Right Triangle Trigonometry

As we studied in "Intro to the Trigonometric Functions: Part 1," if we put the same angle in the center of two circles of different radii, we can construct two *similar triangles*; see Figure 1.

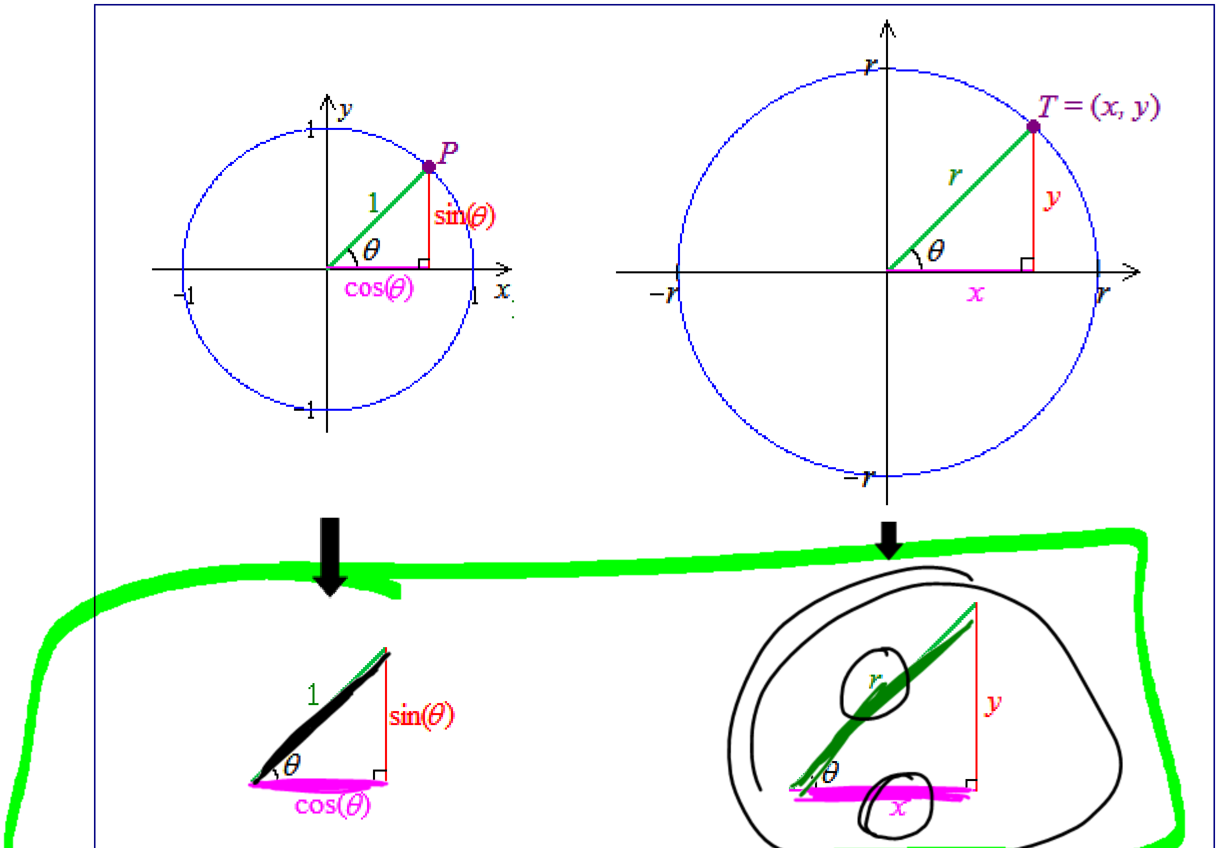

Figure 1: The angle θ in both a unit circle and in a circle of radius r, inducing similar right triangles.

We can use these similar triangles to obtain the following ratios (which we can use to derive expressions for $\sin(\theta)$ and $\cos(\theta)$):

$$\frac{\sin(\theta)}{1} = \frac{y}{r} \qquad \Rightarrow \boxed{\sin(\theta) = \frac{y}{r}}$$

$$\frac{\cos(\theta)}{1} = \frac{x}{r} \qquad \Rightarrow \boxed{\cos(\theta) = \frac{x}{r}}$$

To help remember these ratios, it's best to imagine yourself standing at angle θ looking into the triangle. Then the side labeled y is on the **opposite** side of the triangle while the side labeled x is **adjacent** to you. We use these descriptions (as well as the fact that the side labeled r is the **hypotenuse** of the triangle) to refer to the sides of the triangles in Figure 1.

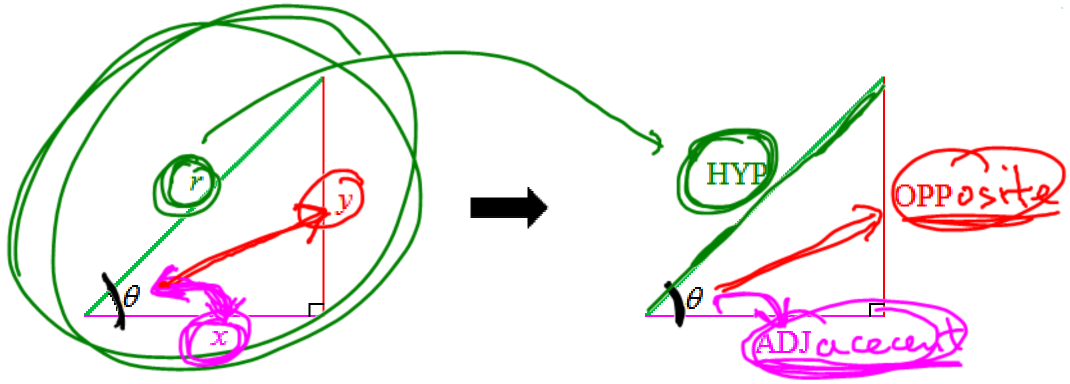

Figure 2: We use the terms **opposite** (or **OPP**), **adjacent** (or **ADJ**), and **hypotenuse** (or **HYP**) to refer to the sides of a right triangle.

DEFINITION: If θ is the angle given in the right triangles in Figure 2, then

$$\sin(\theta) = \frac{y}{r} = \frac{\text{OPP}}{\text{HYP}} \qquad \cos(\theta) = \frac{x}{r} = \frac{\text{ADJ}}{\text{HYP}} \qquad \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{\text{OPP}/\cancel{\text{HYP}}}{\text{ADJ}/\cancel{\text{HYP}}} = \frac{\text{OPP}}{\text{ADJ}}$$

Consequently, the other trigonometric functions can be defined as follows:

$$\cot(\theta) = \frac{1}{\tan(\theta)} = \frac{\cos(\theta)}{\sin(\theta)} \qquad \sec(\theta) = \frac{1}{\cos(\theta)} \qquad \csc(\theta) = \frac{1}{\sin(\theta)}$$

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EXAMPLE 1: Find value for all six trigonometric functions of the angle α given in the right triangle in Figure 3. (The triangle may not be drawn to scale.)

$$\sin(\alpha) = \frac{OPP}{HYP} = \frac{9}{15} = \frac{3}{5}$$

$$\cos(\alpha) = \frac{ADJ}{HYP} = \frac{12}{15} = \frac{4}{5}$$

$$\tan(\alpha) = \frac{OPP}{ADJ} = \frac{9}{12} = \frac{3}{4}$$

$$\cot(\alpha) = \frac{1}{\tan(\alpha)} = \frac{4}{3}$$

$$\sec(\alpha) = \frac{1}{\cos(\theta)} = \frac{5}{4}$$

$$\csc(\alpha) = \frac{1}{\sin(\theta)} = \frac{5}{3}$$

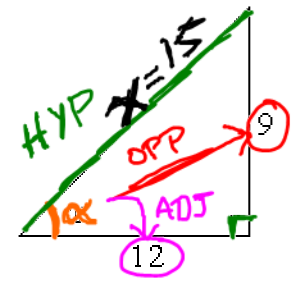

Figure 3

Use Pythagoras to find x:

$$x^2 = 12^2 + 9^2$$

$$\Rightarrow x^2 = 144 + 81$$

$$\Rightarrow x^2 = 225$$

$$\Rightarrow x = \sqrt{225} = \sqrt{25 \cdot 9} = 5 \cdot 3 = 15$$

We can use the trigonometric functions, along with the Pythagorean Theorem to "**solve a right triangle**," i.e., find the missing side-lengths and missing angle-measures for a triangle.

EXAMPLE 2: Solve the right triangle given in Figure 4 by finding A, b, and c. (The triangle might not be drawn to scale.)

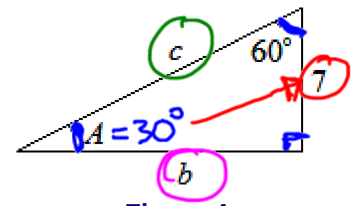

Figure 4

$$A + \underbrace{60^\circ + 90^\circ}_{150^\circ} = 180^\circ$$

$$\boxed{A = 30^\circ}$$

Use pythagoran to find b:

$$b^2 + 7^2 = c^2$$

$$b^2 + 7^2 = 14^2$$

$$b^2 = 14^2 - 7^2$$

$$b^2 = 7^2 \cdot 2^2 - 7^2$$

$$b^2 = 7^2(2^2 - 1)$$

$$b^2 = 7^2 \cdot 3$$

$$\boxed{b = \sqrt{7^2 \cdot 3} = 7\sqrt{3}}$$

$$c \cdot \sin(30^\circ) = \frac{7}{\cancel{c}} \cdot \cancel{c}$$

$$\frac{c \cdot \cancel{\sin(30^\circ)}}{\cancel{\sin(30^\circ)}} = \frac{7}{\sin(30^\circ)}$$

$$c = \frac{7}{\sin(30^\circ)}$$

$$\boxed{c = \frac{7}{1/2} = 14}$$

EXAMPLE 3: Solve the right triangle given in Figure 5 by finding c, α, and β. (The triangle might not be drawn to scale.)

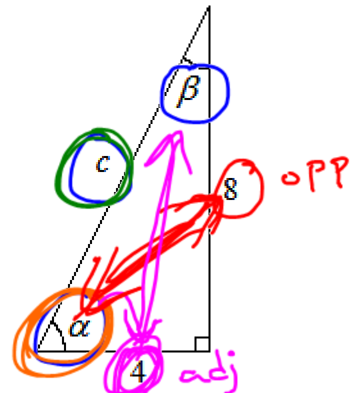

Figure 5

$$c^2 = 4^2 + 8^2$$
$$\Rightarrow c^2 = 16 + 64$$
$$\Rightarrow c^2 = 80$$
$$\Rightarrow c = \sqrt{80}$$
$$c = \sqrt{4 \cdot 20}$$
$$c = 2\sqrt{4 \cdot 5}$$
$$\boxed{c = 4\sqrt{5}}$$

$$\tan(\alpha) = \frac{\text{opp}}{\text{adj}}$$
$$\tan(\alpha) = \frac{8}{4}$$
$$\tan(\alpha) = 2$$
$$\tan^{-1}(\tan(\alpha)) = \tan^{-1}(2)$$
$$\alpha = \tan^{-1}(2)$$
$$\boxed{\alpha \approx 63.43^\circ}$$

$$\beta + \alpha + 90^\circ = 180^\circ$$
$$\beta + 63.43^\circ + 90 \approx 180^\circ$$
$$\beta \approx 180^\circ - 90^\circ - 63.43^\circ$$
$$\boxed{\beta \approx 26.57^\circ}$$