Right Triangle Trigonometry

As we studied in “Intro to the Trigonometric Functions: Part 1,” if we put the same angle in the center of two circles of different radii, we can construct two *similar triangles*; see Figure 1.

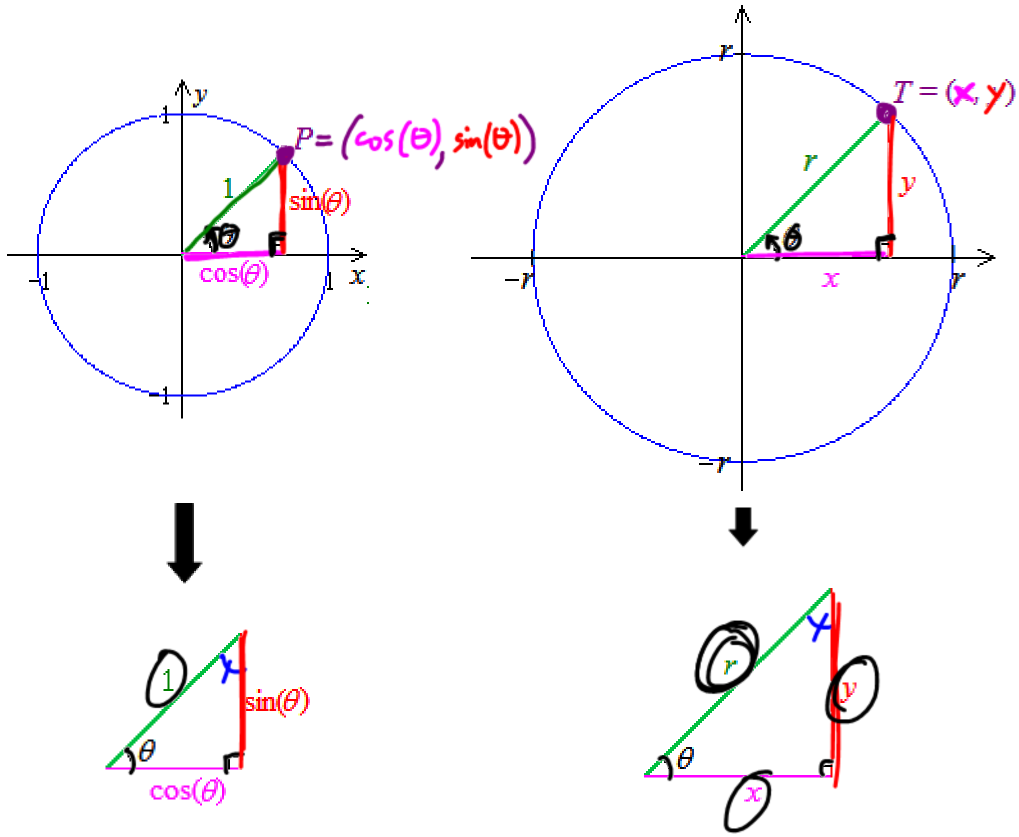

Figure 1: The angle θ in both a unit circle and in a circle of radius r, inducing similar right triangles.

We can use these similar triangles to obtain the following ratios (which we can use to derive expressions for $\sin(\theta)$ and $\cos(\theta)$):

$$\frac{\cos(\theta)}{1} = \frac{x}{r} \qquad\qquad \frac{\sin(\theta)}{1} = \frac{y}{r}$$

$$\Rightarrow \cos(\theta) = \frac{x}{r} \qquad\qquad \Rightarrow \sin(\theta) = \frac{y}{r}$$

To help remember these ratios, it’s best to imagine yourself standing at angle θ looking into the triangle. Then the side labeled y is on the **opposite** side of the triangle while the side labeled x is **adjacent** to you. We use these descriptions (as well as the fact that the side labeled r is the **hypotenuse** of the triangle) to refer to the sides of the triangles in Figure 1.

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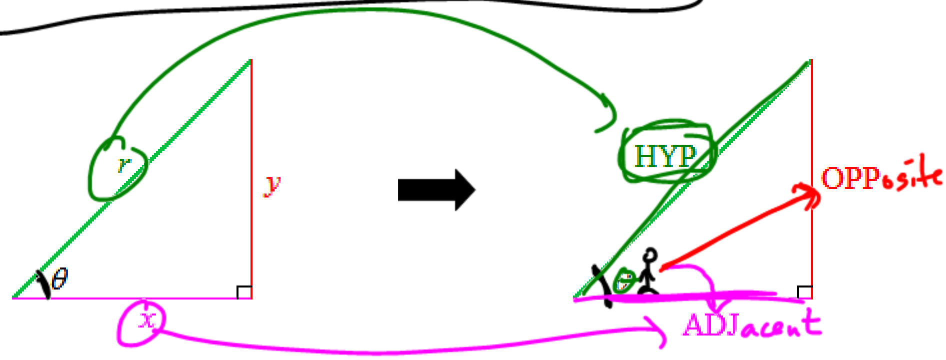

Figure 2: We use the terms **opposite** (or **OPP**), **adjacent** (or **ADJ**), and **hypotenuse** (or **HYP**) to refer to the sides of a right triangle.

DEFINITION: If θ is the angle given in the right triangles in Figure 2, then

$$\sin(\theta) = \frac{y}{r} = \frac{\text{OPP}}{\text{HYP}} \qquad \cos(\theta) = \frac{x}{r} = \frac{\text{ADJ}}{\text{HYP}} \qquad \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{\text{OPP}/\cancel{\text{HYP}}}{\text{ADJ}/\cancel{\text{HYP}}} = \frac{\text{OPP}}{\text{ADJ}}$$

Consequently, the other trigonometric functions can be defined as follows:

$$\cot(\theta) = \frac{1}{\tan(\theta)} \qquad \sec(\theta) = \frac{1}{\cos(\theta)} = \frac{\text{HYP}}{\text{ADJ}} \qquad \csc(\theta) = \frac{1}{\sin(\theta)} = \frac{\text{HYP}}{\text{OPP}}$$

EXAMPLE 1: Find value for all six trigonometric functions of the angle α given in the right triangle in Figure 3. (The triangle may not be drawn to scale.)

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$$\sin(\alpha) = \frac{9}{15} = \frac{3}{5}$$

$$\cos(\alpha) = \frac{12}{15} = \frac{4}{5}$$

$$\tan(\alpha) = \frac{9}{12} = \frac{3}{4}$$

$$\sec(\alpha) = \frac{1}{\cos(\alpha)} = \frac{1}{4/5} = 5/4$$

$$\csc(\alpha) = \frac{1}{\sin(\alpha)} = \frac{1}{3/5} = 5/3$$

$$\cot(\alpha) = \frac{1}{\tan(\alpha)} = \frac{4}{3}$$

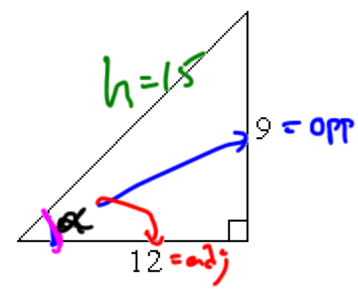

Figure 3

Use Pythagoras:

$$h^2 = 12^2 + 9^2$$

$$\Rightarrow h = \sqrt{144 + 81} = \sqrt{225} = \sqrt{9 \cdot 25} = 3 \cdot 5 = 15$$

We can use the trigonometric functions, along with the Pythagorean Theorem to "**solve a right triangle**," i.e., find the missing side-lengths and missing angle-measures for a triangle.

EXAMPLE 2: Solve the right triangle given in Figure 4 by finding A, b, and c. (The triangle might not be drawn to scale.)

Find A using 180° rule:

$$A + 60^\circ + 90^\circ = 180^\circ$$
$$A = 30^\circ$$

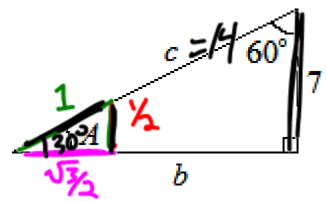

Figure 4

$$\underbrace{\sin(30^\circ)}_{} = \frac{7}{c}$$
$$\boxed{\frac{1}{2} = \frac{7}{c}}$$
$$c = 14$$

Find b using Pythagoras:

$$b^2 + 7^2 = 14^2$$
$$b^2 = 14^2 - 7^2$$
$$b = \sqrt{14^2 - 7^2}$$
$$= \sqrt{(7\cdot 2)^2 - 7^2}$$
$$= \sqrt{7^2 \cdot 2^2 - 7^2 \cdot 1}$$
$$= \sqrt{7^2 \cdot (2^2 - 1)}$$
$$= \sqrt{7^2 \cdot 3}$$
$$= 7\sqrt{3}$$

Check: $\cos(30^\circ) = \frac{7\sqrt{3}}{14} = \frac{\sqrt{3}}{2}$

EXAMPLE 3: Solve the right triangle given in Figure 5 by finding c, α, and β. (The triangle might not be drawn to scale.)

Use Pythagoras to find c:

$$c^2 = 4^2 + 8^2$$
$$c^2 = 16 + 64$$
$$c^2 = 80$$
$$c = \sqrt{80}$$
$$= \sqrt{2 \cdot 2 \cdot 2 \cdot 2 \cdot 5}$$
$$= 4\sqrt{5} > 8$$

$$\tan(\alpha) = \frac{8}{4}$$
$$\alpha = \tan^{-1}(2)$$
$$\approx 63.4^\circ$$

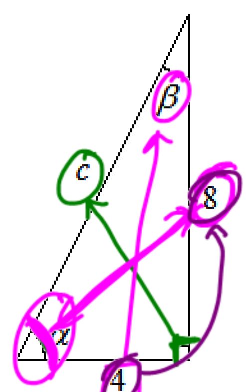

Figure 5

$$\alpha + \beta + 90^\circ = 180^\circ$$
$$63.4^\circ + \beta + 90^\circ \approx 180^\circ$$
$$\beta \approx 26.6^\circ$$