Commutative & Associative Property of *

$(a+b)+c = a+(b+c)$
$= a+(c+b)$
$= (c+b)+a$
$= a+b+c$

Proving Trigonometric Identities

This quarter we've studied many important trigonometric identities. Because these identities are so useful, it is worthwhile to learn (or memorize) many of them (e.g., the Pythagorean Identity). But there are many other identities that aren't particularly important (so they aren't worth memorizing) but they exist and provide us an opportunity to learn another skill: proving mathematical statements. Today we will *prove trig identities*.

Proving statements is a *big* part of math – in a way, it *is* math! There are a great variety of different types of statements, and a correspondingly great variety of "techniques" for proving statements. We'll only focus on proving trig identities using one technique for proving trig identities: we'll focus on a *direct* proof that relies on the ***transitive property of equality***:

$\boxed{2+3+4}$

The Transitive Property of Equality

Suppose that $a, b, c \in \mathbb{R}$. If $a = b$ and $b = c$, then $a = c$.

This property may seem obvious but it's powerful: it's the property that allows us to "relax our rules" regarding equal signs. "Equal" is, by definition, a ***bi**nary operation*, so it's defined on **two** terms, so "$\boxed{a = b}$" is defined but "$a = b = c$" isn't (initially) defined: the transitive property tells us that a "triple-equation" like this is meaningful the transitive property allows us to "streamline" the cumbersome argument in **(1)** as showing in the efficient argument in **(2)**.

(1)

$\boxed{a=b}$
& $\boxed{b=c}$
"therefore" $\therefore a=c$

(2)

$a = b$
$\quad = c$

To prove trig identities we'll use the argument-structure in **(2)**. This method works well for trig identities since they consist of two expressions that are always equals so, if we can *literally* show that the two sides of the trig identity are equal, we'll have a rock-solid proof. More importantly, the skill we're practicing (e.g., manipulating an expression "a" into a different-but-equal expression "c") is important in Calculus.

EXAMPLE 1: Prove the identity $\sin(x) = \dfrac{\tan(x)}{\sec(x)}$.

"Start with more complicated side"

$$\frac{\tan(x)}{\sec(x)} = \frac{\frac{\sin(x)}{\cos(x)}}{\frac{1}{\cos(x)}}$$

$$= \frac{\sin(x)}{\cancel{\cos(x)}} \cdot \frac{\cancel{\cos(x)}}{1}$$

$$= \frac{\sin(x)}{1}$$

$$= \sin(x)$$

Therefore, $\sin(x) = \dfrac{\tan(x)}{\sec(x)}$ is an identity

EXAMPLE 2: Prove the identity $\csc(x)\cos(x) = \cot(x)$

explore other side:

$$\csc(x)\cdot\cos(x) = \frac{1}{\sin(x)}\cdot\frac{\cos(x)}{1}$$

$$= \frac{\cos(x)}{\sin(x)}$$

$$= \cot(x)$$

$$\cot(x) = \frac{\cos(x)}{\sin(x)}$$

$$= \cos(x)\cdot\frac{1}{\sin(x)}$$

$$= \cos(x)\cdot\csc(x)$$

$$= \csc(x)\cdot\cos(x)$$

EXAMPLE 3: Prove the identity $\cot(x) + \tan(x) = \csc(x)\sec(x)$

$$\cot(x) + \tan(x) = \frac{\cos(x)}{\sin(x)} + \frac{\sin(x)}{\cos(x)}$$

$$= \frac{\cos(x)}{\sin(x)}\cdot\frac{\cos(x)}{\cos(x)} + \frac{\sin(x)}{\cos(x)}\cdot\frac{\sin(x)}{\sin(x)}$$

$$= \frac{\cos^2(x) + \sin^2(x)}{\sin(x)\cdot\cos(x)}$$

$$= \frac{1}{\sin(x)\cos(x)}$$

$$= \frac{1}{\sin(x)}\cdot\frac{1}{\cos(x)}$$

$$= \csc(x)\cdot\sec(x)$$

$$\csc(x)\sec(x) = \frac{1}{\sin(x)}\cdot\frac{1}{\cos(x)}$$

$$= \frac{1}{\sin(x)\cos(x)}$$

Conjugates: $a+b$ & $a-b$ are conjugates.

$(a+b)\cdot(a-b) = a^2 - b^2$ which is a difference of squares.

EXAMPLE 4: Prove the identity $\dfrac{1}{1-\cos(t)} + \dfrac{1}{1+\cos(t)} = 2\csc^2(t)$

$$\frac{1}{1-\cos(t)} + \frac{1}{1+\cos(t)} = \frac{1}{1-\cos(t)}\cdot\left(\frac{1+\cos(t)}{1+\cos(t)}\right) + \frac{1}{1+\cos(t)}\cdot\left(\frac{1-\cos(t)}{1-\cos(t)}\right)$$

$$= \frac{1+\cancel{\cos(t)} + 1-\cancel{\cos(t)}}{(1-\cos(t))(1+\cos(t))}$$

$$= \frac{2}{(1-\cos^2(t))}$$

$$= \frac{2}{\sin^2(t)}$$

$$= 2\cdot\frac{1}{\sin^2(t)}$$

$$= 2\csc^2(t)$$

$\sin^2(x) + \cos^2(x) = 1$

$\cos^2(x) = 1 - \sin^2(x)$

$\sin^2(x) = 1 - \cos^2(x)$

$2\csc^2(x) = 2\cdot\left(\frac{1}{\sin(x)}\right)^2$

$= 2\cdot\frac{1}{\sin^2(x)}$

$= \frac{2}{\sin^2(x)}$

EXAMPLE 5: Prove the identity $\dfrac{\cos(\theta)}{1-\sin(\theta)} = \dfrac{1+\sin(\theta)}{\cos(\theta)}$.

$$\frac{\cos(\theta)}{1-\sin(\theta)} \cdot \frac{1+\sin(\theta)}{1+\sin(\theta)}$$

$$= \frac{\cos(\theta)\cdot(1+\sin(\theta))}{1-\sin^2(\theta)}$$

$$= \frac{\cancel{\cos(\theta)}\cdot(1+\sin(\theta))}{\cos^{\cancel{2}}(\theta)}$$

$$= \boxed{\frac{1+\sin(\theta)}{\cos(\theta)}}$$

$$\frac{1+\sin(\theta)}{\cos(\theta)} = \frac{1+\sin(\theta)}{\cos(\theta)} \cdot \frac{\cos(\theta)}{\cos(\theta)}$$

$$= \frac{(1+\sin(\theta))\cdot\cos(\theta)}{\cos^2(\theta)}$$

$$= \frac{(1+\sin(\theta))\cos(\theta)}{1-\sin^2(\theta)}$$

$$= \frac{(\cancel{1+\sin(\theta)})\cos(\theta)}{(1-\sin(\theta))(\cancel{1+\sin(\theta)})}$$

$$= \frac{\cos(\theta)}{1-\sin(\theta)}$$

$$\boxed{\sin^2(\theta)+\cos^2(\theta)=1}$$

$$\cos^2(\theta) = 1-\sin^2(\theta)$$
$$\sin^2(\theta) = 1-\cos^2(\theta)$$

?

$$\cancel{(1-\sin(\theta))}\cos(\theta)\,\frac{\cos(\theta)}{1-\sin(\theta)} \overset{?}{=} \frac{1+\sin(\theta)}{\cancel{\cos(\theta)}}\cdot\cos(\theta)\cdot(1-\sin(\theta))$$

$$\cos^2(\theta) = (1+\sin(\theta))(1-\sin(\theta))$$

$$\cos^2(\theta) = 1-\sin^2(\theta)$$