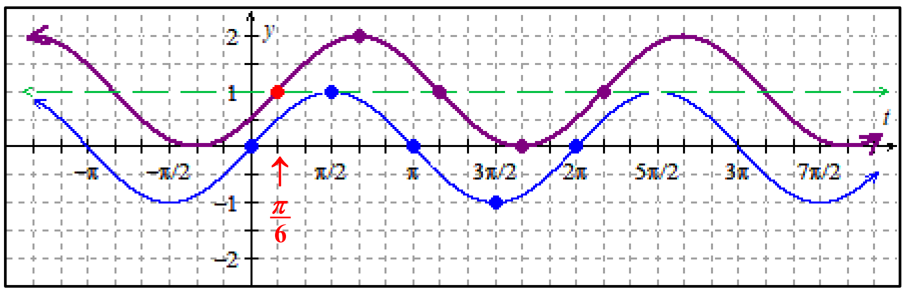
SOLUTIONS: Practice Worksheet: Graphs of Trig Functions

1. The graph of $y = \sin(t)$ is given below with **five key points** emphasized. As we know, the graph of $f(t) = \sin\left(t - \frac{\pi}{6}\right) + 1$ is a *transformation* of $y = \sin(t)$. Use what we know about graph transformations from MTH 111 to "transform" the **five key points** on $y = \sin(t)$ and then connect these points in order to construct a graph of $y = f(t)$.

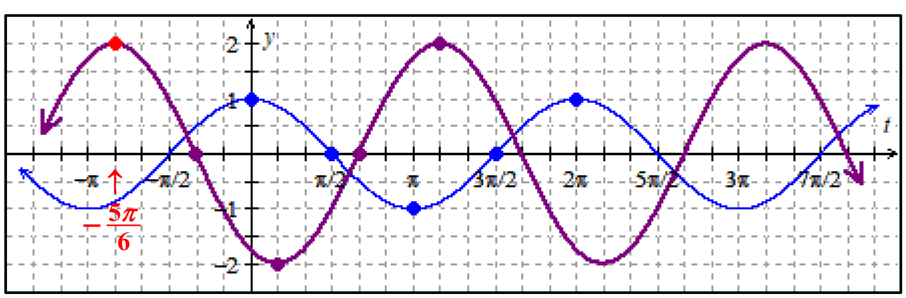
Compared with $y = \sin(t)$, $f(t) = \sin\left(t - \frac{\pi}{6}\right) + 1$ is shifted right $\frac{2\pi}{3}$ units and shifted up 1 unit. Notice that $f(t) = \sin\left(t - \frac{\pi}{6}\right) + 1$ appears to be a sine wave that "starts" at $t = \frac{\pi}{6}$ that has a midline of $y = 1$



Graphs of $y = \sin(t)$ and $y = \sin\left(t - \frac{\pi}{6}\right) + 1$.

2. The graph of $y = \cos(t)$ is given below with **five key points** emphasized. As we know, the graph of $g(t) = 2\cos\left(t + \frac{5\pi}{6}\right)$ is a *transformation* of $y = \cos(t)$. Use what we know about graph transformations from MTH 111 to "transform" the **five key points** on $y = \cos(t)$ and then connect these points in order to construct a graph of $y = g(t)$.

Compared with $y = \cos(t)$, $g(t) = 2\cos\left(t + \frac{5\pi}{6}\right)$ is shifted left $\frac{5\pi}{6}$ units and stretched vertically by a factor of 2.



Graphs of $y = \cos(t)$ and $y = 2\cos\left(t + \frac{5\pi}{6}\right)$.

3. Scale the axes on the given coordinate plane an appropriately for a graph of $y = \sin(2t) + 3$ and then draw a graph of $y = \sin(2t) + 3$ by first plotting the points where the graph will intersect the midline and the points where the graph will reach maximum and minimum values, and then connect these points with an appropriately curved sinusoidal wave.

Compared with $y = \sin(t)$, $y = \sin(2t) + 3$ is compressed horizontally by a factor of $\frac{1}{2}$ and shifted up 3 units.

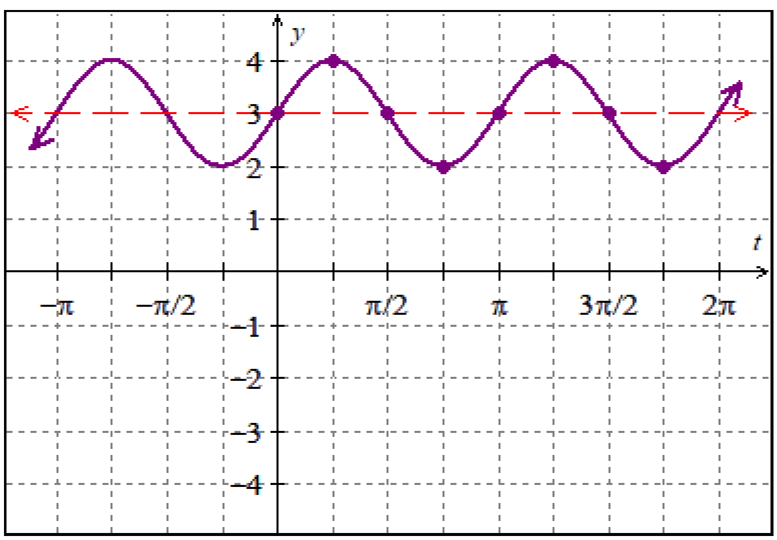
The horizontal compression means that the function has a period of

$$2\pi \cdot \frac{1}{2} = \pi \text{ units.}$$

The vertical shift means that the function has a midline of $y = 3$.

Since the algebraic rule for the function doesn't involve a vertical stretch, its amplitude is 1 unit.

Since the algebraic rule for the function doesn't involve a horizontal shift, its graph will appear like a sine wave "starts" at $t = 0$, i.e., its y-intercept will occur on the midline and the graph will be increasing there.



A graph of $y = \sin(2t) + 3$.

4. Scale the axes on the given coordinate plane an appropriately for a graph of $y = 3\cos(\pi t)$ and then draw a graph of $y = 3\cos(\pi t)$ by first plotting the points where the graph will intersect the midline and the points where the graph will reach maximum and minimum values, and then connect these points with an appropriately curved sinusoidal wave.

Compared with $y = \cos(t)$, $y = 3\cos(\pi t)$ is compressed horizontally by a factor of $\frac{1}{\pi}$ and stretched vertically by a factor of 3.

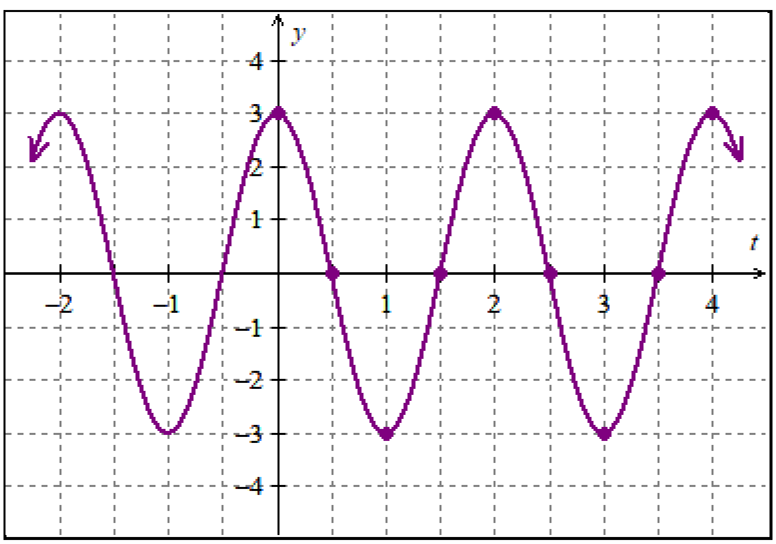
The horizontal compression means that the function has a period of

$$2\pi \cdot \frac{1}{\pi} = 2 \text{ units.}$$

The vertical stretch means that the function's amplitude is 3 units.

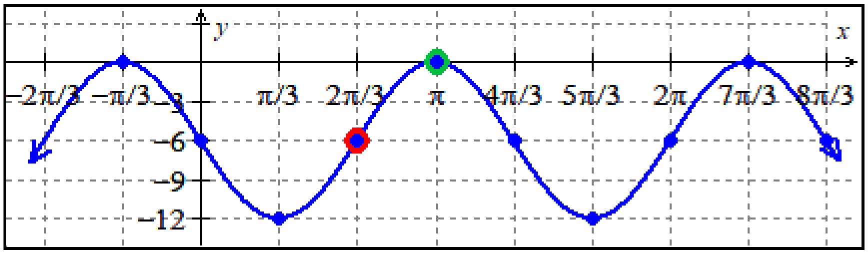
Since the algebraic rule for the function doesn't involve a vertical shift, it must have a midline of $y = 0$.

Since the algebraic rule for the function doesn't involve a horizontal shift, its graph will appear like a cosine wave that "starts" at $t = 0$; i.e., its y-intercept will occur at a maximum value for the function.



A graph of $y = 3\cos(\pi t)$.

5. Find two different algebraic rules for the sinusoidal function $y = p(x)$ graphed below. One of your rules should involve sine and the other should involve cosine.



A graph of $y = p(x)$.

First let's write a rule involving sine, so our rule will have the form $p(x) = A\sin\left(\omega(x-h)\right) + k$ and we need to determine the values of A, ω, h, and k.

- The midline is the line midway between the function's maximum and minimum output values. The function's maximum output value is 0 and its minimum output value is -12. Since -6 is the average of these values, the midline is $y=-6$ so $k=-6$.
- The amplitude is the distance between the function's maximum output value, 0, and its midline $y=-6$, which is 6 units. Therefore, $|A| = 6$.
- The function completes one period between $x = \frac{2\pi}{3}$ and $x = 2\pi$. Thus, the period of the function is $2\pi - \frac{2\pi}{3} = \frac{4\pi}{3}$. To find ω we need to solve $\frac{4\pi}{3} = 2\pi \cdot \frac{1}{\omega}$:

$$\frac{4\pi}{3} = 2\pi \cdot \frac{1}{\omega}$$
$$\Rightarrow \quad \omega = 2\pi \cdot \frac{1}{4\pi/3}$$
$$\Rightarrow \quad \omega = 2\pi \cdot \frac{3}{4\pi}$$
$$\Rightarrow \quad \omega = \frac{3}{2}$$

- Near the y-axis, the graph of $y = \sin(x)$ is increasing and passes through its midline, so we need to look for a spot in the graph of $y = p(x)$ where it shows this behavior, and one such spot is at $x = \frac{2\pi}{3}$ (this point has been highlighted in **red** in the graph above) so we can so consider this graph a sine wave shifted right $\frac{2\pi}{3}$ units and use $h = \frac{2\pi}{3}$.

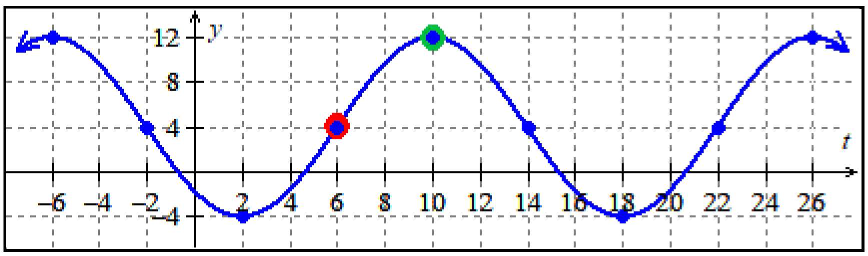
Therefore, an algebraic rule for the graphed function is $p(x) = 6\sin\left(\frac{3}{2}\left(x - \frac{2\pi}{3}\right)\right) - 6$.

Now we'll write a rule involving cosine.

Since we want to use cosine to construct our rule, it will have the form $p(x) = A\cos\big(\omega(x-h)\big) + k$. Since the amplitude, period, and midline aren't dependent on whether we use sine or cosine in our algebraic rule, we can use the same values for A, ω, and k that we used above. So we only need to determine an appropriate horizontal shift, h, that works for cosine. Near the y-axis, the graph of $y = \cos(t)$ is reaches its maximum value, so we need to look for a spot in the graph of $y = p(x)$ where it shows this behavior, and one such spot is at $x = \pi$ (this point has been highlighted in **green** in the graph above) so we can consider this graph a cosine wave shifted right π units and use $h = \pi$.

Therefore, an algebraic rule for the graphed function is $p(x) = 6\cos\left(\frac{3}{2}(x-\pi)\right) - 6$.

6. Find two different algebraic rules for the sinusoidal function $y = q(t)$ graphed below. One of your rules should involve sine and the other should involve cosine.



A graph of $y = q(t)$.

First let's write a rule involving sine, so our rule will have the form $q(t) = A\sin\big(\omega(t-h)\big) + k$ and we need to determine the values of A, ω, h, and k.

- The midline is the line midway between the function's maximum and minimum output values. The function's maximum output value is 12 and its minimum output value is -4. Since 4 is the average of these values, the midline is $y = 4$ so $k = 4$.
- The amplitude is the distance between the function's maximum output value, 12, and its midline $y = 4$, which is 8 units. Therefore, $|A| = 8$.
- The function completes one period between $t = 6$ and $t = 22$. Thus, the period of the function is $22 - 6 = 16$. To find ω we need to solve $16 = 2\pi \cdot \frac{1}{\omega}$:

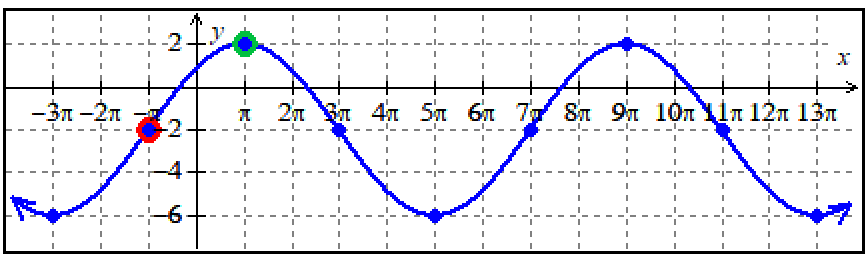
$$16 = 2\pi \cdot \frac{1}{\omega}$$
$$\Rightarrow \quad \omega = 2\pi \cdot \frac{1}{16} = \frac{\pi}{8}$$

- Near the y-axis, the graph of $y = \sin(t)$ is increasing and passes through its midline, so we need to look for a spot in the graph of $y = q(t)$ where it shows this behavior, and one such spot is at $t = 6$ (this point has been highlighted in **red** in the graph above) so we can consider this graph a sine wave shifted right 6 units and use $h = 6$.

- Therefore, an algebraic rule for the graphed function is $q(t) = 8\sin\left(\frac{\pi}{8}(t-6)\right) + 4$.

Now we'll write a rule involving cosine, so our rule will have the form $q(t) = A\cos\left(\omega(t-h)\right) + k$. Since the amplitude, period, and midline aren't dependent on whether we use sine or cosine in our algebraic rule, we can use the same values for A, ω, and k that we used above. So we only need to determine an appropriate horizontal shift, h, that works for cosine. Near the y-axis, the graph of $y = \cos(t)$ is reaches its maximum value, so we need to look for a spot in the graph of $y = q(t)$ where it shows this behavior, and one such spot is at $t = 10$ (this point has been highlighted in **green** in the graph above) so we can consider this graph a cosine wave shifted right 10 units and use $h = 10$. Therefore, an algebraic rule for the graphed function is $q(t) = 8\cos\left(\frac{\pi}{8}(t-10)\right) + 4$.

7. Find two different algebraic rules for the sinusoidal function $y = m(x)$ graphed below. One of your rules should involve sine and the other should involve cosine.



A graph of $y = m(x)$.

First let's write a rule involving sine, so our rule will have the form $m(x) = A\sin\left(\omega(x-h)\right) + k$ and we need to determine the values of A, ω, h, and k.

- The midline is the line midway between the function's maximum and minimum output values. The function's maximum output value is 2 and its minimum output value is -6. Since -2 is the average of these values, the midline is $y = -2$ so $k = -2$.

- The amplitude is the distance between the function's maximum output value, 2, and its midline $y = -2$, which is 4 units. Therefore, $|A| = 4$.

- The function completes one period between $x = \pi$ and $x = 9\pi$. Thus, the period of the function is $9\pi - \pi = 8\pi$. To find ω we need to solve $8\pi = 2\pi \cdot \frac{1}{\omega}$:

$$8\pi = 2\pi \cdot \frac{1}{\omega}$$
$$\Rightarrow \quad \omega = 2\pi \cdot \frac{1}{8\pi}$$
$$\Rightarrow \quad \omega = \frac{1}{4}$$

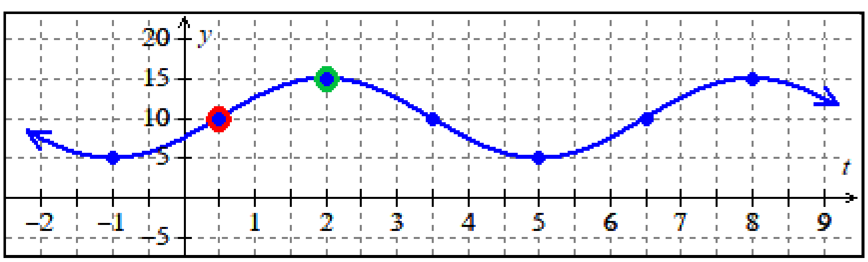
- Near the y-axis, the graph of $y = \sin(x)$ is increasing and passes through its midline, so we need to look for a spot in the graph of $y = m(x)$ where it shows this behavior, and one such spot is at $x = -\pi$ (this point has been highlighted in **red** in the graph above) so we can so consider this graph a sine wave shifted left π units and use $h = -\pi$.

Therefore, an algebraic rule for the graphed function is $m(x) = 4\sin\left(\frac{1}{4}(x - (-\pi))\right) - 2$ which simplifies as $m(x) = 4\sin\left(\frac{1}{4}(x + \pi)\right) - 2$.

Now we'll write a rule involving cosine, so our rule will have the form $m(x) = A\cos\left(\omega(x - h)\right) + k$. Since the amplitude, period, and midline aren't dependent on whether we use sine or cosine in our algebraic rule, we can use the same values for A, ω, and k that we used above. So we only need to determine an appropriate horizontal shift, h, that works for cosine. Near the y-axis, the graph of $y = \cos(t)$ is reaches its maximum value, so we need to look for a spot in the graph of $y = m(x)$ where it shows this behavior, and one such spot is at $x = \pi$ (this point has been highlighted in **green** in the graph above) so we can consider this graph a cosine wave shifted right π units and use $h = \pi$.

Therefore, an algebraic rule for the graphed function is $m(x) = 4\cos\left(\frac{1}{4}(x - \pi)\right) - 2$.

8. Find two different algebraic rules for the sinusoidal function $y = n(t)$ graphed below. One of your rules should involve sine and the other should involve cosine.



A graph of $y = n(t)$.

First let's write a rule involving sine, so our rule will have the form $n(t) = A\sin\left(\omega(t - h)\right) + k$ and we need to determine the values of A, ω, h, and k.

- The midline is the line midway between the function's maximum and minimum output values. The function's maximum output value is 15 and its minimum output value is 5. Since 10 is the average of these values, the midline is $y = 10$ so $k = 10$.

- The amplitude is the distance between the function's maximum output value, 15, and its midline $y = 10$, which is 5 units. Therefore, $|A| = 5$.

- The function completes one period between $t = 2$ and $t = 8$. Thus, the period of the function is $8 - 2 = 6$. To find ω we need to solve $6 = 2\pi \cdot \frac{1}{\omega}$:

$$6 = 2\pi \cdot \frac{1}{\omega}$$
$$\Rightarrow \quad \omega = 2\pi \cdot \frac{1}{6} = \frac{\pi}{3}$$

- Near the y-axis, the graph of $y = \sin(t)$ is increasing and passes through its midline, so we need to look for a spot in the graph of $y = n(t)$ where it shows this behavior, and one such spot is at $t = \frac{1}{2}$ (this point has been highlighted in **red** in the graph above) so we can consider this graph a sine wave shifted right $\frac{1}{2}$ of a unit and use $h = \frac{1}{2}$.

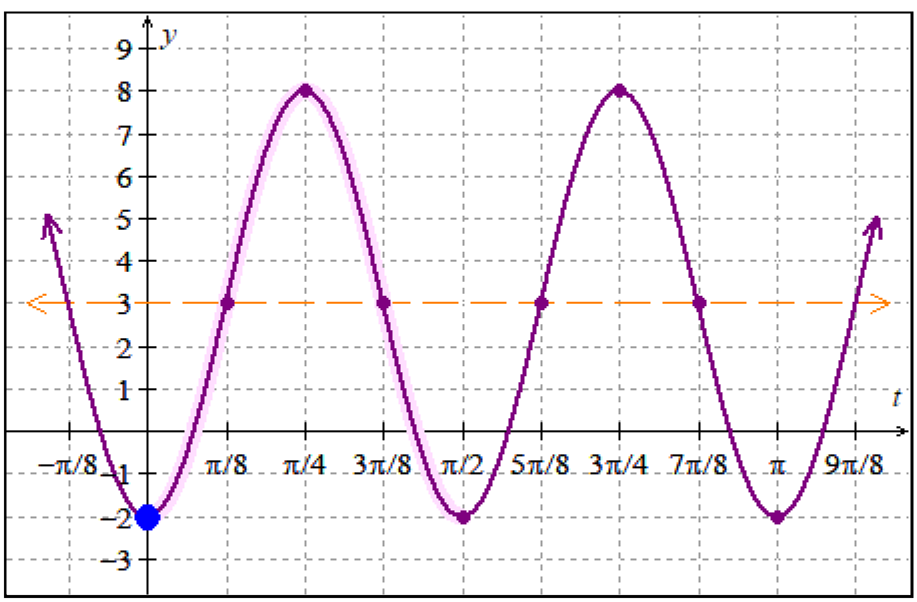
- Therefore, an algebraic rule for the graphed function is $n(t) = 5\sin\left(\frac{\pi}{3}\left(t - \frac{1}{2}\right)\right) + 10$.

Now we'll write a rule involving cosine, so our rule will have the form $n(t) = A\cos\left(\omega(t - h)\right) + k$. Since the amplitude, period, and midline aren't dependent on whether we use sine or cosine in our algebraic rule, we can use the same values for A, ω, and k that we used above. So we only need to determine an appropriate horizontal shift, h, that works for cosine. Near the y-axis, the graph of $y = \cos(t)$ is reaches its maximum value, so we need to look for a spot in the graph of $y = n(t)$ where it shows this behavior, and one such spot is at $t = 2$ (this point has been highlighted in **green** in the graph above) so we can consider this graph a cosine wave shifted right 2 units and use $h = 2$. Therefore, an algebraic rule for the graphed function is $n(t) = 5\cos\left(\frac{\pi}{3}(t - 2)\right) + 10$.

9. Draw a graph of at least two periods of the functions in **a–f** below by first *plotting the points* where the graph will intersect the midline and *plotting the points* where the graph will reach maximum and minimum values, and then *connect these points* with an appropriately curved sinusoidal wave. List the period, midline, and amplitude of each function. (Be sure to label the scale on the axes of your graph.)

a. $f(t) = -5\cos(4t) + 3$

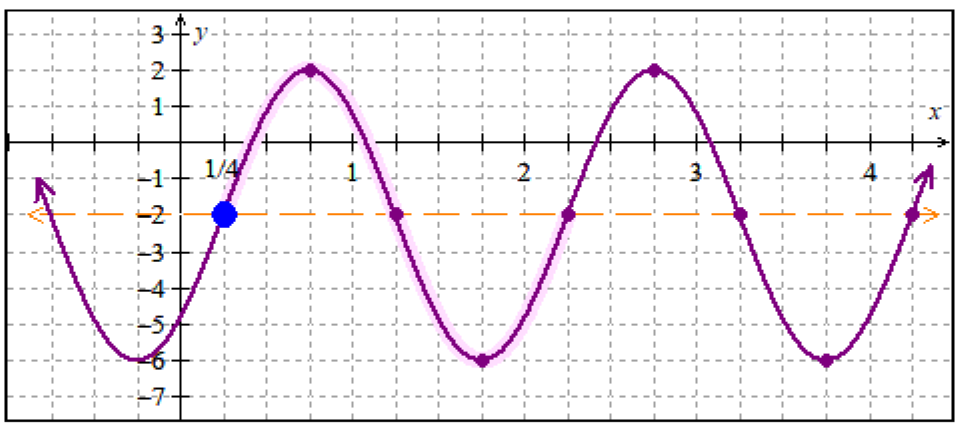
- $|A| = |-5| = 5$ so the **amplitude** is 5 units. Since $A < 0$, we'll need to draw a "reflected cosine wave".
- $k = 3$ so the **midline** is $y = 3$.
- $\omega = 4$ so the **period** is $2\pi \cdot \frac{1}{4} = \frac{\pi}{2}$ units.
- There is no horizontal shift so we'll "start" a reflected cosine wave on the y-axis and make sure it has the appropriate midline, amplitude, and period; we've highlighted the "first" period in pink. (Since this is a "reflected cosine wave", it needs to start at a minimum output value rather than at its maximum output value like $y = \cos(t)$.)



A graph of $f(t) = -5\cos(4t) + 3$.

b. $g(x) = 4\sin\left(\pi\left(x - \frac{1}{4}\right)\right) - 2$

- $|A| = |4| = 4$ so the **amplitude** is 4 units.
- $k = -2$ so the **midline** is $y = -2$.
- $\omega = \pi$ so the **period** is $2\pi \cdot \frac{1}{\pi} = 2$ units.
- $h = \frac{1}{4}$ so the horizontal shift is $\frac{1}{4}$ units to the right, so we'll "start" a sine wave at $x = \frac{1}{4}$ and make sure it has the appropriate midline, amplitude, and period; we've highlighted the "first" period in pink.

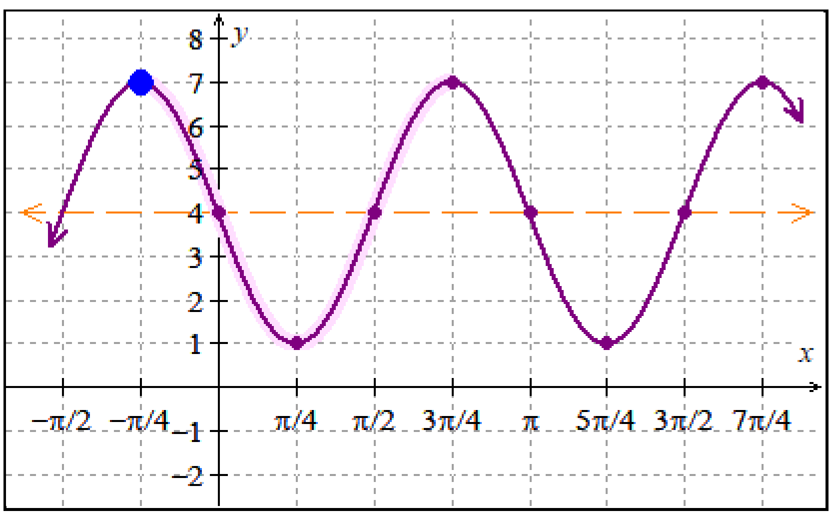


A graph of $g(x) = 4\sin\left(\pi\left(x - \frac{1}{4}\right)\right) - 2$.

c. $G(x) = 3\cos\left(2x + \frac{\pi}{2}\right) + 4$

$$= 3\cos\left(2\left(x - \left(-\frac{\pi}{4}\right)\right)\right) + 4$$

- $|A| = |3| = 3$ so the **amplitude** is 3 units.
- $k = 4$ so the **midline** is $y = 4$.
- $\omega = 2$ so the **period** is $2\pi \cdot \frac{1}{2} = \pi$ units.
- $h = -\frac{\pi}{4}$ so the horizontal shift is $\frac{\pi}{4}$ units to the left, so we'll "start" a cosine wave at $x = -\frac{\pi}{4}$ and make sure it has the appropriate midline, amplitude, and period; we've highlighted the "first" period in pink.

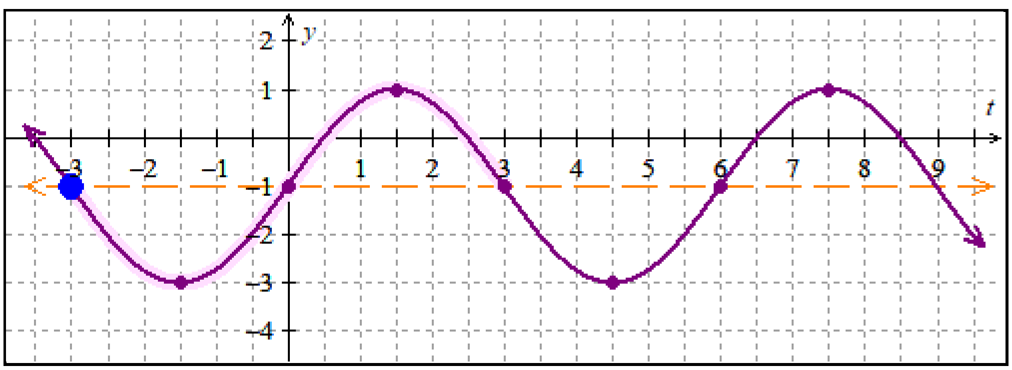


A graph of $G(x) = 3\cos\left(2x + \frac{\pi}{2}\right) + 4$.

d. $F(t) = -2\sin\left(\frac{\pi}{3}t + \pi\right) - 1$

$= -2\sin\left(\frac{\pi}{3}\left(t - (-3)\right)\right) - 1$

- $|A| = |-2| = 2$ so the **amplitude** is 2 units. Since $A < 0$, we'll need to draw a "reflected sine wave".
- $k = -1$ so the **midline** is $y = -1$.
- $\omega = \frac{\pi}{3}$ so the **period** is $2\pi \cdot \frac{1}{\pi/3} = 6$ units.
- $h = -3$ so the horizontal shift is 3 units to the left, so we'll "start" a reflected sine wave at $t = -3$ and make sure it has the appropriate midline, amplitude, and period; we've highlighted the "first" period in pink. (Since this is a "reflected sine wave", it needs to travel *down* from its starting point at its midline.)

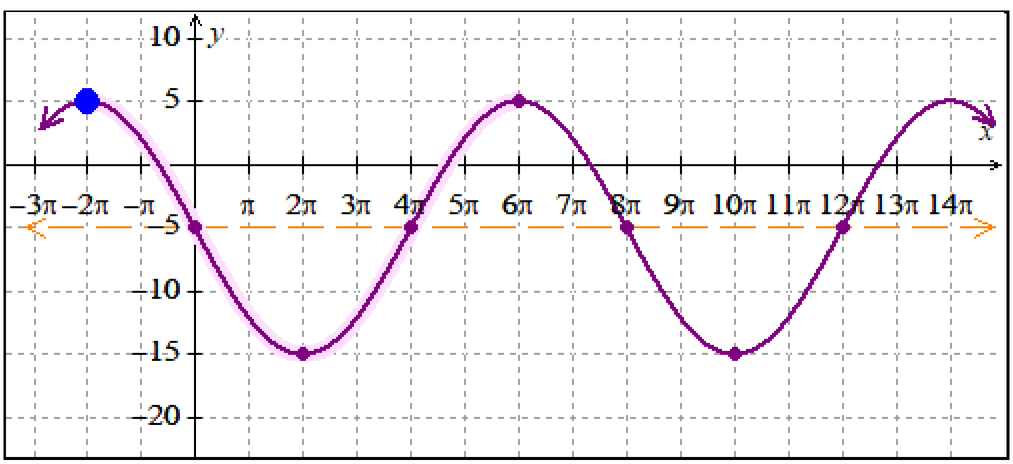


A graph of $F(t) = -2\sin\left(\frac{\pi}{3}t + \pi\right) - 1$.

e. $p(x) = 10\cos\left(\dfrac{x + 2\pi}{4}\right) - 5$

$= 10\cos\left(\frac{1}{4}\left(x - (-2\pi)\right)\right) - 5$

- $|A| = |10| = 10$ so the **amplitude** is 10 units.
- $k = -5$ so the **midline** is $y = -5$.
- $\omega = \frac{1}{4}$ so the **period** is $2\pi \cdot \frac{1}{1/4} = 8\pi$ units.
- $h = -2\pi$ so the horizontal shift is 2π units to the left, so we'll "start" a cosine wave at $x = -2\pi$ and make sure it has the appropriate midline, amplitude, and period; we've highlighted the "first" period in pink.

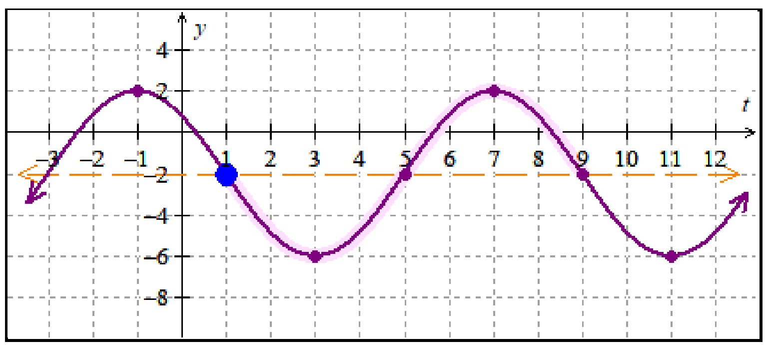


A graph of $p(x) = 10\cos\left(\dfrac{x + 2\pi}{4}\right) - 5$.

f. $q(t) = -4\sin\left(\frac{\pi}{4}t - \frac{\pi}{4}\right) - 2$

$= -4\sin\left(\frac{\pi}{4}(t - 1)\right) - 2$

- $|A| = |-4| = 4$ so the **amplitude** is 4 units. Since $A < 0$, we'll need to draw a "reflected sine wave".
- $k = -2$ so the **midline** is $y = -2$.
- $\omega = \frac{\pi}{4}$ so the **period** is $2\pi \cdot \frac{1}{\pi/4} = 8$ units.
- $h = 1$ so the horizontal shift is 1 units to the left, so we'll "start" a reflected sine wave at $t = 1$ and make sure it has the appropriate midline, amplitude, and period; we've highlighted the "first" period in pink. (Since this is a "reflected sine wave", it needs to travel *down* from its starting point at its midline.)



A graph of $q(t) = -4\sin\left(\frac{\pi}{4}t - \frac{\pi}{4}\right) - 2$.

10. Describe the values that are **not** in the domain of $y = \tan(t)$.

Since $\tan(t) = \frac{\sin(t)}{\cos(t)}$, $y = \tan(t)$ is undefined where $\cos(t) = 0$. Recall that $\cos(t) = 0$ when $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$ and all angles coterminal with these two angles. One way to express these values is like this:

$$\left\{t \mid t = \frac{\pi}{2} + 2k\pi \text{ for all } k \in \mathbb{Z}\right\} \cup \left\{t \mid t = \frac{3\pi}{2} + 2k\pi \text{ for all } k \in \mathbb{Z}\right\}$$

This set can be simplified and expressed as:

$$\left\{t \mid t = \frac{\pi}{2} + k\pi \text{ for all } k \in \mathbb{Z}\right\}.$$

These are the values that are **not** in the domain of $y = \tan(t)$.

11. Describe the values that are **not** in the domain of $y = \sec(t)$.

Since $\sec(t) = \frac{1}{\cos(t)}$, $y = \sec(t)$ is undefined where $\cos(t) = 0$. In #12, we observed that these values can be represented by the following set:

$$\left\{ t \mid t = \frac{\pi}{2} + k\pi \text{ for all } k \in \mathbb{Z} \right\}.$$

These are the values that are **not** in the domain of $y = \sec(t)$.

12. Describe the values that are **not** in the domain of $y = \cot(t)$.

Since $\cot(t) = \frac{\cos(t)}{\sin(t)}$, $y = \cot(t)$ is undefined where $\sin(t) = 0$. Recall that $\sin(t) = 0$ when $t = 0$, when $t = \pi$, and all angles coterminal with these two angles. One way to express these values is like this:

$$\left\{ t \mid t = 2k\pi \text{ for all } k \in \mathbb{Z} \right\} \cup \left\{ t \mid t = \pi + 2k\pi \text{ for all } k \in \mathbb{Z} \right\}$$

This set can be simplified and expressed as:

$$\left\{ t \mid t = k\pi \text{ for all } k \in \mathbb{Z} \right\}.$$

These are the values that are **not** in the domain of $y = \cot(t)$.

13. Describe the values that are **not** in the domain of $y = \csc(t)$.

Since $\csc(t) = \frac{1}{\sin(t)}$, $y = \csc(t)$ is undefined where $\sin(t) = 0$. In #14, we observed that these values can be represented by the following set:

$$\left\{ t \mid t = k\pi \text{ for all } k \in \mathbb{Z} \right\}.$$

These are the values that are **not** in the domain of $y = \csc(t)$.