SOLUTIONS: Practice Worksheet: Non-right Triangles

1. 1. $B = 118^\circ$, $C = 37^\circ$, $a = 5$

First, let's draw a sketch of the situation:

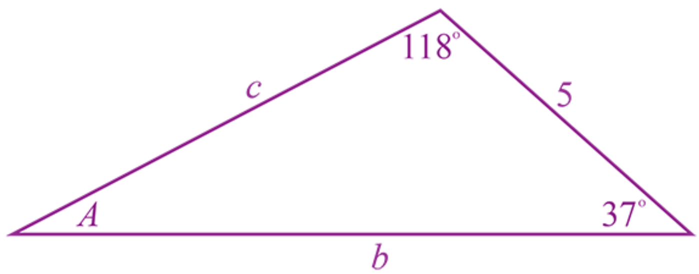


Figure 1

We can use the fact that the sum of the angles in a triangle is 180° to find A:

$$\begin{aligned} A &= 180^\circ - 118^\circ - 37^\circ \\ &= 25^\circ \end{aligned}$$

Now we have an "angle and side opposite" pair to work with so we can use the Law of Sines to find b and c:

$$\frac{b}{\sin(118^\circ)} = \frac{5}{\sin(A)}$$

$$\Rightarrow \qquad b = \frac{5 \cdot \sin(118^\circ)}{\sin(25^\circ)}$$

$$\Rightarrow \qquad b \approx 10.45$$

and

$$\frac{c}{\sin(37^\circ)} = \frac{5}{\sin(A)}$$

$$\Rightarrow \qquad c = \frac{5 \cdot \sin(37^\circ)}{\sin(25^\circ)}$$

$$\Rightarrow \qquad c \approx 7.12$$

1. b. $B = 76^\circ$, $a = 8$, $c = 6$

First, let's draw a sketch of the situation:

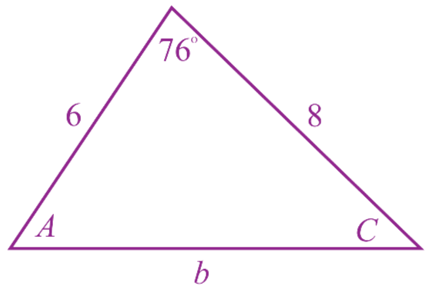


Figure 2

As with **4.a.**, we aren't given an "angle and side opposite" pair to work with so we cannot employ the Law of Sines but we are given the length of the two sides that form the angle $B = 76^\circ$ so we can use the Law of Cosines to find b:

$$\begin{aligned} & b^2 = 6^2 + 8^2 - 2\cdot 6\cdot 8\cdot\cos(76^\circ) \\ \Rightarrow\quad & b^2 = 100 - 96\cdot\cos(76^\circ) \\ \Rightarrow\quad & b = \sqrt{100 - 96\cdot\cos(76^\circ)} \\ \Rightarrow\quad & b \approx 8.76 \end{aligned}$$

Now that we know the length of side b, we have an "angle and side opposite" pair to work with so we can use the Law of Sines to find either of the remaining missing angles; let's aim for C since it's opposite a smaller side than is A. (Note that it's always better to use the Law of Sines to find the smaller angle since the inverse-sine function cannot produce angles larger than 90°.)

$$\begin{aligned} & \frac{\sin(C)}{6} = \frac{\sin(76^\circ)}{b} \\ \Rightarrow\quad & \sin(C) \approx \frac{6\cdot\sin(76^\circ)}{8.76} \\ \Rightarrow\quad & C \approx \sin^{-1}\left(\frac{6\cdot\sin(76^\circ)}{8.76}\right) \\ \Rightarrow\quad & C \approx 41.65^\circ \end{aligned}$$

Finally, we can use the fact that the sum of the angles in a triangle is 180° to find A:

$$\begin{aligned} A &= 180^\circ - 76^\circ - C \\ &\approx 180^\circ - 76^\circ - 41.65^\circ \\ &\approx 62.35^\circ \end{aligned}$$

1. 2. $a = 5$, $b = 6$, $c = 7$

First, let's draw a sketch of the situation:

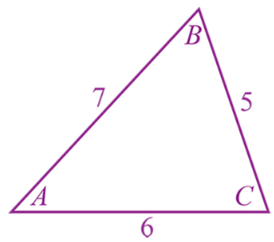


Figure 3

We aren't given any angle measures so we don't have any "angle and side opposite" pairs to work with so we cannot employ the Law of Sines, so we'll have to start with the Law of Cosines which we can use to find any of the angles but let's aim for A:

$$\begin{aligned} & 5^2 = 7^2 + 6^2 - 2\cdot 7\cdot 6\cdot\cos(A) \\ \Rightarrow\quad & 25 = 85 - 84\cdot\cos(A) \\ \Rightarrow\quad & 84\cdot\cos(A) = 60 \\ \Rightarrow\quad & \cos(A) = \tfrac{60}{84} \\ \Rightarrow\quad & A = \cos^{-1}\left(\tfrac{5}{7}\right) \\ \Rightarrow\quad & \approx 44.42^\circ \end{aligned}$$

Now that we know the measure of angle A, we have an "angle and side opposite" pair to work with so we can use the Law of Sines to find either of the remaining missing angles; let's aim for B since it's opposite a smaller side than is C. (Note that it's always better to use the Law of Sines to find the smaller angle since the inverse-sine function cannot produce angles larger than 90°.)

$$\begin{aligned} & \frac{\sin(B)}{6} = \frac{\sin(A)}{5} \\ \Rightarrow\quad & \sin(B) \approx \frac{6\cdot\sin(44.42^\circ)}{5} \\ \Rightarrow\quad & B \approx \sin^{-1}\left(\frac{6\cdot\sin(44.42^\circ)}{5}\right) \\ \Rightarrow\quad & B \approx 57.13^\circ \end{aligned}$$

Finally, we can use the fact that the sum of the angles in a triangle is 180° to find C:

$$\begin{aligned} C &= 180^\circ - A - B \\ &\approx 180^\circ - 44.42^\circ - 57.13^\circ \\ &\approx 78.45^\circ \end{aligned}$$

1. d. $A = 62^\circ$, $B = 70^\circ$, $b = 10$

First, let's draw a sketch of the situation:

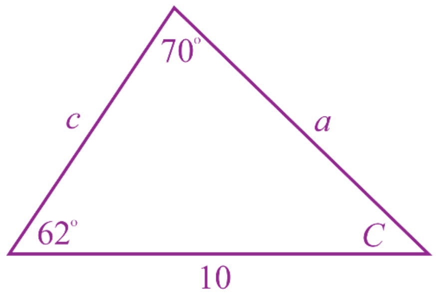


Figure 4

First, we can use the fact that the sum of the angles in a triangle is 180° to find C:

$$\begin{aligned} C &= 180^\circ - 62^\circ - 70^\circ \\ &= 48^\circ \end{aligned}$$

We have an "angle and side opposite" pair to work with so we can use the Law of Sines to find a and c:

$$\frac{a}{\sin(62^\circ)} = \frac{10}{\sin(70^\circ)}$$

$$\Rightarrow \quad a = \frac{10 \cdot \sin(62^\circ)}{\sin(70^\circ)}$$

$$\Rightarrow \quad a \approx 9.4$$

and

$$\frac{c}{\sin(C)} = \frac{10}{\sin(70^\circ)}$$

$$\Rightarrow \quad \frac{c}{\sin(48^\circ)} = \frac{10}{\sin(70^\circ)}$$

$$\Rightarrow \quad c = \frac{10 \cdot \sin(48^\circ)}{\sin(70^\circ)}$$

$$\Rightarrow \quad c \approx 7.9$$

1. e. $A = 64^\circ$, $C = 76^\circ$, $b = 9$

First, let's draw a sketch of the situation:

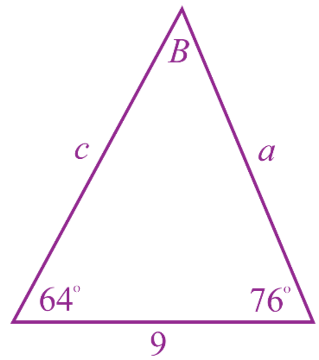


Figure 5

First, we can use the fact that the sum of the angles in a triangle is 180° to find B:

$$\begin{aligned} B &= 180^\circ - 64^\circ - 76^\circ \\ &= 40^\circ \end{aligned}$$

Now we have an "angle and side opposite" pair to work with so we can use the Law of Sines to find a and c:

$$\begin{aligned} \frac{a}{\sin(64^\circ)} &= \frac{9}{\sin(B)} \\ \frac{a}{\sin(64^\circ)} &= \frac{9}{\sin(40^\circ)} \\ \Rightarrow \quad a &= \frac{9 \cdot \sin(64^\circ)}{\sin(40^\circ)} \\ \Rightarrow \quad a &\approx 12.58 \end{aligned}$$

and

$$\begin{aligned} \frac{c}{\sin(76^\circ)} &= \frac{9}{\sin(B)} \\ \frac{c}{\sin(76^\circ)} &= \frac{9}{\sin(40^\circ)} \\ \Rightarrow \quad c &= \frac{9 \cdot \sin(76^\circ)}{\sin(40^\circ)} \\ \Rightarrow \quad c &\approx 13.59 \end{aligned}$$

1. f. $c = 40^\circ$, $a = 9$, $b = 13$

First, let's draw a sketch of the situation:

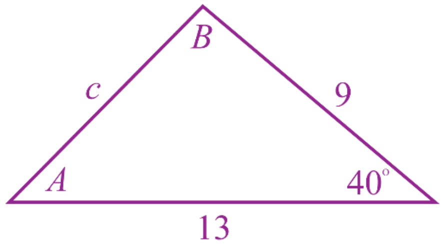


Figure 6

As with **7.b.**, we aren't given an "angle and side opposite" pair to work with so we cannot employ the Law of Sines but we are given the length of the two sides that form the angle $C = 50^\circ$ so we can use the Law of Cosines to find c:

$$\begin{aligned} & c^2 = 9^2 + 13^2 - 2\cdot 9\cdot 13\cdot \cos(40^\circ) \\ \Rightarrow\ & c^2 = 250 - 234\cdot\cos(40^\circ) \\ \Rightarrow\ & c = \sqrt{250 - 234\cdot\cos(40^\circ)} \\ \Rightarrow\ & c \approx 8.41 \end{aligned}$$

Now that we know the length of side c, we have an "angle and side opposite" pair to work with so we can use the Law of Sines to find either of the remaining missing angles; let's aim for A since it's opposite a smaller side than is B. (Note that it's always better to use the Law of Sines to find the smaller angle since the inverse-sine function cannot produce angles larger than 90°. **In this case, if you try to B first using the Law of Sines, you will obtain an incorrect measure of B !!**)

$$\begin{aligned} & \frac{\sin(A)}{9} = \frac{\sin(40^\circ)}{c} \\ \Rightarrow\ & \sin(A) \approx \frac{9\cdot\sin(40^\circ)}{8.41} \\ \Rightarrow\ & A \approx \sin^{-1}\left(\frac{9\cdot\sin(40^\circ)}{8.41}\right) \\ \Rightarrow\ & A \approx 43.46^\circ \end{aligned}$$

Finally, we can use the fact that the sum of the angles in a triangle is 180° to find B:

$$\begin{aligned} B &= 180^\circ - 40^\circ - A \\ &\approx 180^\circ - 40^\circ - 43.46^\circ \\ &\approx 96.54^\circ \end{aligned}$$

1. g. $A = 40^\circ$, $a = 11$, $c = 8$

First, let's draw a sketch of the situation:

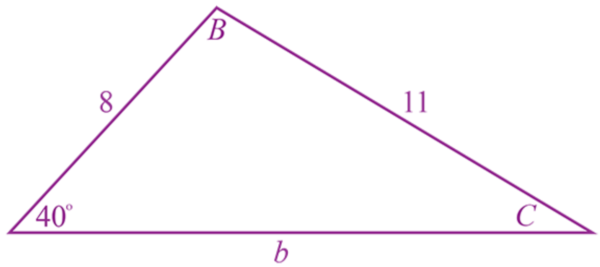


Figure 7

To solve the triangle, we can start by using the Law of Sines to find C:

$$\frac{\sin(C)}{8} = \frac{\sin(40^\circ)}{11}$$

$$\Rightarrow \quad \sin(C) = \frac{8 \cdot \sin(40^\circ)}{11}$$

$$\Rightarrow \quad C = \sin^{-1}\left(\frac{8 \cdot \sin(40^\circ)}{11}\right) \approx 27.87^\circ$$

Now we can use the fact that the sum of the angles in a triangle is 180° to find B:

$$\begin{aligned} B &= 180^\circ - 40^\circ - C \\ &\approx 180^\circ - 40^\circ - 27.87^\circ \\ &\approx 112.13^\circ \end{aligned}$$

Finally, we can use the Law of Sines to find b:

$$\frac{b}{\sin(B)} = \frac{11}{\sin(40^\circ)}$$

$$\Rightarrow \quad b = \frac{11 \cdot \sin(B)}{\sin(40^\circ)}$$

$$\Rightarrow \quad b \approx \frac{11 \cdot \sin(112.13^\circ)}{\sin(40^\circ)} \approx 15.85$$

1. h. $A = 28^\circ$, $a = 7$, $c = 12$

First, let's draw a sketch of the situation:

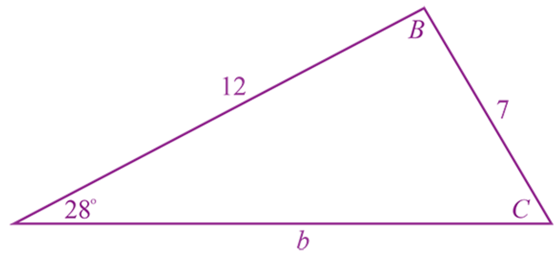


Figure 8

Since $7 < 12$, the side of length 7 units can pivot at angle B without changing any of the known info, i.e., there are two different triangles that satisfy the given information. In Figure 15, we've drawn both of these triangles, one green and one red. (Compare this situation to **1a**, **1b**, **1c**, and **1d** where any attempt to pivot any of the sides would distort some of the given information so such pivoting isn't possible and there's only one possible triangle.)

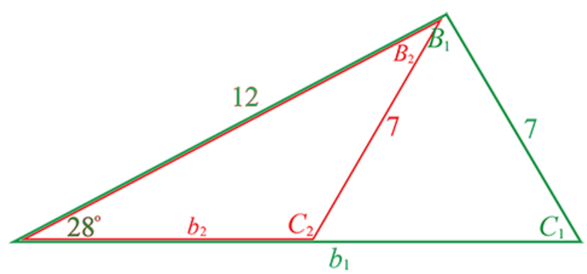


Figure 9

Now we can find the values of C_1 and C_2 using the Law of Sines. At the start, we'll represent the angle with the generic symbol "C" and wait to use subscripts until we're differentiating between the two possible values.

$$\frac{\sin(C)}{12} = \frac{\sin(28^\circ)}{7}$$

$$\Rightarrow \quad \sin(C) = \frac{12 \cdot \sin(28^\circ)}{7}$$

$$\Rightarrow \quad C = \sin^{-1}\left(\frac{12 \cdot \sin(28^\circ)}{7}\right)$$

Now we can start distinguishing between C_1 and C_2 so that we can find these two angles. The largest angle that inverse-sine can output is 90° but C_2 in the red triangle in Figure 15 is clearly greater than 90°: to find C_2 we'll need to employ the identity $\sin(\theta) = \sin(180^\circ - \theta)$:

$$C_1 = \sin^{-1}\left(\frac{12\cdot\sin(28^\circ)}{7}\right) \approx 53.59^\circ \qquad \text{or} \qquad C_2 = 180^\circ - \sin^{-1}\left(\frac{12\cdot\sin(28^\circ)}{7}\right) \approx 126.41^\circ$$

To finish, we can solve for the two possible triangles:

Possibility 1: The **Green** Triangle

$$C_1 \approx 53.59^\circ$$

$$\begin{aligned} B_1 &= 180^\circ - 28^\circ - C_1 \\ &\approx 180^\circ - 28^\circ - 53.59^\circ \\ &\approx 98.41^\circ \end{aligned}$$

$$\frac{b_1}{\sin(B_1)} = \frac{7}{\sin(28^\circ)}$$

$$\Rightarrow \quad b_1 = \frac{7\cdot\sin(B_1)}{\sin(28^\circ)}$$

$$\Rightarrow \quad b_1 \approx \frac{7\cdot\sin(98.41^\circ)}{\sin(28^\circ)} \approx 14.75$$

Possibility 2: The **Red** Triangle

$$C_2 \approx 126.41^\circ$$

$$\begin{aligned} B_2 &= 180^\circ - 28^\circ - C_2 \\ &\approx 180^\circ - 28^\circ - 126.41^\circ \\ &\approx 25.59^\circ \end{aligned}$$

$$\frac{b_2}{\sin(B_2)} = \frac{7}{\sin(28^\circ)}$$

$$\Rightarrow \quad b_2 = \frac{7\cdot\sin(B_2)}{\sin(28^\circ)}$$

$$\Rightarrow \quad b_2 \approx \frac{7\cdot\sin(25.59^\circ)}{\sin(28^\circ)} \approx 6.44$$

1. i. $C = 67^\circ$, $a = 8$, $c = 5$

First, let's draw a sketch of the situation – the sketch shows an "impossible to create triangle" but below we'll use math to discover that this is in fact the case.

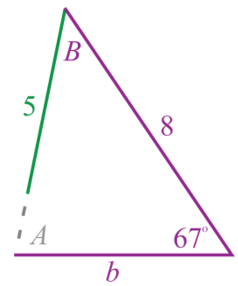


Figure 10

Approaching the situation without assuming it's impossible, we would aim for finding since we have an "angle and side opposite" pair to work with, and we know "a".

$$\frac{\sin(A)}{8} = \frac{\sin(67^\circ)}{5}$$

$$\Rightarrow \qquad A = \sin^{-1}\left(\frac{8 \cdot \sin(67^\circ)}{5}\right)$$

If you ask a calculator to give you this value, it will tell you it's "**undefined**" – it turns out this is because the given conditions are impossible.

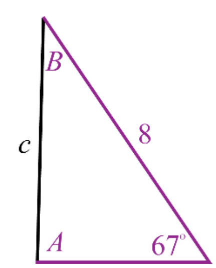
The reason $\sin^{-1}\left(\frac{8 \cdot \sin(67^\circ)}{5}\right)$ is undefined is that $\frac{8 \cdot \sin(67^\circ)}{5}$ is greater than 1 (so it's outside the domain of arcsine), and the reason that this fraction is "too big" is that "5" is "too small" – if the denominator of the fraction were larger, the fraction would be smaller, and then it could be less than 1 and inside the domain of arcsine.

To verify that "$c = 5$" is too short for this side in a triangle with the other given conditions, let's determine the minimum length of this side. The shortest distance between a line an any point not on the line will be in a direction *perpendicular* to the line – this means that we'll want to imagine if A were a right angle (i.e., 90°) and see how long c would be, and that is the smallest possible value for the length of c.

$$\sin(67^\circ) = \frac{c}{8}$$

$$\Rightarrow \qquad c = 8 \cdot \sin(67^\circ)$$

$$\Rightarrow \qquad c \approx 7.36$$



So c needs to be *at least* 7.36 or it isn't long enough, so the given "$c = 5$" results in an impossible situation

2. As you know, a triangle has three sides and three angles: these are the "six components of a triangle." Notice that in each part of the previous problem (#7) you are given the measurements of **three** of these six components. Sometimes (like in part (a)) you are given the lengths of all three sides; usually you are given a combination of sides and angles; but you aren't ever given all three angle measures: contemplate, discuss, and explain why.

The three angles in a triangle establish the *shape* of the triangle but, without knowing the length of at least one of the sides, we know nothing about the *size* of the triangle. For example, if the three facts we know about a triangle are that it has angles 30°, 60°, and 90°, the triangle could be any one of *infinitely many* triangles that have these three angles; below are three *different* triangles that all have the same three angles 30°, 60°, and 90°.