Solutions to "Some Additional Practice for the Midterm Exam"

1. a. Convert $24°$ into radians.

$$24° = 24° \cdot \frac{2\pi \text{ rad.}}{360°}$$

$$= \frac{2\pi}{15} \text{ rad.}$$

b. Convert $\frac{3}{2}$ radians into degrees.

$$\frac{3}{2} \text{ rad.} = \frac{3}{2} \text{ rad.} \cdot \frac{360°}{2\pi \text{ rad.}}$$

$$= \frac{540°}{2\pi}$$

$$= \frac{270°}{\pi}$$

2. Find the arc-length spanned by an angle measuring $24°$ on a circle of radius 30 feet.

In part **a** of #1 we found that $24° = \frac{2\pi}{15}$ rad. Thus,

$$s = r\,\theta$$

$$= 30 \cdot \frac{2\pi}{15}$$

$$= 4\pi$$

and we see that the arc-length spanned by an angle measuring $24°$ on a circle of radius 30 feet is 4π feet.

3. Evaluate the following expressions:

a. $\sin(225°) = -\sin(45°)$ (since $45°$ is the reference angle for $225°$ and sine is negative in the 3rd quadrant)

$$= -\frac{\sqrt{2}}{2}$$

b. $\sin\left(\frac{5\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right)$ 　　(since $\frac{\pi}{3}$ is the reference angle for $\frac{5\pi}{3}$

　　　　　　　　　　　　　　　and sine is negative in the 4th quadrant)

$= -\frac{\sqrt{3}}{2}$

c. $\cos(300°) = \cos(60°)$ 　　(since $60°$ is the reference angle for $300°$)

$= \frac{1}{2}$

d. $\cos\left(\frac{17\pi}{6}\right) = \cos\left(\frac{5\pi}{6}\right)$ 　　(since $\frac{5\pi}{6}$ is coterminal with $\frac{17\pi}{6}$)

$= -\cos\left(\frac{\pi}{6}\right)$ 　　(since $\frac{\pi}{6}$ is the reference angle for $\frac{5\pi}{6}$

　　　　　　　　　　　　　　and cosine is negative in the 2nd quadrant)

$= -\frac{\sqrt{3}}{2}$

e. $\tan\left(\frac{3\pi}{4}\right) = \frac{\sin\left(\frac{3\pi}{4}\right)}{\cos\left(\frac{3\pi}{4}\right)}$

$= \frac{\sin\left(\frac{\pi}{4}\right)}{-\cos\left(\frac{\pi}{4}\right)}$ 　　(since $\frac{\pi}{4}$ is the reference angle for $\frac{3\pi}{4}$

　　　　　　　　　　　　and cosine is negative in the 2nd quadrant)

$= \frac{\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}}$

$= -1$

f. $\sec\left(3600°\right) = \frac{1}{\cos\left(3600°\right)}$

$= \frac{1}{\cos\left(360°\right)}$ 　　(since $360°$ is coterminal with $3600°$)

$= \frac{1}{1}$

$= 1$

4. If $\sin(\theta) = \frac{3}{4}$ and θ is in the second quadrant, find the exact value of the following expressions.

a. $\cos(\theta)$

We can use the Pythagorean identity to find $\cos(\theta)$:

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

$$\Rightarrow \left(\tfrac{3}{4}\right)^2 + \cos^2(\theta) = 1$$

$$\Rightarrow \cos^2(\theta) = 1 - \tfrac{9}{16}$$

$$\Rightarrow \cos^2(\theta) = \tfrac{9}{16}$$

$$\Rightarrow \cos(\theta) = -\tfrac{\sqrt{7}}{4} \quad \text{(note that we take the negative square root since cosine is negative in the second quadrant)}$$

b. $\sin(\theta + 2\pi) = \sin(\theta)$ (since the period of $y = \sin(\theta)$ is 2π)

$$= \tfrac{3}{4}$$

c. $\sin(-\theta) = -\sin(\theta)$ (due to the symmetry of a circle and the definition of sine)

$$= -\tfrac{3}{4}$$

d. $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$

$$= \dfrac{\tfrac{3}{4}}{-\tfrac{\sqrt{7}}{4}}$$

$$= -\dfrac{3}{\sqrt{7}}$$

e. $\csc(\theta) = \dfrac{1}{\sin(\theta)}$

$$= \dfrac{1}{\tfrac{3}{4}}$$

$$= \tfrac{4}{3}$$

f. $\sec(\theta) = \dfrac{1}{\cos(\theta)}$

$ = \dfrac{1}{-\sqrt{7}\big/4}$

$ = -\dfrac{4}{\sqrt{7}}$

5. Find all of the solutions to the following equations on the interval $[0, 2\pi)$:

a. $2\cos(\theta) = 1$

$$2\cos(\theta) = 1$$
$$\Rightarrow \quad \cos(\theta) = \tfrac{1}{2}$$

In the first quadrant, we know that $\theta = \frac{\pi}{3}$ is a solution.

In the second and third quadrant cosine is negative, so there are no solutions.

In the fourth quadrant we know that $\theta = \frac{5\pi}{3}$ is a solution.

Thus the solution set on the interval $[0, 2\pi)$ is $\left\{\frac{\pi}{3}, \frac{5\pi}{3}\right\}$.

b. $\sin(2\theta) + 3 = 4$

$$\sin(2\theta) + 3 = 4$$
$$\Rightarrow \quad \sin(2\theta) = 1$$

Since $\sin(x) = 1$ when $x = \frac{\pi}{2} + 2k\pi$ (where $k \in \mathbb{Z}$), we need:

$$2\theta = \tfrac{\pi}{2} + 2k\pi$$

$$\Rightarrow \theta = \frac{\frac{\pi}{2} + 2k\pi}{2}$$

$$\Rightarrow \theta = \tfrac{\pi}{4} + k\pi$$

Since $0 \le \theta < 2\pi$, we can only take $k = 0$ and $k = 1$ in the statement above. Thus, the solution set is $\left\{\frac{\pi}{4}, \frac{5\pi}{4}\right\}$.

6. Find *all* of the solutions to the following equations:

a. $3\sin(x) + 4 = 5$

$$3\sin(x) + 4 = 5$$
$$\Rightarrow \quad 3\sin(x) = 1$$
$$\Rightarrow \quad \sin(x) = \frac{1}{3}$$

Since this sine value isn't one we are familiar with, we will need to use the inverse sine function (i.e., arcsine function).

$$\sin(x) = \frac{1}{3}$$
$$\Rightarrow \quad x = \sin^{-1}\left(\frac{1}{3}\right) + 2k\pi,$$
$$\text{or } x = \pi - \sin^{-1}\left(\frac{1}{3}\right) + 2k\pi \quad \text{where } k \in \mathbb{Z}$$

b. $7 + 3\sqrt{2}\cos(4t) = 4$

$$7 + 3\sqrt{2}\cos(4t) = 4$$
$$\Rightarrow \quad 3\sqrt{2}\cos(4t) = -3$$
$$\Rightarrow \quad \cos(4t) = \frac{-3}{3\sqrt{2}}$$
$$\Rightarrow \quad \cos(4t) = -\frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \begin{cases} 4t = \frac{3\pi}{4} + 2k\pi, \ k \in \mathbb{Z} \\ \text{OR} \\ 4t = -\frac{3\pi}{4} + 2k\pi, \ k \in \mathbb{Z} \end{cases}$$

$$\Rightarrow \quad \begin{cases} t = \frac{3\pi}{16} + \frac{k\pi}{2}, \ k \in \mathbb{Z} \\ \text{OR} \\ t = -\frac{3\pi}{16} + \frac{k\pi}{2}, \ k \in \mathbb{Z} \end{cases}$$

7. Use the sine and cosine functions to find the coordinates of the point P in Figure 1 that is specified by $\frac{7\pi}{6}$ on the circumference of a circle of radius 3 units.
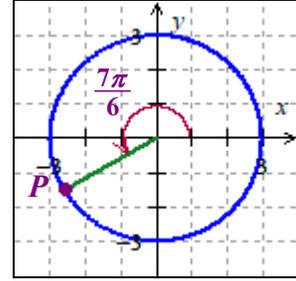

Figure 1

Coordinates of P:

$$\left(3\cos\left(\tfrac{7\pi}{6}\right),\ 3\sin\left(\tfrac{7\pi}{6}\right)\right) = \left(3\cdot\left(-\tfrac{\sqrt{3}}{2}\right),\ 3\cdot\left(-\tfrac{1}{2}\right)\right)$$

$$= \left(-\tfrac{3\sqrt{3}}{2},\ -\tfrac{3}{2}\right)$$

8. Sketch a graph of $g(t) = 3\sin\left(2t + \tfrac{\pi}{4}\right) - 2$. State the period, midline, and amplitude.

First, let's write the function in "standard form":

$$g(t) = 3\sin\left(2t + \tfrac{\pi}{4}\right) - 2$$

$$= 3\sin\left(2\left(t + \tfrac{\pi}{8}\right)\right) - 2$$

Thus, this is a function of the form $g(t) = A\sin\left(\omega(t - h)\right) + k$ where $A = 3$, $\omega = 2$, $h = -\tfrac{\pi}{8}$, and $k = -2$. Since $A = 3$ the function has **amplitude 3 units**. Using the fact that $\omega = 2$, we can find the period: $\text{period} = \tfrac{2\pi}{\omega} = \tfrac{2\pi}{2} = \pi$.

So the **period is π units**. Since $k = -2$, the **midline is $y = -2$**. Since $h = -\tfrac{\pi}{8}$, we need to "start" our sine wave at $t = -\tfrac{\pi}{8}$, i.e., shift the wave left $\tfrac{\pi}{8}$ units. Below is a graph of $g(t) = 3\sin\left(2\left(t + \tfrac{\pi}{8}\right)\right) - 2$.
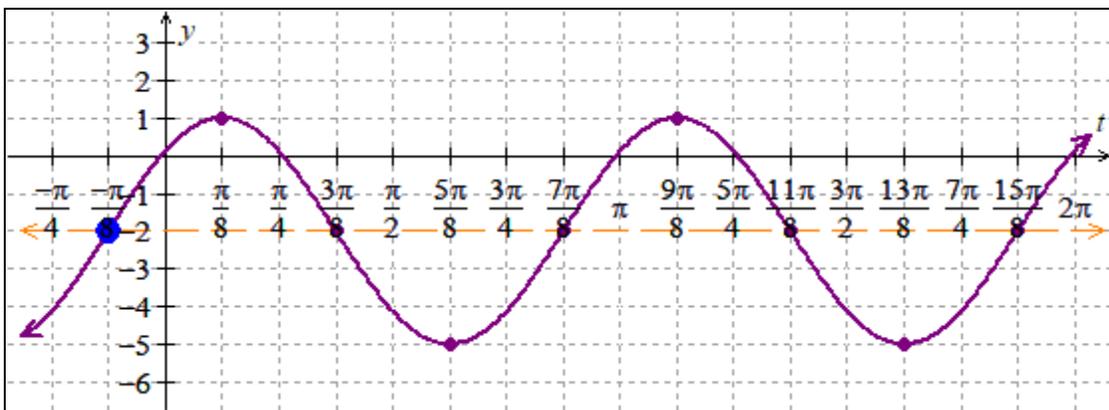
A graph of $g(t) = 3\sin\left(2t + \tfrac{\pi}{4}\right) - 2 = 3\sin\left(2\left(t + \tfrac{\pi}{8}\right)\right) - 2$.

9. Find four algebraic rules (one using positive sine, one using negative sine, one using positive cosine, and one using negative cosine) for the function $y = f(t)$ graphed in Figure 2.
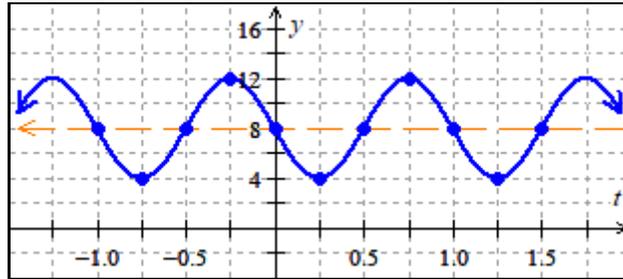

Figure 2: Graph of $y = f(t)$.

Whether we use sine or cosine for our rule, it will have form $f(t) = \pm |A| \sin(\omega(t - h)) + k$ or $f(t) = \pm |A| \cos(\omega(t - h)) + k$ so we can start by determining the values of $|A|$, ω, and k.

- the amplitude is 4 units so we know that $|A| = 4$
- the midline is $y = 8$ so we know that $k = 8$
- the period is 1 unit so we know that ω must satisfy $1 = 2\pi \cdot \frac{1}{\omega}$. Thus,

$$1 = 2\pi \cdot \frac{1}{\omega}$$

$$\Rightarrow \omega = \frac{2\pi}{1} = 2\pi$$

Since a positive sine wave "starts" at $t = 0.5$, we can use $h = 0.5$ to construct a rule:

$$f(t) = 4\sin(2\pi(t - 0.5)) + 8$$

Since a negative (reflected) sine wave "starts" at $t = 0$, we can use $h = 0$:

$$f(t) = -4\sin(2\pi \cdot t) + 8$$

Since a positive cosine wave "starts" at $t = -0.25$, we can use $h = -0.25$:

$$f(t) = 4\cos(2\pi(t + 0.25)) + 8$$

Since a negative (reflected) cosine wave "starts" at $t = 0.25$, we can use $h = 0.25$:

$$f(t) = -4\cos(2\pi(t - 0.25)) + 8$$

(Note that there are many other possible answers.)

b. Use one of your answer to part **a** to find exact solutions to $f(t) = 10$.

$$f(t) = 10$$
$$\Rightarrow \quad -4\sin(2\pi t) + 8 = 10$$
$$\Rightarrow \quad -4\sin(2\pi t) = 2$$
$$\Rightarrow \quad \sin(2\pi t) = -\frac{1}{2}$$
$$\Rightarrow 2\pi t = -\frac{\pi}{6} + 2k\pi \quad \text{or} \quad 2\pi t = \frac{7\pi}{6} + 2k\pi, \quad k \in \mathbb{Z}$$
$$\Rightarrow \quad t = \frac{-\frac{\pi}{6} + 2k\pi}{2\pi} \quad \text{or} \quad t = \frac{\frac{7\pi}{6} + 2k\pi}{2\pi}, \quad k \in \mathbb{Z}$$
$$\Rightarrow \quad t = -\frac{1}{12} + k \quad \text{or} \quad t = \frac{7}{12} + k, \quad k \in \mathbb{Z}$$

10. Evaluate the following:

a. $\sin\left(\cos^{-1}\left(-\frac{1}{2}\right)\right)$

$$\sin\left(\cos^{-1}\left(-\frac{1}{2}\right)\right) = \sin\left(\frac{2\pi}{3}\right) \quad \text{(since the range of } y = \cos^{-1}(x) \text{ is } [0, \pi])$$
$$= \frac{\sqrt{3}}{2}$$

b. $\sin^{-1}\left(\sin\left(\frac{7\pi}{4}\right)\right)$

$$\sin^{-1}\left(\sin\left(\frac{7\pi}{4}\right)\right) = \sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$$
$$= -\frac{\pi}{4} \quad \text{(since the range of } y = \sin^{-1}(x) \text{ is } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right])$$

c. $\cos^{-1}\left(\sin\left(\frac{4\pi}{3}\right)\right)$

$$\cos^{-1}\left(\sin\left(\frac{4\pi}{3}\right)\right) = \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$$
$$= \frac{5\pi}{6} \quad \text{(since the range of } y = \cos^{-1}(x) \text{ is } [0, \pi])$$