Solutions to "Midterm Exam"

(2 ea.) **1.** Circle **T** for *true* or **F** for *false*. (You don't need to justify your answers.)

a. **F** $\cos^{-1}\left(\cos\left(-\frac{\pi}{4}\right)\right) = -\frac{\pi}{4}$.

($-\frac{\pi}{4}$ isn't in the range of the inverse cosine function)

b. **T** $\sin(-t) = -\sin(t)$ for all t.

(sine is an ODD function, i.e., it's symmetric about the origin)

c. **T** An angle of measure 1° is smaller than an angle of measure 1 radian.

($1^\circ < 1 \text{ rad} \approx 57.3^\circ$)

d. **F** An angle of measure 30° in a circle of radius 1 unit is smaller than an angle of measure 30° in a circle of radius 2 units.

($30^\circ = 30^\circ$ no matter what circle it's in)

(7.5) **2.** What is the length of the arc spanned by an angle of 100° in a circle of radius 24 feet? [Provide a completely simplified, exact numerical value.]

To compute the arc-length we can use the formula $s = r \cdot \theta$, but we need θ in radians to use this formula:

$$\begin{aligned} \theta = 100^\circ &= 100^\circ \cdot \frac{2\pi}{360^\circ} \\ &= \frac{200\pi}{360} \\ &= \frac{5\pi}{9} \end{aligned}$$

Now we can calculate the arc-length:

$$\begin{aligned} s = r \cdot \theta &= (24 \text{ feet}) \cdot \left(\frac{5\pi}{9}\right) \\ &= \frac{3 \cdot 8 \cdot 5\pi}{3 \cdot 3} \text{ feet} \\ &= \frac{40\pi}{3} \text{ feet} \end{aligned}$$

Thus, the desired arc-length is $\frac{40\pi}{3}$ feet.

(2.5 ea.) **3.** Find the ***exact*** value for each of the following expressions. Be sure to use proper notation to communicate your answer, i.e., link the given expression and your answer with an equal sign. If the given expression is undefined write, "*The expression is undefined*." An example has been provided to clarify how your response should look.

ex. $\sin(0)$

$$\sin(0) = 0$$

(Write all of this to communicate what "$\sin(0)$" *equals*.)

a. $\sin\left(\frac{\pi}{4}\right)$.

$$\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

b. $\cos(\pi)$.

$$\cos(\pi) = -1$$

c. $\sin\left(\frac{5\pi}{6}\right)$.

$$\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$$

d. $\cos\left(-\frac{5\pi}{6}\right)$.

$$\cos\left(-\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$

e. $\sin\left(\frac{13\pi}{4}\right)$.

$$\sin\left(\frac{13\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

f. $\cos\left(\frac{4\pi}{3}\right)$.

$$\cos\left(\frac{4\pi}{3}\right) = -\frac{1}{2}$$

g. $\sec\left(\frac{11\pi}{6}\right)$.

$$\sec\left(\frac{11\pi}{6}\right) = \frac{1}{\cos\left(\frac{11\pi}{6}\right)} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

h. $\csc\left(\frac{\pi}{3}\right)$.

$$\csc\left(\frac{\pi}{3}\right) = \frac{1}{\sin\left(\frac{\pi}{3}\right)} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

i. $\tan\left(\frac{3\pi}{2}\right)$.

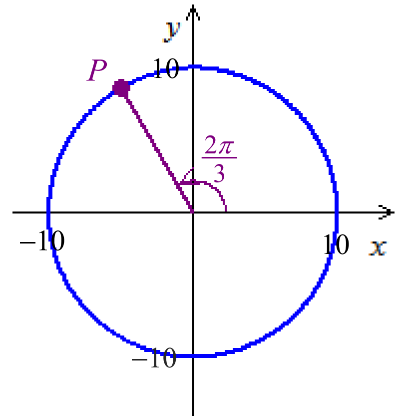
$$\tan\left(\frac{3\pi}{2}\right) = \frac{\sin\left(\frac{3\pi}{2}\right)}{\cos\left(\frac{3\pi}{2}\right)}.$$

Since $\cos\left(\frac{3\pi}{2}\right) = 0$, $\tan\left(\frac{3\pi}{2}\right)$ is undefined.

j. $\tan\left(\frac{5\pi}{3}\right)$.

$$\tan\left(\frac{5\pi}{3}\right) = \frac{\sin\left(\frac{5\pi}{3}\right)}{\cos\left(\frac{5\pi}{3}\right)} = \frac{-\sqrt{3}/2}{1/2} = -\sqrt{3}$$

(7.5) **4.** Use the sine and cosine functions to find the exact coordinates of point P specified by the angle $\frac{2\pi}{3}$ on the circumference of a circle of radius 10 units. Be sure to ***show your use of sine and cosine*** and to provide completely simplified, exact numerical values.



Coordinates of point P:

$$\left(12\cos\left(\tfrac{2\pi}{3}\right),\, 12\sin\left(\tfrac{2\pi}{3}\right)\right) = \left(12\cdot\left(-\tfrac{1}{2}\right),\, 12\cdot\left(\tfrac{\sqrt{3}}{2}\right)\right)$$
$$= \left(-6,\ 6\sqrt{3}\right)$$

5. If $\cos(\theta) = \frac{\sqrt{14}}{5}$ and $\frac{3\pi}{2} < \theta < 2\pi$ (i.e. θ is in the fourth quadrant), find exact numerical values for the expressions given below. [Be sure to compose conclusions that *directly* communicate the values of the given expressions. See the example given in #3.]

(6) **a.** $\sin(\theta)$.

$$\sin^2(\theta) + \cos^2(\theta) = 1$$
$$\Rightarrow \sin^2(\theta) + \left(\frac{\sqrt{14}}{5}\right)^2 = 1$$
$$\Rightarrow \sin^2(\theta) = 1 - \frac{14}{25}$$
$$\Rightarrow \sin^2(\theta) = \frac{11}{25}$$
$$\Rightarrow \sin(\theta) = -\frac{\sqrt{11}}{5}$$

(note that we take the negative square root since sine is negative in the fourth quadrant)

(4) **b.** $\tan(\theta)$.

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{-\sqrt{11}/5}{\sqrt{14}/5} = -\frac{\sqrt{11}}{\sqrt{14}}$$

(4) **c.** $\sec(\theta)$.

$$\sec(\theta) = \frac{1}{\cos(\theta)} = \frac{1}{\sqrt{14}/5} = \frac{5}{\sqrt{14}}\cdot\frac{\sqrt{14}}{\sqrt{14}} = \frac{5\sqrt{14}}{14}$$

(7.5) **6.** Find ***all*** of the solutions of the following trigonometric equation. [Provide completely simplified, exact numerical values.]

$$10\sin(4\theta) + 5 = 0$$

$$10\sin(4\theta) + 5 = 0$$

$$\Rightarrow \quad 10\sin(4\theta) = -5$$

$$\Rightarrow \quad \sin(4\theta) = -\tfrac{1}{2}$$

$$\Rightarrow \quad 4\theta = \tfrac{7\pi}{6} + 2k\pi \quad \text{or} \quad 4\theta = \pi - \tfrac{7\pi}{6} + 2k\pi, \; k \in \mathbb{Z}$$

$$\Rightarrow \quad \theta = \tfrac{7\pi}{24} + \tfrac{k\pi}{2} \quad \text{or} \quad \theta = -\tfrac{\pi}{24} + \tfrac{k\pi}{2}, \; k \in \mathbb{Z}$$

[Note that there are other ways that the solutions can be represented. For example, the solutions represented by $\theta = \frac{7\pi}{24} + \frac{k\pi}{2}$ can instead represented by $\theta = -\frac{5\pi}{24} + \frac{k\pi}{2}$, and the solutions represented by $\theta = -\frac{\pi}{24} + \frac{k\pi}{2}$ can instead be represented by $\theta = \frac{11\pi}{24} + \frac{k\pi}{2}$.]

(7.5) **7.** Find all of the solutions of the following trigonometric equation **on the interval $[0, 2\pi)$**. [Provide completely simplified, exact numerical values.]

$$6\cos(3x) + 2 = -4$$

$$6\cos(3x) + 2 = -4$$

$$\Rightarrow \quad 6\cos(3x) = -6$$

$$\Rightarrow \quad \cos(3x) = -1$$

$$\Rightarrow \quad 3x = \pi + 2k\pi, \quad k \in \mathbb{Z}$$

(there's only one "family" of solutions since the output "-1" only occurs once in each period)

$$\Rightarrow \quad x = \tfrac{\pi}{3} + \tfrac{2k\pi}{3}, \quad k \in \mathbb{Z}$$

Since we need $x \in [0, 2\pi)$, we can see that the solution set is:

$$\left\{\tfrac{\pi}{3}, \tfrac{\pi}{3} + \tfrac{2\cdot 1\pi}{3}, \tfrac{\pi}{3} + \tfrac{2\cdot 2\pi}{3}\right\} = \left\{\tfrac{\pi}{3}, \pi, \tfrac{5\pi}{3}\right\}.$$

(4 ea.) **8.** Evaluate the following expressions. [Be sure to compose conclusions that *directly* communicate what the given expressions equal and to provide completely simplified exact numerical values.]

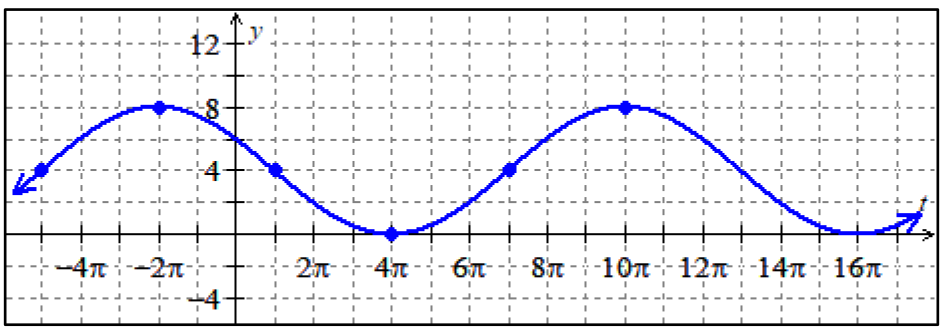
a. $\sin\left(\cos^{-1}\left(-\frac{1}{2}\right)\right)$

$$\sin\left(\cos^{-1}\left(-\tfrac{1}{2}\right)\right) = \sin\left(\tfrac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

b. $\cos^{-1}\left(\cos\left(\frac{5\pi}{4}\right)\right)$

$$\cos^{-1}\left(\cos\left(\tfrac{5\pi}{4}\right)\right) = \cos^{-1}\left(-\tfrac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$$

(7.5) **9.** Find a possible algebraic rule for the sinusoidal function f graphed below. Note that the following points are on the graph: $(-2\pi, 8)$, $(\pi, 4)$, $(4\pi, 0)$, $(7\pi, 4)$, and $(10\pi, 8)$.



The graph of $y = f(t)$.

There are many possibilities. Here we'll write a rule involving cosine, so our rule will have the form $f(t) = A\cos\left(\omega(t-h)\right) + k$.

- The function's maximum output value is 8 and its minimum output value is 0: since 4 is the average of these values, the midline is $y = 4$ so $k = 4$.
- The amplitude is the distance between function's maximum output value, 8, and its midline, $y = 4$. This is 4 units so $|A| = 4$.
- The function completes one period between $t = -2\pi$ and $t = 10\pi$. Thus, the period of the function is $10\pi - (-2\pi) = 12\pi$ units. To find ω we need to solve $12\pi = 2\pi \cdot \frac{1}{\omega}$:

$$12\pi = 2\pi \cdot \frac{1}{\omega}$$
$$\Rightarrow \quad \omega = 2\pi \cdot \frac{1}{12\pi} = \frac{1}{6}$$

- Since a natural cosine graph "starts" at $t = -2\pi$, we can use $h = -2\pi$.

Therefore, an algebraic rule for the graphed function is $f(t) = 4\cos\left(\frac{1}{6}(t + 2\pi)\right) + 4$.

(7.5) **10.** Determine the period, midline, and amplitude of the function $g(t) = 3\cos\left(\pi t + \frac{\pi}{2}\right) - 2$ and then use that information to draw a graph of **at least two periods** of the function.

- **LIST THE *PERIOD*, *MIDLINE*, AND *AMPLITUDE* OF THE FUNCTION.**
- **LABEL THE *SCALE* OF YOUR GRAPH *ON THE AXES*!**
- **ACCURATELY *PLOT POINTS* WHERE THE FUNCTION CROSSES THE *MIDLINE* AND WHERE THE FUNCTION REACHES *MAXIMUMS* & *MINIMUMS*, AND THEN CONNECT THOSE POINTS.**

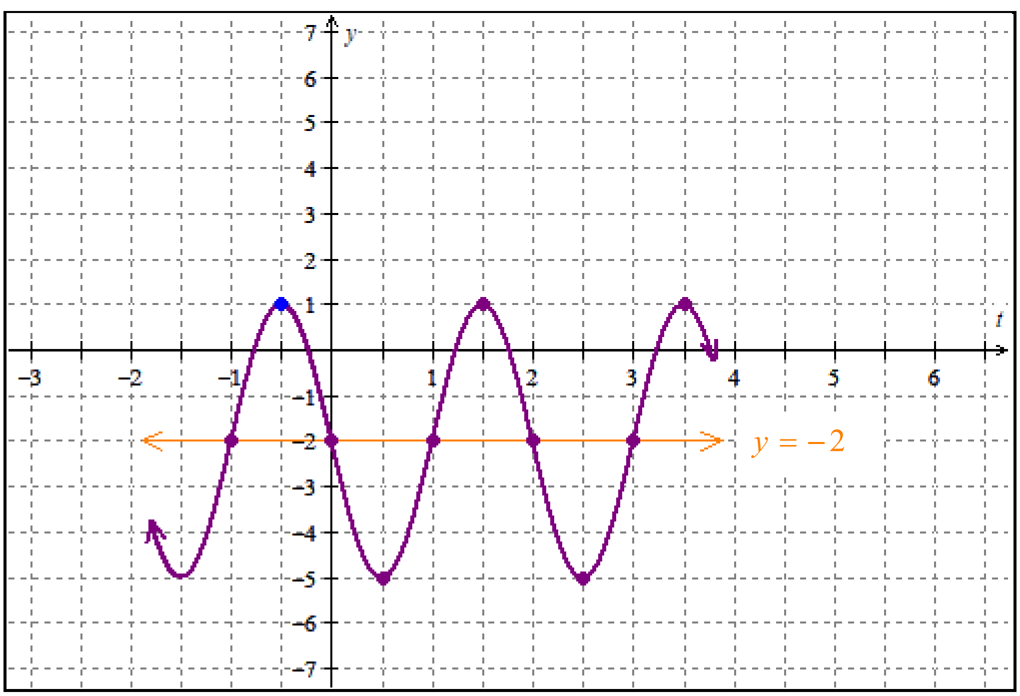
Since

$$g(t) = 3\cos\left(\pi t + \frac{\pi}{2}\right) - 2$$
$$= 3\cos\left(\pi\left(t + \frac{1}{2}\right)\right) - 2,$$

we can conclude that:

- the **amplitude** is $|A| = |-3| = 3$
- the **midline** is $y = -2$
- the **period** is $2\pi \cdot \frac{1}{\pi} = 2$

The horizontal shift is $\frac{1}{2}$ of a unit to the left, i.e., $h = -\frac{1}{2}$, so we'll "start" a cosine wave at $t = -\frac{1}{2}$ and make sure it has the appropriate midline, amplitude, and period.



A graph of $g(t) = 3\cos\left(\pi t + \frac{\pi}{2}\right) - 2$