7}{10}\right)^2 = 1 \qquad \text{(since } \cos(\theta) = \tfrac{7}{10}\text{)}$$

$$\Rightarrow \sin^2(\theta) = 1 - \frac{49}{100}$$

$$\Rightarrow \sin(\theta) = -\frac{\sqrt{51}}{10}$$

(note that we take the negative square root since sine is negative in the 4th quadrant)

Thus,

$$\begin{aligned}\sin(2\theta) &= 2\sin(\theta)\cos(\theta) \\ &= 2\left(-\frac{\sqrt{51}}{10}\right)\left(\frac{7}{10}\right) \qquad \text{(since } \sin(\theta) = -\tfrac{\sqrt{51}}{10} \text{ and } \cos(\theta) = \tfrac{7}{10}\text{)} \\ &= -\frac{14\sqrt{51}}{100}\end{aligned}$$

b. $\cos(2\theta)$.

We could use any one of the three double-angle identities for cosine but we'll choose the identity that only involves cosine since we are given the cosine value in the question. (If we use given info rather than info that we've discovered in part **a.**, we avoid the possibility of using incorrect info if we made mistakes in part **a.**)

$$\begin{aligned}\cos(2\theta) &= 2\cos^2(\theta) - 1 \\ &= 2\cdot\left(\frac{7}{10}\right)^2 - 1 \qquad \text{(since } \cos(\theta) = \tfrac{7}{10}\text{)} \\ &= 2\cdot\frac{49}{100} - 1 \\ &= \frac{98}{100} - \frac{100}{100} \\ &= -\frac{2}{100} \\ &= -\frac{1}{50}\end{aligned}$$

c. $\cos\left(\frac{\theta}{2}\right)$.

The half-angle identity for cosine is $\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos(\theta)}{2}}$; and the "$\pm$" in the identity suggests that we need to determine which sign is correct in this case. Since

$$\begin{aligned} &\frac{3\pi}{2} < \theta < 2\pi \\ \Rightarrow\ &\frac{3\pi}{2\cdot 2} < \frac{\theta}{2} < \frac{2\pi}{2} \\ \Rightarrow\ &\frac{3\pi}{4} < \frac{\theta}{2} < \pi, \end{aligned}$$

we see that $\frac{\theta}{2}$ is in the 2nd quadrant so $\cos\left(\frac{\theta}{2}\right) < 0$; thus, we'll use the negative value:

$$\begin{aligned} \cos\left(\frac{\theta}{2}\right) &= -\sqrt{\frac{1+\cos(\theta)}{2}} \\ &= -\sqrt{\frac{1+{}^{7}\!/_{10}}{2}} \qquad \text{(since } \cos(\theta) = \tfrac{7}{10}\text{)} \\ &= -\sqrt{\frac{17}{10}\cdot\frac{1}{2}} \\ &= -\sqrt{\frac{17}{20}} \\ &= -\frac{\sqrt{17}}{2\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}} = -\frac{\sqrt{85}}{2\cdot 5} = -\frac{\sqrt{85}}{10} \end{aligned}$$

5. Suppose that $\sin(x) = \frac{9}{11}$ and $\frac{\pi}{2} < x < \pi$. Calculate the *exact value* of each of the following using an appropriate **double-angle** or **half-angle identity**. (These identities are included on the Identities and Formulas Reference Sheet.)

a. $\sin(2x)$. [Hint: first find $\cos(x)$]

Since the double-angle identity for sine involves $\cos(x)$ let's first find this using the Pythagorean identity:

$$\sin^2(x) + \cos^2(x) = 1$$

$$\Rightarrow \left(\frac{9}{11}\right)^2 + \cos^2(x) = 1 \qquad \text{(since } \sin(x) = \tfrac{9}{11}\text{)}$$

$$\Rightarrow \cos^2(x) = 1 - \frac{81}{121}$$

$$\Rightarrow \cos^2(x) = \frac{40}{121}$$

$$\Rightarrow \cos(x) = -\sqrt{\frac{40}{121}} \qquad \text{(note that we take the negative square root since cosine is negative in the 2nd quadrant)}$$

$$\Rightarrow \cos(x) = -\frac{2\sqrt{10}}{11}$$

Thus,

$$\sin(2x) = 2\sin(x)\cos(x)$$

$$= 2\left(\frac{9}{11}\right)\left(-\frac{2\sqrt{10}}{11}\right) \qquad \text{(since } \sin(x) = \tfrac{9}{11} \text{ and } \cos(x) = -\tfrac{2\sqrt{10}}{11}\text{)}$$

$$= -\frac{36\sqrt{10}}{121}$$

b. $\cos(2x)$.

We could use any one of the three double-angle identities for cosine but we'll choose the identity that only involves sine since we are given the sine value in the question. (If we use given info rather than info that we've discovered in part **a.**, we avoid the possibility of using incorrect info if we made mistakes in **a.**)

$$\cos(2x) = 1 - 2\sin^2(x)$$

$$= 1 - 2\cdot\left(\frac{9}{11}\right)^2 \qquad \text{(since } \sin(x) = \tfrac{9}{11}\text{)}$$

$$= 1 - 2\cdot\frac{81}{121}$$

$$= \frac{121}{121} - \frac{162}{121}$$

$$= -\frac{41}{121}$$

c. $\sin\left(\frac{x}{2}\right)$.

The half-angle identity for sine is $\sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1-\cos(x)}{2}}$, and the "$\pm$" in the identity suggests that we need to determine which sign is correct in this case. Since

$$\begin{aligned} &\frac{\pi}{2} < x < \pi \\ \Rightarrow\ &\frac{\pi/2}{2} < \frac{x}{2} < \frac{\pi}{2} \\ \Rightarrow\ &\frac{\pi}{4} < \frac{x}{2} < \frac{\pi}{2}, \end{aligned}$$

we see that $\frac{x}{2}$ is in the 1st quadrant so $\sin\left(\frac{x}{2}\right) > 0$, so we need the positive value:

$$\begin{aligned} \sin\left(\frac{x}{2}\right) &= +\sqrt{\frac{1-\cos(x)}{2}} \\ &= \sqrt{\frac{1-\left(-\frac{2\sqrt{10}}{11}\right)}{2}} \quad \text{(since } \cos(x) = -\frac{2\sqrt{10}}{11}\text{)} \\ &= \sqrt{\frac{1}{2}+\frac{\sqrt{10}}{11}} \\ &= \sqrt{\frac{11+2\sqrt{10}}{22}} \end{aligned}$$

6. Prove the following identities using the double-angle identities for sine and cosine included on the Identities and Formulas Reference Sheet. (Be sure to organize your proof as shown in the Online Lecture Notes and class notes videos.)

a. $\tan(2x) = \dfrac{2\tan(x)}{1-\tan^2(x)}$

$$\tan(2x) = \frac{\sin(2x)}{\cos(2x)}$$

$$= \frac{2\sin(x)\cos(x)}{\cos^2(x)-\sin^2(x)}$$

(since $\sin(2x) = 2\sin(x)\cos(x)$ and $\cos(2x) = \cos^2(x) - \cos^2(x)$)

$$= \frac{2\sin(x)\cos(x)}{\cos^2(x)-\sin^2(x)} \cdot \frac{\frac{1}{\cos^2(x)}}{\frac{1}{\cos^2(x)}}$$

(I'm trying this since I can predict that it will give me the denominator I need)

$$= \frac{\frac{2\sin(x)\cos(x)}{\cos^2(x)}}{\frac{\cos^2(x)}{\cos^2(x)} - \frac{\sin^2(x)}{\cos^2(x)}}$$

$$= \frac{\frac{2\sin(x)}{\cos(x)}}{1-\tan^2(x)}$$

$$= \frac{2\tan(x)}{1-\tan^2(x)}$$

b. $\dfrac{1-\cos(2t)}{\sin(2t)} = \tan(t)$

Let's start with the left side of the identity since it's "more complicated" – it involves the sine and cosine of a doubled-angle so we can start by using the double angle identities for since and cosine:

$$\frac{1-\cos(2t)}{\sin(2t)} = \frac{1-\left(1-2\sin^2(t)\right)}{2\sin(t)\cos(t)}$$

(since $\cos(2t) = 1-2\sin^2(t)$ and $\sin(2t) = 2\sin(t)\cos(t)$)

$$= \frac{2\sin^2(t)}{2\sin(t)\cos(t)}$$

$$= \frac{\cancel{2}\sin^{\cancel{2}}(t)}{\cancel{2}\,\cancel{\sin(t)}\cos(t)}$$

$$= \frac{\sin(t)}{\cos(t)}$$

$$= \tan(t)$$