SOLUTIONS: Some Additional Practice: Final Exam

1. Suppose that $\sin(\alpha) = \frac{5}{13}$ and $\cos(\beta) = \frac{3}{5}$, and where $0 < \alpha < \frac{\pi}{2}$ and $\frac{3\pi}{2} < \beta < 2\pi$.

a. Find the exact value of $\sin(\alpha + \beta)$.

To find $\sin(\alpha + \beta)$ we'll need to use the sine-of-a-sum identity:

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)$$

We'll need $\cos(\alpha)$ and $\sin(\beta)$ to use this formula. Let's use the Pythagorean identity, and start by finding $\cos(\alpha)$:

$$\sin^2(\alpha) + \cos^2(\alpha) = 1$$
$$\Rightarrow \left(\frac{5}{13}\right)^2 + \cos^2(\alpha) = 1$$
$$\Rightarrow \frac{25}{169} + \cos^2(\alpha) = 1$$
$$\Rightarrow \cos^2(\alpha) = 1 - \frac{25}{169}$$
$$\Rightarrow \cos^2(\alpha) = \frac{144}{169}$$
$$\Rightarrow \cos(\alpha) = \frac{12}{13}$$

We take the positive square root of $\frac{144}{169}$ since $0 < \alpha < \frac{\pi}{2}$, i.e., α is in the quadrant 1.

Now let's use the Pythagorean identity to find $\sin(\beta)$:

$$\sin^2(\beta) + \cos^2(\beta) = 1$$
$$\Rightarrow \sin^2(\beta) + \left(\frac{3}{5}\right)^2 = 1$$
$$\Rightarrow \sin^2(\beta) + \frac{9}{25} = 1$$
$$\Rightarrow \sin^2(\beta) = 1 - \frac{9}{25}$$
$$\Rightarrow \sin^2(\beta) = \frac{16}{25}$$
$$\Rightarrow \sin(\beta) = -\frac{4}{5}$$

We take the negative square root of $\frac{16}{25}$ since $\frac{3\pi}{2} < \beta < 2\pi$, i.e., β is in the quadrant 4.

Therefore,

$$\begin{aligned}\sin(\alpha + \beta) &= \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)\\ &= \frac{5}{13}\cdot\frac{3}{5} + \left(-\frac{4}{5}\right)\cdot\frac{12}{13}\\ &= \frac{15}{65} - \frac{48}{65}\\ &= -\frac{33}{65}\end{aligned}$$

b. Find the exact value of $\cos(\alpha - \beta)$.

To find $\cos(\alpha - \beta)$ we'll need to use the cosine-of-a-difference identity:

$$\cos(\alpha - \beta) = \sin(\alpha)\sin(\beta) + \cos(\alpha)\cos(\beta)$$

We can use the values we found in part (a):

$$\begin{aligned}\cos(\alpha - \beta) &= \sin(\alpha)\sin(\beta) + \cos(\alpha)\cos(\beta)\\ &= \frac{5}{13}\cdot\left(-\frac{4}{5}\right) + \frac{12}{13}\cdot\frac{3}{5}\\ &= \frac{16}{65}\end{aligned}$$

c. Find the exact value of $\sin(2\beta)$.

We can use the double-angle identity:

$$\sin(2\beta) = 2\sin(\beta)\cos(\beta).$$

(Note that we can use the value of $\cos(\beta)$ that we found in part (a).)

$$\begin{aligned}\sin(2\beta) &= 2\sin(\beta)\cos(\beta)\\ &= 2\cdot\left(-\frac{4}{5}\right)\cdot\frac{3}{5}\\ &= -\frac{24}{25}\end{aligned}$$

d. Find the exact value of $\cos(2\beta)$.

We can use the double-angle identity $\cos(2\beta) = 1 - 2\sin^2(\beta)$:

$$\begin{aligned}\cos(2\beta) &= 1 - 2\sin^2(\beta)\\ &= 1 - 2\cdot\left(-\frac{4}{5}\right)^2\\ &= 1 - \frac{32}{25}\\ &= -\frac{7}{25}\end{aligned}$$

e. Find the exact value of $\sin\left(\frac{\beta}{2}\right)$.

Recall that $\frac{3\pi}{2} < \beta < 2\pi$. Thus,

$$\frac{\frac{3\pi}{2}}{2} < \frac{\beta}{2} < \frac{2\pi}{2}$$
$$\Rightarrow \frac{3\pi}{4} < \frac{\beta}{2} < \pi.$$

Since $\frac{3\pi}{4} < \frac{\beta}{2} < \pi$, we know that $\frac{\beta}{2}$ is in quadrant 2 so $\sin\left(\frac{\beta}{2}\right) > 0$; therefore, we need to choose the *positive* square root in the half-angle identity:

$$\begin{aligned}\sin\left(\frac{\beta}{2}\right) &= +\sqrt{\frac{1-\cos(\beta)}{2}} \\ &= \sqrt{\frac{1-\frac{3}{5}}{2}} \\ &= \sqrt{\frac{2}{5}\cdot\frac{1}{2}} \\ &= \sqrt{\frac{1}{5}} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}\end{aligned}$$

f. Find the exact value of $\cos\left(\frac{\beta}{2}\right)$.

In (e) we discovered that $\frac{\beta}{2}$ is in quadrant 2 so $\cos\left(\frac{\beta}{2}\right) < 0$; therefore, we need to choose the *negative* square root in the half-angle identity:

$$\begin{aligned}\cos\left(\frac{\beta}{2}\right) &= -\sqrt{\frac{1+\cos(\beta)}{2}} \\ &= -\sqrt{\frac{1+\frac{3}{5}}{2}} \\ &= -\sqrt{\frac{8}{5}\cdot\frac{1}{2}} \\ &= -\sqrt{\frac{4}{5}} \\ &= -\frac{2}{\sqrt{5}} = -\frac{2\sqrt{5}}{5}\end{aligned}$$

2. Prove the following identities.

a. $\tan(x) + \cot(x) = \sec(x)\csc(x)$

$$\begin{aligned}
\tan(x) + \cot(x) &= \frac{\sin(x)}{\cos(x)} + \frac{\cos(x)}{\sin(x)} \\
&= \frac{\sin(x)}{\cos(x)} \cdot \frac{\sin(x)}{\sin(x)} + \frac{\cos(x)}{\sin(x)} \cdot \frac{\cos(x)}{\cos(x)} \\
&= \frac{\sin^2(x)}{\cos(x)\sin(x)} + \frac{\cos^2(x)}{\sin(x)\cos(x)} \\
&= \frac{\sin^2(x) + \cos^2(x)}{\cos(x)\sin(x)} \\
&= \frac{1}{\cos(x)\sin(x)} \\
&= \frac{1}{\cos(x)} \cdot \frac{1}{\sin(x)} \\
&= \sec(x)\csc(x)
\end{aligned}$$

b. $\tan^2(x) - \sin^2(x) = \tan^2(x)\sin^2(x)$

$$\begin{aligned}
\tan^2(x) - \sin^2(x) &= \frac{\sin^2(x)}{\cos^2(x)} - \sin^2(x) \cdot \frac{\cos^2(x)}{\cos^2(x)} \\
&= \frac{\sin^2(x) - \sin^2(x)\cos^2(x)}{\cos^2(x)} \\
&= \frac{\sin^2(x) \cdot \left(1 - \cos^2(x)\right)}{\cos^2(x)} \\
&= \frac{\sin^2(x) \cdot \sin^2(x)}{\cos^2(x)} \\
&= \frac{\sin^2(x)}{\cos^2(x)} \cdot \sin^2(x) \\
&= \tan^2(x)\sin^2(x)
\end{aligned}$$

c. $\cos(2x) = \cos^4(x) - \sin^4(x)$

$$\begin{aligned}\cos^4(x) - \sin^4(x) &= \left(\cos^2(x) - \sin^2(x)\right)\left(\cos^2(x) + \sin^2(x)\right) \\ &= \left(\cos^2(x) - \sin^2(x)\right)\cdot 1 \\ &= \cos^2(x) - \sin^2(x) \\ &= \cos(2x)\end{aligned}$$

3. Convert the following polar ordered pairs into Cartesian (i.e., rectangular) coordinates.

a. $\left(3, \frac{\pi}{2}\right)$

The polar coordinates (r, θ) correspond to the Cartesian coordinates $(x, y) = \left(r\cos(\theta), r\sin(\theta)\right)$. So

$$\begin{aligned}x &= 3\cos\left(\tfrac{\pi}{2}\right) \\ &= 0\end{aligned} \qquad \text{and} \qquad \begin{aligned}y &= 3\sin\left(\tfrac{\pi}{2}\right) \\ &= 3\end{aligned}$$

Therefore, the polar ordered pair $\left(3, \frac{\pi}{2}\right)$ corresponds to the Cartesian ordered pair $(0, 3)$.

b. $\left(\pi, \frac{5\pi}{3}\right)$

The polar coordinates (r, θ) correspond to the Cartesian coords. $(x, y) = \left(r\cos(\theta), r\sin(\theta)\right)$. So

$$\begin{aligned}x &= \pi\cos\left(\tfrac{5\pi}{3}\right) \\ &= \pi\cdot\tfrac{1}{2} \\ &= \tfrac{\pi}{2}\end{aligned} \qquad \text{and} \qquad \begin{aligned}y &= \pi\sin\left(\tfrac{5\pi}{3}\right) \\ &= \pi\cdot -\tfrac{\sqrt{3}}{2} \\ &= -\tfrac{\pi\sqrt{3}}{2}\end{aligned}$$

Therefore, the polar ordered pair $\left(\pi, \frac{5\pi}{3}\right)$ corresponds to the Cartesian ordered pair $\left(\frac{\pi}{2}, -\frac{\pi\sqrt{3}}{2}\right)$.

c. $(10, -10^{\circ})$

The polar coordinates (r, θ) correspond to the Cartesian coords. $(x, y) = (r\cos(\theta), r\sin(\theta))$. So

$$x = 10\cos(-10^{\circ}) \approx 9.85 \quad \text{and} \quad y = 10\sin(-10^{\circ}) \approx -1.74$$

Therefore, the polar ordered pair $(10, -10^{\circ})$ corresponds to the Cartesian ordered pair $\left(10\cos(10^{\circ}), 10\sin(10^{\circ})\right) \approx (9.85, -1.74)$.

4. Convert the following Cartesian (i.e., rectangular) ordered pairs into polar coordinates.

a. $(10, -10)$

We need to convert Cartesian coordinates (x, y) into polar coordinates (r, θ). The distance from the origin to the point, r, is given by $r = \sqrt{x^2 + y^2}$:

$$\begin{aligned} r &= \sqrt{10^2 + (-10)^2} \\ &= 10\sqrt{2} \end{aligned}$$

To find θ (which should be in the 4th quadrant since that's where the given point sits), we can use the formula $\theta = \tan^{-1}\left(\frac{y}{x}\right)$:

$$\begin{aligned} \theta &= \tan^{-1}\left(\frac{10}{-10}\right) \\ &= \tan^{-1}(-1) \\ &= -\tfrac{\pi}{4} \qquad \text{(since } \tan\left(-\tfrac{\pi}{4}\right) = -1\text{)} \end{aligned}$$

Therefore, the Cartesian ordered pair $(10, -10)$ corresponds to the polar ordered pair $\left(10\sqrt{2}, -\frac{\pi}{4}\right)$.

b. $(-3, 0)$

We need to convert Cartesian coordinates (x, y) into polar coordinates (r, θ). Clearly $r = 3$ and $\theta = \pi$ (since the point lies on the negative x-axis). Therefore, the Cartesian ordered pair $(-3, 0)$ corresponds to the polar ordered pair $(3, \pi)$.

c. $\left(-8, -8\sqrt{3}\right)$

We need to convert Cartesian coordinates (x, y) into polar coordinates (r, θ). The distance from the origin to the point, r, is given by $r = \sqrt{x^2 + y^2}$:

$$\begin{aligned} r &= \sqrt{(-8)^2 + (-8\sqrt{3})^2} \\ &= \sqrt{256} \\ &= 16 \end{aligned}$$

To find θ (which should be in the 3rd quadrant since that's where the given point sits), we can use the formula $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ – but we'll need to add π to the result since the range of arctangent is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, i.e., arctangent can only give us angles in quadrants 1 or 4 but our point is in quadrant 2, so we need to add a half-revolution to rotate the angle into the correct quadrant:

$$\begin{aligned} \theta &= \tan^{-1}\left(\frac{-8\sqrt{3}}{-8}\right) + \pi \\ &= \tan^{-1}(\sqrt{3}) + \pi \\ &= \frac{\pi}{3} + \pi \qquad \text{(since } \tan\left(\frac{\pi}{3}\right) = \sqrt{3}\text{)} \\ &= \frac{4\pi}{3} \end{aligned}$$

So the Cartesian ordered pair $\left(8, -8\sqrt{3}\right)$ corresponds to the polar ordered pair $\left(16, \frac{4\pi}{3}\right)$.

5. Find an equation involving polar coordinates whose graph is equivalent to the Cartesian equation $y = 3x - 1$.

We can use the fact that $x = r\cos(\theta)$ and $y = r\sin(\theta)$:

$$\begin{aligned} & y = 3x - 1 \\ \Rightarrow\ & r\sin(\theta) = 3r\cos(\theta) - 1 \\ \Rightarrow\ & 1 = 3r\cos(\theta) - r\sin(\theta) \\ \Rightarrow\ & 1 = r\left(3\cos(\theta) - \sin(\theta)\right) \\ \Rightarrow\ & r = \frac{1}{3\cos(\theta) - \sin(\theta)} \end{aligned}$$

6. Find an equation involving polar coordinates whose graph is equivalent to the Cartesian equation $y = x^2$.

To convert from Cartesian to polar we can use the fact that

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \end{aligned} :$$

Thus,

$$\begin{aligned} & y = x^2 \\ \Rightarrow\ & r\sin(\theta) = \left(r\cos(\theta)\right)^2 \\ \Rightarrow\ & r\sin(\theta) = r^2\cos^2(\theta) \\ \Rightarrow\ & \frac{\sin(\theta)}{\cos^2(\theta)} = \frac{r^2}{r} \\ \Rightarrow\ & r = \frac{\sin(\theta)}{\cos^2(\theta)} \\ \Rightarrow\ & r = \frac{\sin(\theta)}{\cos(\theta)} \cdot \frac{1}{\cos(\theta)} \\ \Rightarrow\ & r = \tan(\theta)\sec(\theta) \end{aligned}$$

(Use the fact that $\sec(\theta) = \frac{1}{\cos(\theta)}$ to graph $r = \tan(\theta)\sec(\theta)$ on your calculator in the "polar" function setting to verify that you get a parabola.)

7. Translate the complex number $z = -3 + 3\sqrt{3} \cdot i$ into its polar form $z = re^{i\theta}$.

The number z has form $z = a + bi$ where $a = -3$ and $b = 3\sqrt{3}$. So

$$\begin{aligned} r &= \sqrt{a^2 + b^2} \\ &= \sqrt{(-3)^2 + \left(3\sqrt{3}\right)^2} \\ &= \sqrt{9 + 27} \\ &= 6 \end{aligned}$$

If we plot z in the Cartesian plane, we see that it lies in the second quadrant, so the angle θ is in the second quadrant, so we'll need to add π to the arctangent value:

$$\begin{aligned}\theta &= \tan^{-1}\left(\frac{b}{a}\right) + \pi \\ &= \tan^{-1}\left(\frac{3\sqrt{3}}{-3}\right) + \pi \\ &= \tan^{-1}\left(-\sqrt{3}\right) + \pi \\ &= \frac{2\pi}{3}\end{aligned}$$

Therefore, $z = 6e^{i\cdot\frac{2\pi}{3}}$.

8. Translate the polar form of the complex number $z = 4e^{i\cdot\frac{5\pi}{6}}$ into its rectangular form $z = a + bi$.

$$\begin{aligned}z &= 4e^{i\cdot\frac{5\pi}{6}} \\ &= 4\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right) \\ &= 4\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) \\ &= -2\sqrt{3} + 2i\end{aligned}$$

9. Determine the magnitude and direction (with respect to the positive x-axis) of the vector $\vec{v} = \langle -3, -7 \rangle$.

Magnitude:

$$\begin{aligned}\|\vec{v}\| &= \sqrt{(-3)^2 + (-7)^2} \\ &= \sqrt{9 + 49} \\ &= \sqrt{58}\end{aligned}$$

Direction:

$$\begin{aligned}\theta &= \tan^{-1}\left(\frac{-7}{-3}\right) \\ &\approx 66.8^\circ + 180^\circ \quad \text{(we add } 180^\circ \text{ since the vector points towards the 3rd quadrant)} \\ &\approx 246.8^\circ\end{aligned}$$

Therefore the vector $\vec{v} = \langle -3, -7 \rangle$ has magnitude $\sqrt{58}$ and direction approximately 246.8° with respect to the positive x-axis.

10. Suppose $\vec{v} = \langle -4, 1 \rangle$ and $\vec{u} = \langle 3, -6 \rangle$.

a. Find $\vec{w} = \vec{v} - 2\vec{u}$.

$$\begin{aligned}\vec{w} &= \vec{v} - 2\vec{u}\\ &= \langle -4, 1 \rangle - 2\cdot\langle 3, -6 \rangle\\ &= \langle -4, 1 \rangle - \langle 6, -12 \rangle\\ &= \langle -4 - 6,\ 1 - (-12) \rangle\\ &= \langle -10, 13 \rangle\end{aligned}$$

b. Use the *dot product* to find the angle between $\vec{v} = \langle -4, 1 \rangle$ and $\vec{u} = \langle 3, -6 \rangle$?

We can use the fact that $\vec{v}\cdot\vec{u} = \|\vec{v}\|\cdot\|\vec{u}\|\cos(\theta)$, where θ is the angle between vectors $\vec{v}$ and $\vec{u}$. First, let's find $\|\vec{v}\|$, $\|\vec{u}\|$, and $\vec{v}\cdot\vec{u}$:

$$\begin{aligned}\|\vec{v}\| &= \sqrt{(-4)^2 + (1)^2}\\ &= \sqrt{16 + 1}\\ &= \sqrt{17}\end{aligned} \qquad \text{and} \qquad \begin{aligned}\|\vec{u}\| &= \sqrt{(3)^2 + (-6)^2}\\ &= \sqrt{9 + 36}\\ &= \sqrt{45}\end{aligned}$$

and

$$\begin{aligned}\vec{v}\cdot\vec{u} &= (-4)\cdot 3 + 1\cdot(-6)\\ &= -12 - 6\\ &= -18.\end{aligned}$$

Thus,

$$\begin{aligned}& \vec{v}\cdot\vec{u} = \|\vec{v}\|\cdot\|\vec{u}\|\cos(\theta)\\ \Rightarrow\quad & -18 = \sqrt{17}\cdot\sqrt{45}\,\cos(\theta)\\ \Rightarrow\quad & \cos(\theta) = \tfrac{-18}{\sqrt{17}\cdot\sqrt{45}}\\ \Rightarrow\quad & \theta = \cos^{-1}\left(\tfrac{-18}{\sqrt{17}\cdot\sqrt{45}}\right) \approx 130.6^\circ\end{aligned}$$

So the angle between vectors $\vec{v}$ and $\vec{u}$ is about 130.6°.

11. **a.** Find the horizontal and vertical components of the vector $\vec{v}$ that starts at the point $P = (5, 6)$ and ends at the point $Q = (2, 2)$.

Since the vector $\vec{v}$ starts at x-coordinate 5 and ends at x-coordinate 2, we see that the horizontal component of the vector is $2 - 5 = -3$.

Since the vector $\vec{v}$ starts at y-coordinate 6 and ends at x-coordinate 2, we see that the vertical component of the vector is $2 - 6 = -4$.

Therefore the vector is $\vec{v} = \langle -3, -4 \rangle$.

b. Find the magnitude, $\|\vec{v}\|$, and the direction (with respect to the positive x-axis) of the vector $\vec{v}$ that you found in part **a**?

Magnitude:

$$\begin{aligned} \|\vec{v}\| &= \sqrt{(-3)^2 + 4^2} \\ &= 5 \end{aligned}$$

Direction:

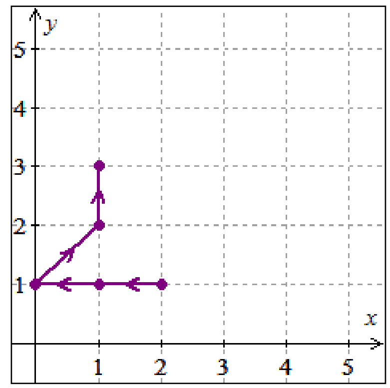
$$\begin{aligned} \theta &= \tan^{-1}\left(\frac{-4}{-3}\right) \\ &\approx 53.13^\circ + 180^\circ \quad \text{(we add } 180^\circ \text{ since the vector points towards the 3rd quadrant)} \\ &\approx 233.13^\circ \end{aligned}$$

Therefore the vector $\vec{v} = \langle -3, -4 \rangle$ has magnitude 5 and direction approximately 233.13° with respect to the positive x-axis.

12. The tables below represent the x- and y-coordinates of the motion of a robot as a function of time, t, in seconds. Sketch the graph of the motion of the robot; use arrows to indicate the direction of travel.

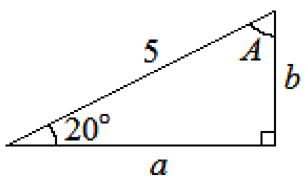
t	$x = f(t)$
0	2
1	1
2	0
3	1
4	1

t	$y = g(t)$
0	1
1	1
2	1
3	2
4	3



13. Find the missing side(s) and missing angle(s) for the triangle given below. (The triangles may not be drawn to scale.)

a.



We can find A using the fact that the sum of the angles in a triangle is 180°:

$$A + 20^\circ + 90^\circ = 180^\circ$$
$$\Rightarrow \quad A = 180^\circ - 20^\circ - 90^\circ$$
$$\Rightarrow \quad A = 70^\circ$$

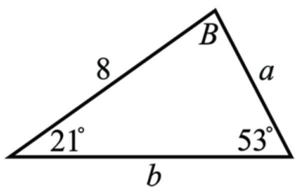
We can find b using sine:

$$\sin(20^\circ) = \frac{b}{5}$$
$$\Rightarrow \quad b = 5\sin(20^\circ)$$
$$\approx 1.71$$

And we can use cosine to find a:

$$\cos(20^\circ) = \frac{a}{5}$$
$$\Rightarrow \quad a = 5\cos(20^\circ)$$
$$\approx 4.698$$

b.



We can find B using the fact that the sum of the angles in a triangle is 180°:

$$21^\circ + B + 53^\circ = 180^\circ$$
$$\Rightarrow \quad B = 180^\circ - 21^\circ - 53^\circ$$
$$\Rightarrow \quad B = 106^\circ$$

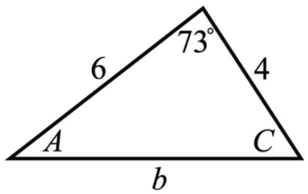
Now we can use the Law of Sines to find b:

$$\frac{8}{\sin(53^\circ)} = \frac{b}{\sin(106^\circ)}$$
$$\Rightarrow \quad b = \frac{8\sin(106^\circ)}{\sin(53^\circ)} \approx 9.62$$

And we can use the Law of Sines again to find a:

$$\frac{8}{\sin(53^\circ)} = \frac{a}{\sin(21^\circ)}$$
$$\Rightarrow \quad a = \frac{8\sin(21^\circ)}{\sin(53^\circ)} \approx 3.59$$

c.



First let's use the Law of Cosines to find b:

$$b^2 = 4^2 + 6^2 - 2\cdot 4\cdot 6\cos(73^\circ)$$
$$\Rightarrow\ b = \sqrt{16 + 36 - 48\cos(73^\circ)}$$
$$\Rightarrow\ b = \sqrt{52 - 48\cos(73^\circ)}$$
$$\Rightarrow\ b \approx 6.16$$

Now we can use the Law of Sines to find A. (We choose to find A first instead of C since A must be smaller than C since A is across a shorter side than C.)

$$\frac{\sin(A)}{4} = \frac{\sin(73^\circ)}{b}$$
$$\Rightarrow\ \sin(A) = \frac{4\sin(73^\circ)}{b}$$
$$\Rightarrow\ A = \sin^{-1}\left(\frac{4\sin(73^\circ)}{b}\right)$$
$$\Rightarrow\ A \approx \sin^{-1}\left(\frac{4\sin(73^\circ)}{5.997}\right)$$
$$\approx 38.39^\circ$$

Finally, we can find C using the fact that the sum of the angles in a triangle is 180°:

$$A + 73^\circ + C = 180^\circ$$
$$\Rightarrow\ C = 180^\circ - 73^\circ - A$$
$$\Rightarrow\ C \approx 180^\circ - 73^\circ - 39.63^\circ$$
$$\Rightarrow\ C \approx 68.61^\circ$$