This packet includes the blank Class Notes documents used for the videos for Weeks 2-10.

Intro to the Trigonometric Functions: Part 1

(these notes correspond to Parts 1 & 2 from my MTH 112 Online Lecture Notes)

As we noticed in the Ferris wheel example, a circle rotating about its center lends itself naturally to the study of periodic functions. In fact, the two most important *trigonometric functions* are defined in terms of a unit circle: the **sine** and **cosine** functions. (Note that a **unit circle** is a circle with a radius of 1 unit; see Figure 1.)

DEFINITIONS:

The **sine function**, denoted $\sin(\theta)$, associates each angle θ with the vertical coordinate (i.e., the y-coordinate) of the point P specified by the angle θ on the circumference of a unit circle.

The **cosine function**, denoted $\cos(\theta)$, associates each angle θ with the horizontal coordinate (i.e., the x-coordinate) of the point P specified by the angle θ on the circumference of a unit circle.

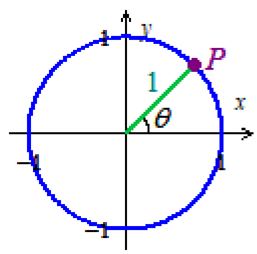


Figure 1: A Unit Circle

There are four other trigonometric functions. These four functions are defined in terms of the sine and cosine functions so first let's get familiar with sine and cosine. In order to enhance our understanding of the sine and cosine functions, we should determine some particular values for the functions and sketch their graphs. The easiest values for us to find are represented by the points where the unit circle intersects the coordinate axes. Use the graph in Figure 2 to complete Table 1. (Keep in mind that cosine represents the horizontal coordinate and sine represents the vertical coordinate.)

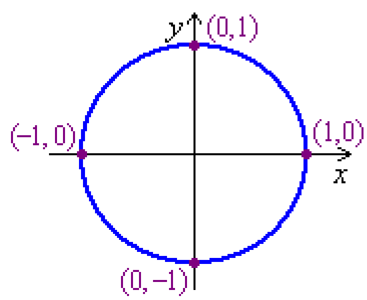
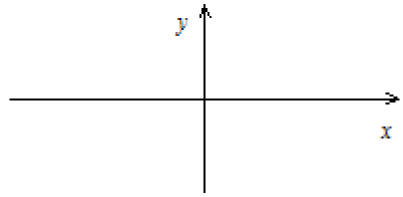


Figure 2

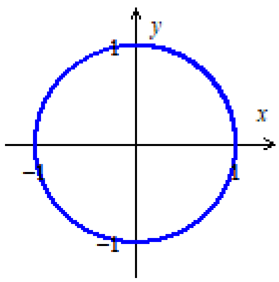
θ (degrees)	0°	90°	180°	270°	360°	450°	540°
θ (radians)							
$\cos(\theta)$							
$\sin(\theta)$							

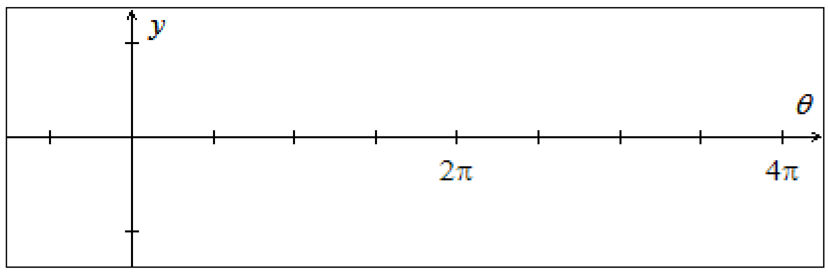
Table 1

Let's determine the signs (positive or negative) of the sine and cosine functions in the four different quadrants of the coordinate plane. Notice that our analysis of the signs of the sine and cosine functions agree with the graphs of the sine and cosine functions.

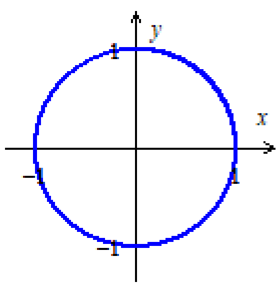


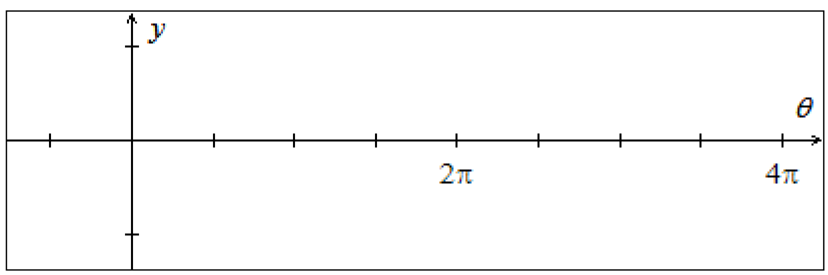
Now, let's plot on coordinate planes the info from Table 1 and sketch graphs of the sine and cosine functions. Be sure to add **scale** to the y-axis and label *all* of the "tics" on the θ-axis.





The graph of $y = \sin(\theta)$.





The graph of $y = \cos(\theta)$.

Notice that angles of measure 2π radians (i.e., $\theta = 360^\circ$) and 0 radians specify the same point on the unit circle: $(1, 0)$. Thus, the sine and cosine values for 2π radians and 0 radians are the same. In general, θ and $\theta + 2\pi$ specify the *same* point on the unit circle, so the sine and cosine values of θ and $\theta + 2\pi$ are the same. Thus, **the period of the sine and cosine functions is 2π radians**.

For all θ, $\sin(\theta) = \sin(\theta + 2\pi)$ and $\cos(\theta) = \cos(\theta + 2\pi)$.

[Let's use Desmos to confirm that we have correct graphs and to discuss what happens when we graph these functions with θ in degrees instead of radians.

Notice that the graphs of $y = \sin(\theta)$ and $y = \cos(\theta)$ are very similar. In fact, if we shift the graph of $y = \sin(\theta)$ ....

This is equation is known as an **identity** since the left and right sides of the equation are *always* identical, no matter what value of θ is used.

DEFINITION: An **identity** is an equation that is true for all values in the domains of the involved expressions.

Earlier in these notes we observed a couple of identities but didn't call them identities. The equations

$$\sin(\theta) = \sin(\theta + 2\pi) \quad \text{and} \quad \cos(\theta) = \cos(\theta + 2\pi)$$

are identities since they are true for *all* values of θ. We can use the definitions and graphs of sine and cosine to determine a few other important identities. (You should spend some time using what you learned in MTH 111 about graph transformations and symmetry to make sense of WHY these identities are true.)

SOME IMPORTANT TRIG IDENTITIES

- $\cos(\theta) = \cos(\theta + 2\pi)$
- $\sin(\theta) = \sin(\theta + 2\pi)$
- $\cos(\theta) = \sin\left(\theta + \frac{\pi}{2}\right)$
- $\sin(\theta) = \cos\left(\theta - \frac{\pi}{2}\right)$
- $\cos(-\theta) = \cos(\theta)$
- $\sin(-\theta) = -\sin(\theta)$
- $\cos(\theta) = \cos(2\pi - \theta)$
- $\sin(\theta) = \sin(\pi - \theta)$

We'll study more identities in these lecture notes, and even more over the next few weeks.

Recall the Pythagorean Theorem from your previous math course-work:

THE PYTHAGOREAN THEOREM:

If the sides of a right triangle (i.e., a triangle with a 90° angle) are labeled like the one given in Figure 3, then $a^2 + b^2 = c^2$.

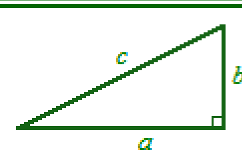


Figure 3

We can use the Pythagorean Theorem along with the definitions of sine and cosine to derive another important identity. In Figure 4, notice how the definitions of sine and cosine naturally lead us to a right triangle with side-lengths $\sin(\theta)$, $\cos(\theta)$, and 1.

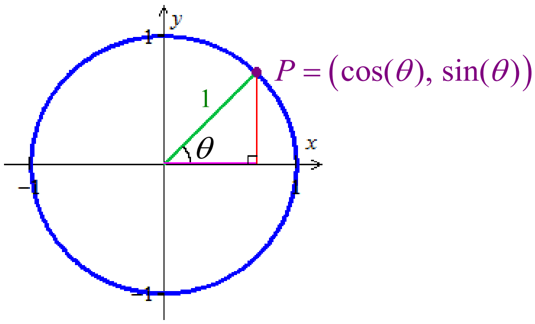


Figure 4

The Pythagorean Identity:

We can generalize the definitions of the sine and cosine functions so that they are applicable to circles of any size, rather than only for unit circles.

DEFINITION: If the point $T = (x, y)$ is specified by the angle θ on the circumference of a circle of radius r as shown in Figure 5 below, then

Notice that if $r = 1$ then this definition $\cos(\theta)$ and $\sin(\theta)$ is equivalent we saw at the beginning of this chapter:

If we solve the equations $\cos(\theta) = \dfrac{x}{r}$ and $\sin(\theta) = \dfrac{y}{r}$ for x and y, respectively, we can obtain the coordinates of a point on the circumference of a circle of any r:

If the point $T = (x, y)$ is specified by the angle θ on the circumference of a circle of radius, r, then

Let's add this information to Figure 5.

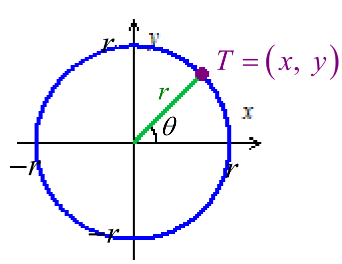


Figure 5

As mentioned at the beginning of these notes, there are four other trigonometric functions. These four functions are defined in terms of the sine and cosine functions:

DEFINITIONS: The **tangent function**, denoted $\tan(\theta)$ is defined by_________________.

The **cotangent function**, denoted $\cot(\theta)$, is defined by_______________.

Consequently:

The **secant function**, denoted $\sec(\theta)$ is defined by__________________.

The **cosecant function**, denoted $\csc(\theta)$ is defined by________________.

We can use the Pythagorean Identity to derive two identities that involve these so-called "other trig functions":

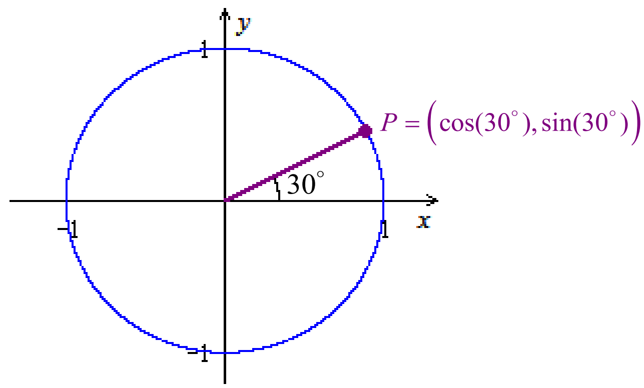
The Pythagorean Identities

EXAMPLE: If $\cos(A) = -\frac{5}{7}$ and $\pi < A < \frac{3\pi}{2}$, find $\sin(A)$, $\tan(A)$, $\cot(A)$, $\sec(A)$, and $\csc(A)$.

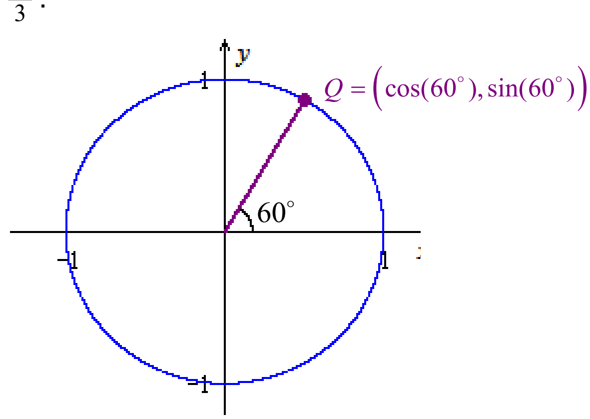
Intro to the Trigonometric Functions: Part 2

(these notes correspond to Parts 3 & 4 from my MTH 112 Online Lecture Notes)

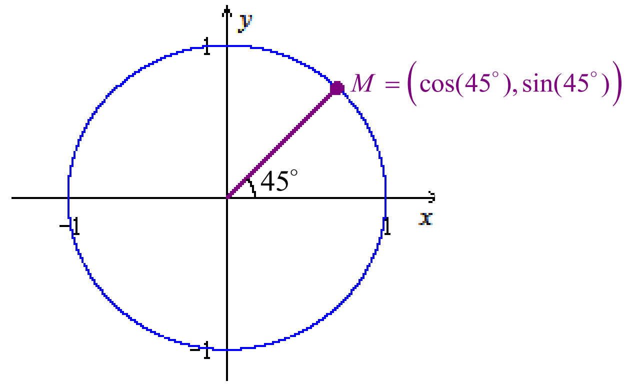
Now let's determine the sine and cosine of some important angles, namely, 30°, 45°, and 60° (i.e., $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$). Let's start with $30^\circ = \frac{\pi}{6}$.



Now let's find the sine and cosine values for $60^\circ = \frac{\pi}{3}$.



Now let's find the sine and cosine values for $45^\circ = \frac{\pi}{4}$:



Since we now know the sine and cosine values of $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$, and since sine and cosine represent, respectively, the vertical and horizontal coordinates of points on the circumference of a unit circle, we now know the coordinates of the points on the circumference of the unit circle specified by the angles $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$; let's label the coordinates on the circle in Fig. 1.

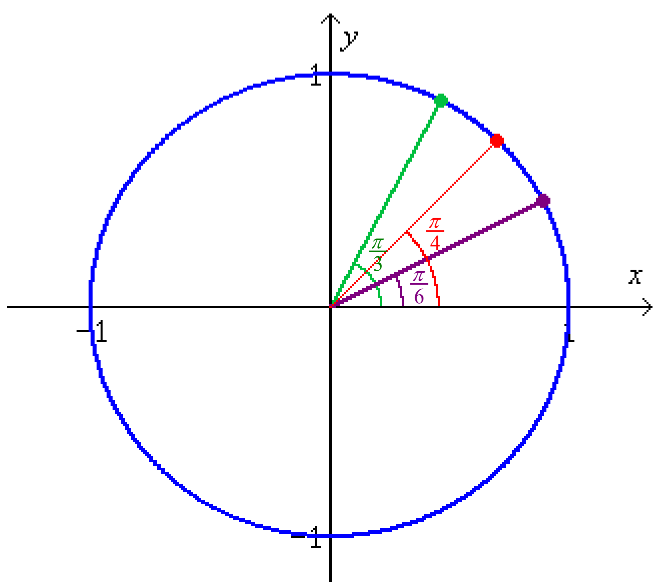


Figure 1

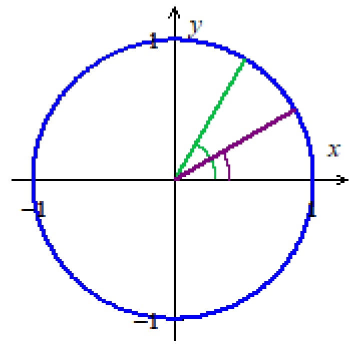
Now let's watch the [video]() in Sect. I, Chapter 3, part 4 of the online lecture notes. (18 minutes)

Finding the sine and cosine of "friendly" angles:

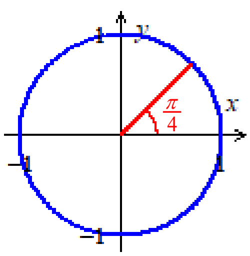
Let's summarize what we've observed about the sine and cosine of multiples of $\frac{\pi}{6}$, $\frac{\pi}{4}$ or $\frac{\pi}{3}$:

Multiples of $\frac{\pi}{4}$:

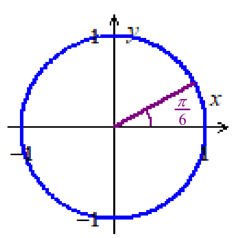
Multiples of $\frac{\pi}{6}$ and $\frac{\pi}{3}$:



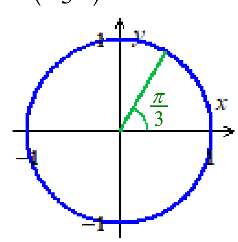
EXAMPLE: Find $\sin\left(\frac{21\pi}{4}\right)$, $\cos\left(\frac{21\pi}{4}\right)$, and $\tan\left(\frac{21\pi}{4}\right)$.



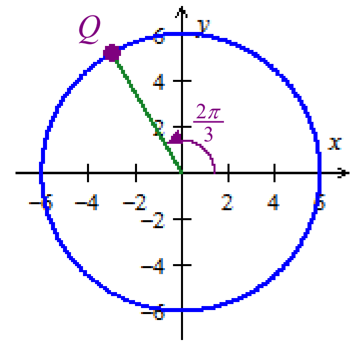
EXAMPLE: Find $\sin\left(\frac{17\pi}{6}\right)$, $\cos\left(\frac{17\pi}{6}\right)$, and $\sec\left(\frac{17\pi}{6}\right)$.



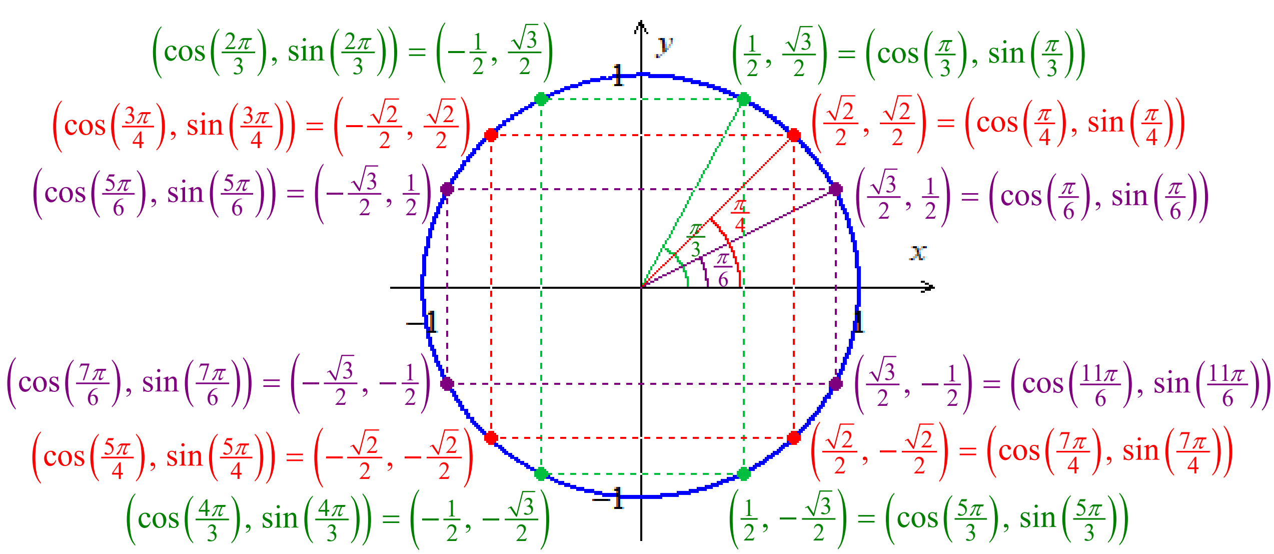
EXAMPLE: Find $\sin\left(\frac{14\pi}{3}\right)$, $\cos\left(\frac{14\pi}{3}\right)$, and $\csc\left(\frac{14\pi}{3}\right)$.



EXAMPLE: A circle with a radius of 6 units is given below. The point Q is specified by the angle $\frac{2\pi}{3}$. Use the sine and cosine function to find the exact coordinates of point Q.



Last slide from video: "Working with Multiples of $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$"



Graphing Sinusoidal Functions

Below is a summary of what is studied in MTH 111 about graph transformations; see Section I, Units 6–8 from my online notes for MTH 111 to review graph transformations.

SUMMARY OF GRAPH TRANSFORMATIONS

Suppose that f and g are functions such that $g(t) = A \cdot f\left(\omega(t-h)\right) + k$ and $A, \omega, h, k \in \mathbb{R}$. In order to transform the graph of the function f into the graph of g:

1st: horizontally stretch/compress the graph of f by a factor of $\frac{1}{|\omega|}$ and, if $\omega < 0$, reflect it about the y-axis. (Stretch if $|\omega| < 1$; compress if $|\omega| > 1$.)

2nd: shift the graph horizontally h units (shift right if $h > 0$; shift left if $h < 0$).

3rd: vertically stretch/compress the graph by a factor of $|A|$ and, if $A < 0$, reflect it about the t-axis. (Stretch if $|A| > 1$; compress if $|A| < 1$.)

4th: shift the graph vertically k units (shift up if k is positive and down if k is negative).

(The *order* in which these transformations are performed *matters*.)

When we apply these graph transformations to the graphs of $y = \sin(t)$ and $y = \cos(t)$ we obtain *sinusoidal functions*:

DEFINITION: A **sinusoidal function** is function f of the form

$$f(t) = A\sin\left(\omega(t-h)\right) + k \quad \text{or} \quad f(t) = A\cos\left(\omega(t-h)\right) + k$$

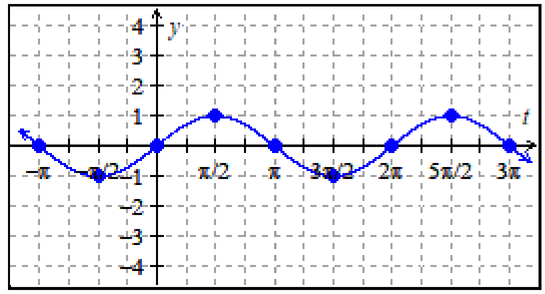
where $A, \omega, h, k \in \mathbb{R}$, $A \neq 0$, and $\omega \neq 0$.

A sinusoidal function of this form has the following properties:

-
-
-
-

(We'll use the examples below to determine the properties for the box above.)

EXAMPLE: The graph of $f(t) = \sin(t)$ is given below. Sketch a graph of $y = 3\sin(t)$.

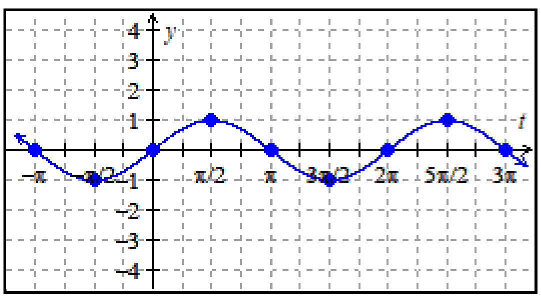


Sketch a graph of $y = 3\sin(t)$.

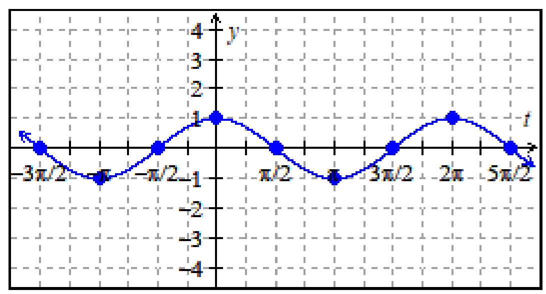
Use Desmos to graph $f(t) = \sin(t)$ & $y = A\sin(t)$ and $g(t) = \cos(t)$ & $y = A\cos(t)$ for various values of $A > 0$; then complete the following sentence:

- The graphs of $y = A\sin(t)$ and $y = A\cos(t)$ have ______________________________.

EXAMPLE: The graph of $f(t) = \sin(t)$ is given below; sketch a graph of $y = -2\sin(t)$; also the graph of $g(t) = \cos(t)$ is given below; sketch a graph of $y = -4\cos(t)$.



Sketch a graph of $y = -2\sin(t)$.

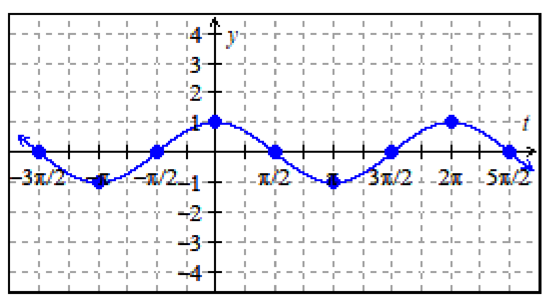


Sketch a graph of $y = -4\cos(t)$.

Observation: The reflected sine wave "starts" __________________________________

while the reflected cosine wave "starts" ______________________________________.

EXAMPLE: The graph of $g(t) = \cos(t)$ is given below. Sketch a graph of $y = \cos(t) + 2$.

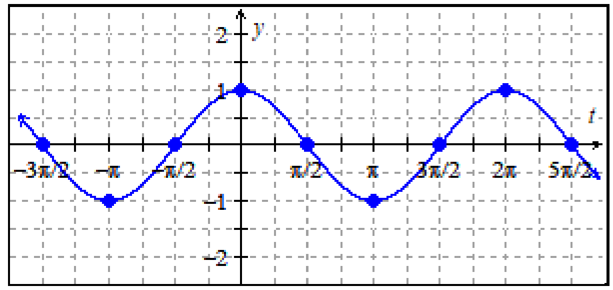


Sketch a graph of $y = \cos(t) + 2$.

Use Desmos to graph $g(t) = \cos(t)$ & $y = \cos(t) + k$ and $f(t) = \sin(t)$ & $y = \sin(t) + k$ for various values of k; then complete the following sentence:

- The graphs of $y = \cos(t) + k$ and $y = \sin(t) + k$ have _______________________.

EXAMPLE: The graph of $g(t) = \cos(t)$ is given below. Sketch a graph of $y = \cos(2t)$.

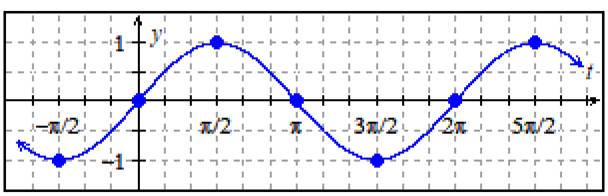


Sketch a graph of $y = \cos(2t)$.

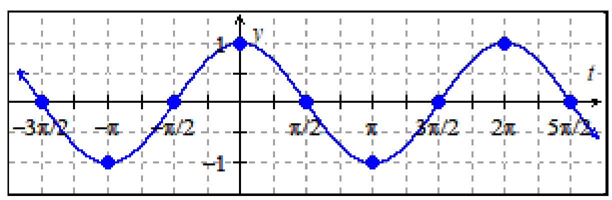
Use Desmos to graph $g(t) = \cos(t)$ & $y = \cos(\omega \cdot t)$ and $f(t) = \sin(t)$ & $y = \sin(\omega \cdot t)$ for various values of $\omega > 0$; then complete the following sentence:

- The graphs of $y = \cos(\omega \cdot t)$ and $y = \sin(\omega \cdot t)$ …

EXAMPLE: The graphs of $f(t) = \sin(t)$ and $g(t) = \cos(t)$ are given below; sketch graphs of $y = \sin\left(t - \frac{\pi}{3}\right)$ and $y = \cos\left(t + \frac{\pi}{4}\right)$.



Sketch a graph of $y = \sin\left(t - \frac{\pi}{3}\right)$.



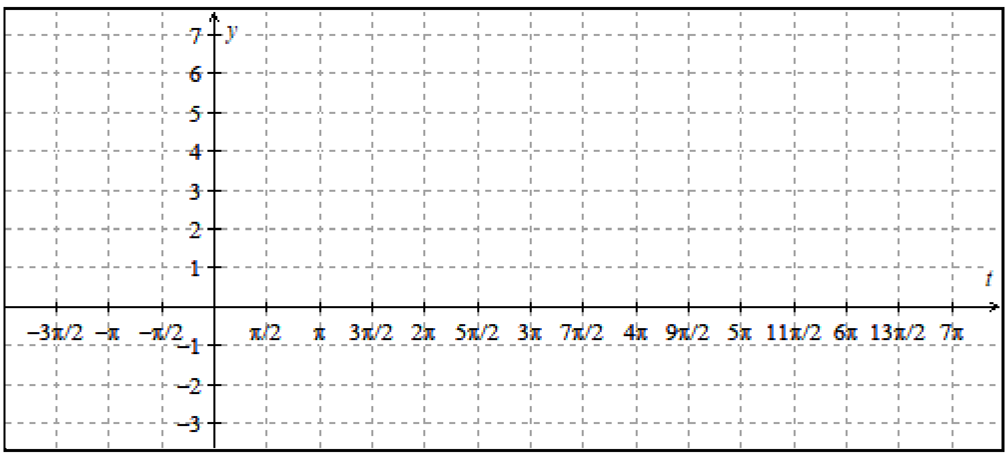
Sketch a graph of $y = \cos\left(t + \frac{\pi}{4}\right)$

EXAMPLE: Use Desmos to compare $p(t) = \cos\left(2t - \frac{\pi}{3}\right)$ and $q(t) = \cos\left(2\left(t - \frac{\pi}{3}\right)\right)$.

Determine the appropriate horizontal shift for each function.

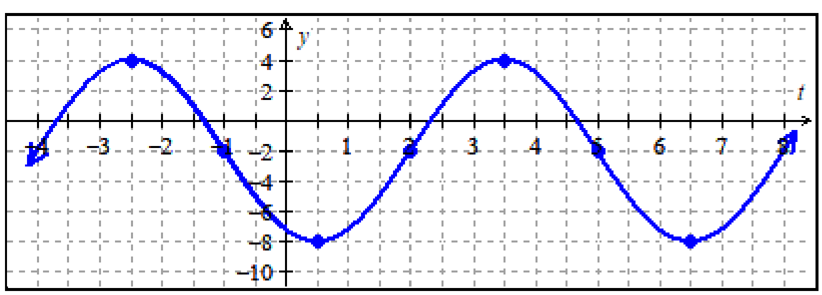
Observation: *To determine the horizontal shift, first factor the input of the trig function.*

EXAMPLE: Sketch a graph of $m(t) = 2\sin\left(\frac{1}{2}t + \frac{\pi}{4}\right) + 3$. State the period, midline, and amplitude of m.



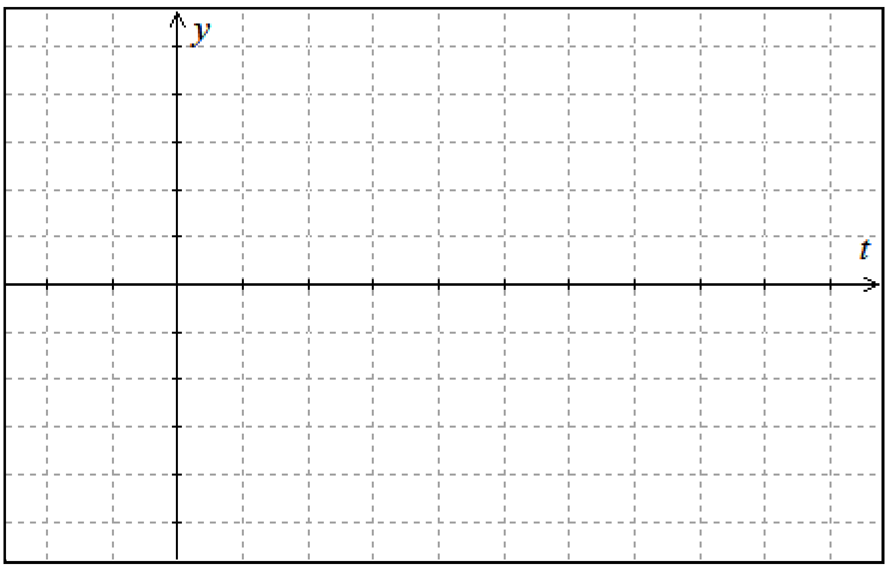
Sketch a graph of $m(t) = 2\sin\left(\frac{1}{2}t + \frac{\pi}{4}\right) + 3$.

EXAMPLE: Find (at least) two algebraic rules (i.e., "formulas"), one involving sine and one involving cosine, for the sinusoidal function n whose graph is given below.



The graph of $y = n(t)$.

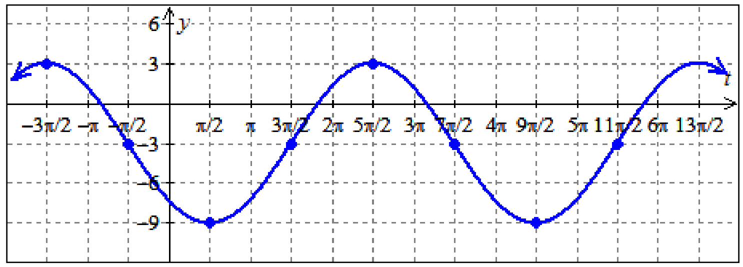
EXAMPLE: Sketch a graph of $f(t) = 2\sin\left(\pi t - \frac{\pi}{4}\right) - 3$ on the coordinate plane below.



Sketch a graph of $f(t) = 2\sin\left(\pi t - \frac{\pi}{4}\right) - 3$.

Additional Examples:

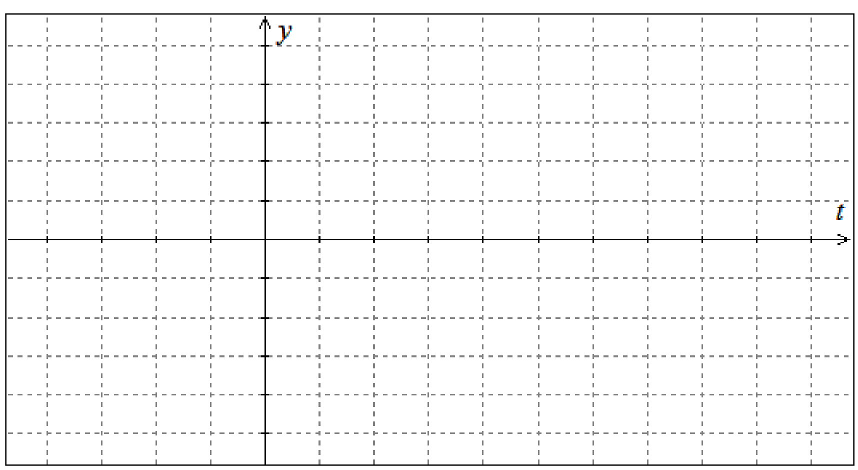
a. Find (at least) two algebraic rules (i.e., "formulas"), one involving sine and one involving cosine, for the sinusoidal function f whose graph is given below.



The graph of $y = f(t)$.

b. Sketch a graph of $g(t) = 3\cos\left(\frac{\pi}{2}t - \frac{\pi}{4}\right) - 1$ on the coordinate plane below.

List the *period*, *amplitude*, *midline*, and *horizontal shift*.



Sketch a graph of $g(t) = 3\cos\left(\frac{\pi}{2}t - \frac{\pi}{4}\right) - 1$.

Graphs of the Other Trigonometric Functions

Recall the definitions of the "other trig functions":

DEFINITIONS: The **tangent function** is defined by____________________________.

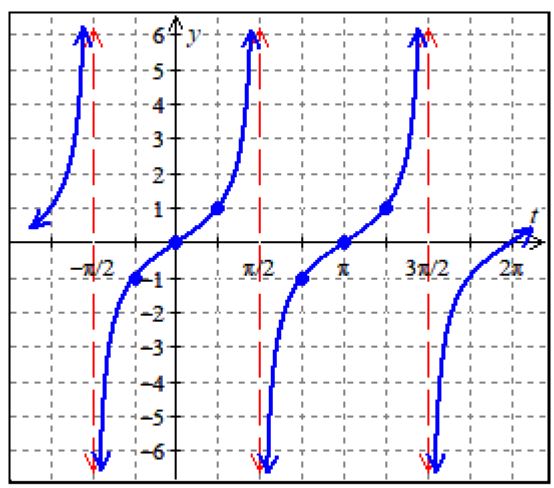
The **cotangent function** is defined by__________________________.

Consequently:

The **secant function** is defined by_____________________________.

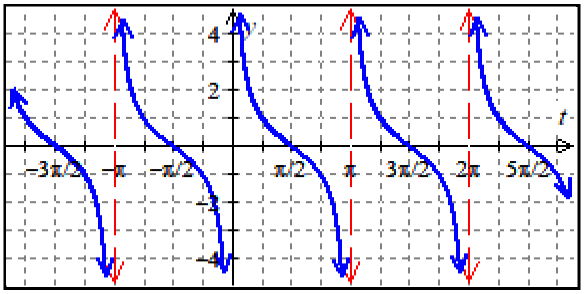
The **cosecant function** is defined by___________________________.

Below is the graph $y = \tan(t)$. Let's discuss why the graph looks like it does.



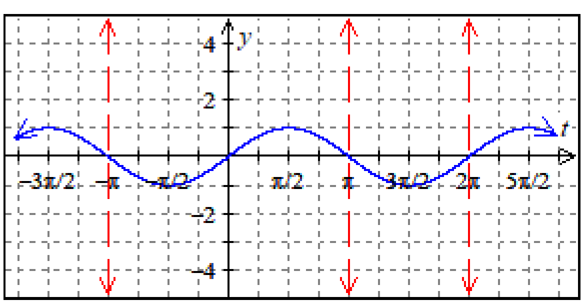
A graph $y = \tan(t)$.

Below is the graph of $y = \cot(t)$. Let's discuss why it looks like it does.

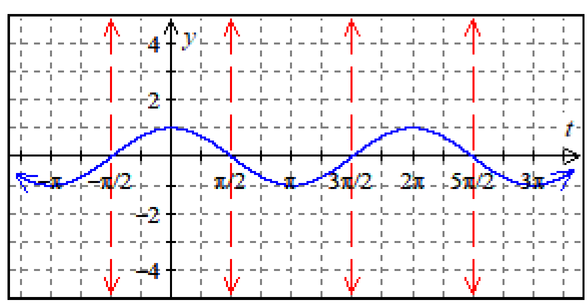


A graph of $y = \cot(t)$.

Let's sketch graphs of $y = \sec(t)$ and $y = \csc(t)$.



Draw a graph of ___________.



Draw graph of ___________.

Inverse Trig Functions

As we studied in MTH 111, the inverse of a function reverses the roles of the inputs and the outputs. (For more info on inverse functions, check out my MTH 111 Online Lecture Notes.)

Suppose that f and f^{-1} are inverses. If $f(a)=b$, then ______________.

Inverse functions are extremely valuable since they "undo" one another and allow us to solve equations. For example, we can solve the equation $x^3 = 10$ by using the inverse of the cubing function, the cube-root function, to "undo" the cubing involved in the equation:

The cubing function has an inverse function because it's **one-to-one**, which means that each output value corresponds to exactly one input value (e.g., the only number with a cube of 8 is 2) – this will allow us to reverse the roles of the inputs and outputs and still have a function. Let's use the graphs below of $y = x^2$ and $y = x^3$ to review one-to-one functions vs. not one-to-one functions.

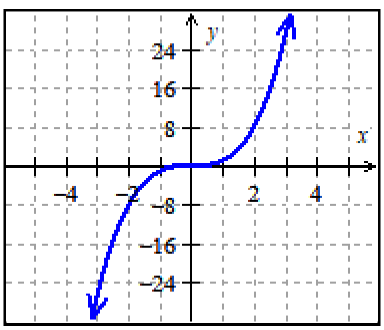


Figure 1: $y = x^3$

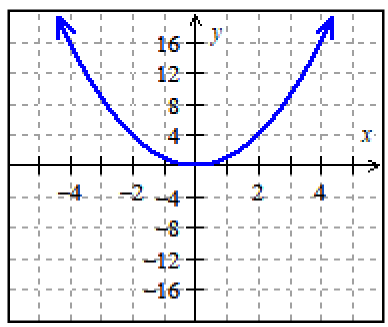


Figure 2: $y = x^2$

Unfortunately, the trig functions aren't one-to-one so, in their natural form, they don't have inverse functions. For example, consider the output $\frac{1}{2}$ for the cosine function: this output corresponds to the inputs $-\frac{\pi}{3}$, $\frac{\pi}{3}$, $\frac{5\pi}{3}$, $\frac{7\pi}{3}$, etc.; see Figure 3.

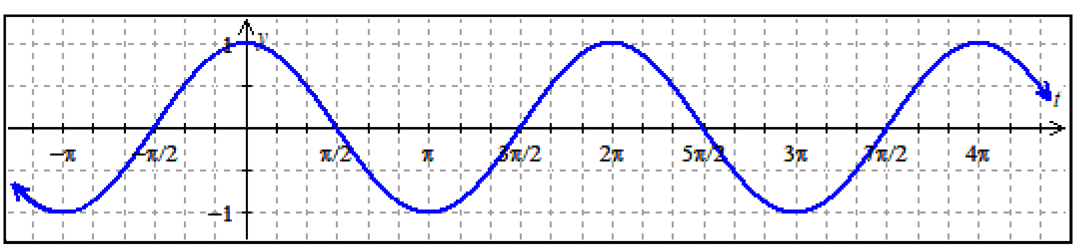
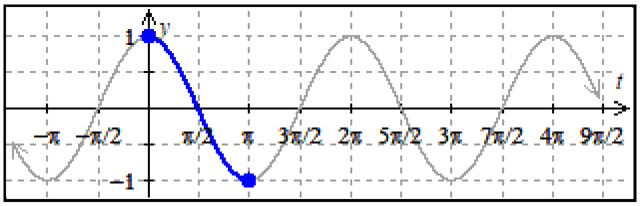
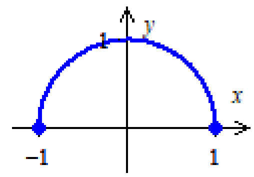


Figure 3: The graph of $y = \cos(t)$.

Since inverse functions are so valuable, we *really* want inverse trig functions, so we need to **restrict the domains** of the functions to intervals on which they are one-to-one, and then we can construct inverse functions.

Let's start by constructing the inverse of the cosine function.





A graph of $y = \cos(t)$.

DEFINITION: The **inverse cosine function**, $y = \cos^{-1}(t)$, is defined by:

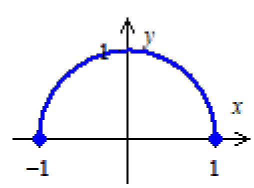
$$\text{If } 0 \le y \le \pi \text{ and } \cos(y) = t\text{, then } y = \cos^{-1}(t).$$

The *domain* of $y = \cos^{-1}(t)$ is ________________ (which is the range of the cosine function) and the *range* of $y = \cos^{-1}(t)$ is ________________.

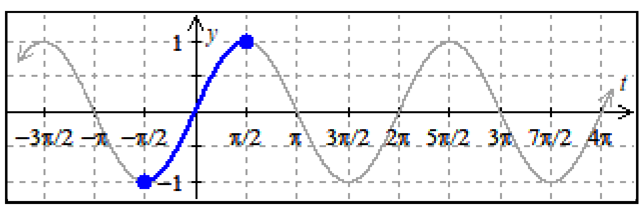
This function is often called **arccosine** and is expressed as $y = \arccos(t)$.

Key Point: Inverse Notation vs. Exponent Notation:

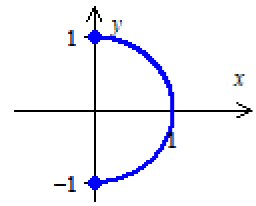
EXAMPLE: Evaluate $\cos^{-1}\left(\frac{1}{2}\right)$.



Now we'll construct the inverse of the sine function.



A graph of $y = \sin(t)$.



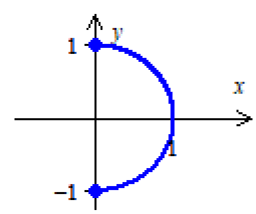
DEFINITION: The **inverse sine function**, $y = \sin^{-1}(t)$, is defined by the following:

$$\text{If } -\frac{\pi}{2} \le y \le \frac{\pi}{2} \text{ and } \sin(y) = t\text{, then } y = \sin^{-1}(t).$$

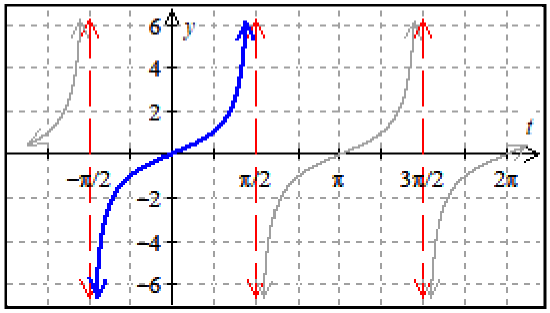
The *domain* of $y = \sin^{-1}(t)$ is ____________________ (which is the range of the sine function) and the *range* of $y = \sin^{-1}(t)$ is _____________________.

This function is often called the **arcsine** and is expressed as $y = \arcsin(t)$.

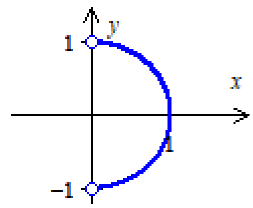
EXAMPLE: Evaluate $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$.



Now let's define the inverse tangent function.



A graph of $y = \tan(t)$.

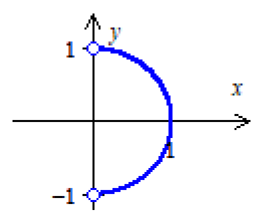


DEFINITION: The **inverse tangent function**, $y = \tan^{-1}(t)$, is defined by:

$$\text{If } -\frac{\pi}{2} < y < \frac{\pi}{2} \text{ and } \tan(y) = t \text{, then } y = \tan^{-1}(t).$$

The *domain* of $y = \tan^{-1}(t)$ is ____________________ (which is the range of the tangent function) and the *range* of $y = \tan^{-1}(t)$ is ______________. This function is often called the **arctangent** and is expressed as $y = \arctan(t)$.

EXAMPLE: Evaluate $\tan^{-1}\left(-\sqrt{3}\right)$.

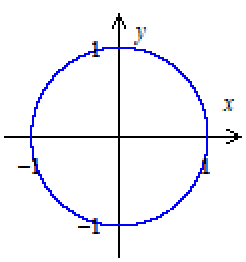


EXAMPLE: Evaluate the following expressions.

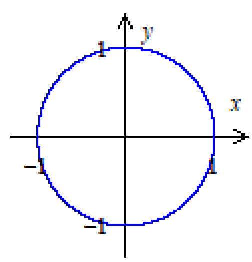
a. $\sin\left(\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)$.

b. $\sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)$.

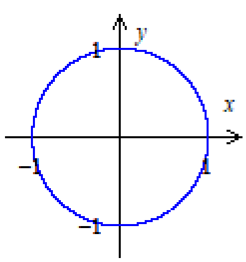
c. $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$.



d. $\cos^{-1}\left(\cos\left(\frac{7\pi}{4}\right)\right)$.

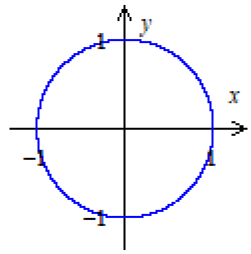


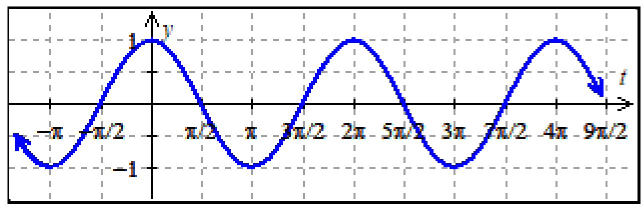
e. $\sin^{-1}\left(\sin\left(\frac{10\pi}{11}\right)\right)$.



Solving Trig Equations

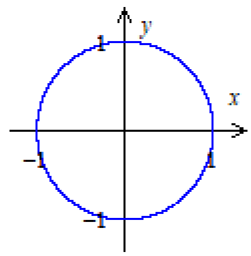
EXAMPLE: Find all of the solutions to the equation $8\cos(t)+3=-1$.

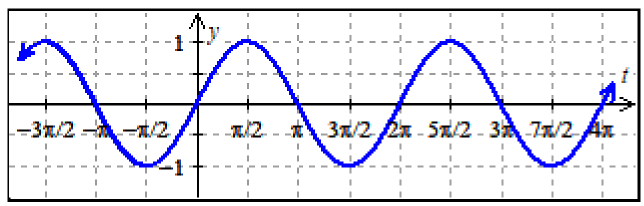




A graph of $y=\cos(t)$.

EXAMPLE: Find all of the solutions to the equation $4\sin(t)+2\sqrt{3}=0$.





A graph of $y=\sin(t)$.

EXAMPLE: First find *all* of the solutions to the equation $2\sqrt{2}\cos(3t) = -2$. Then find the particular solutions on the interval $[0,\ 2\pi)$.

Right Triangle Trigonometry

As we studied in "Intro to the Trigonometric Functions: Part 1," if we put the same angle in the center of two circles of different radii, we can construct two *similar triangles*; see Figure 1.

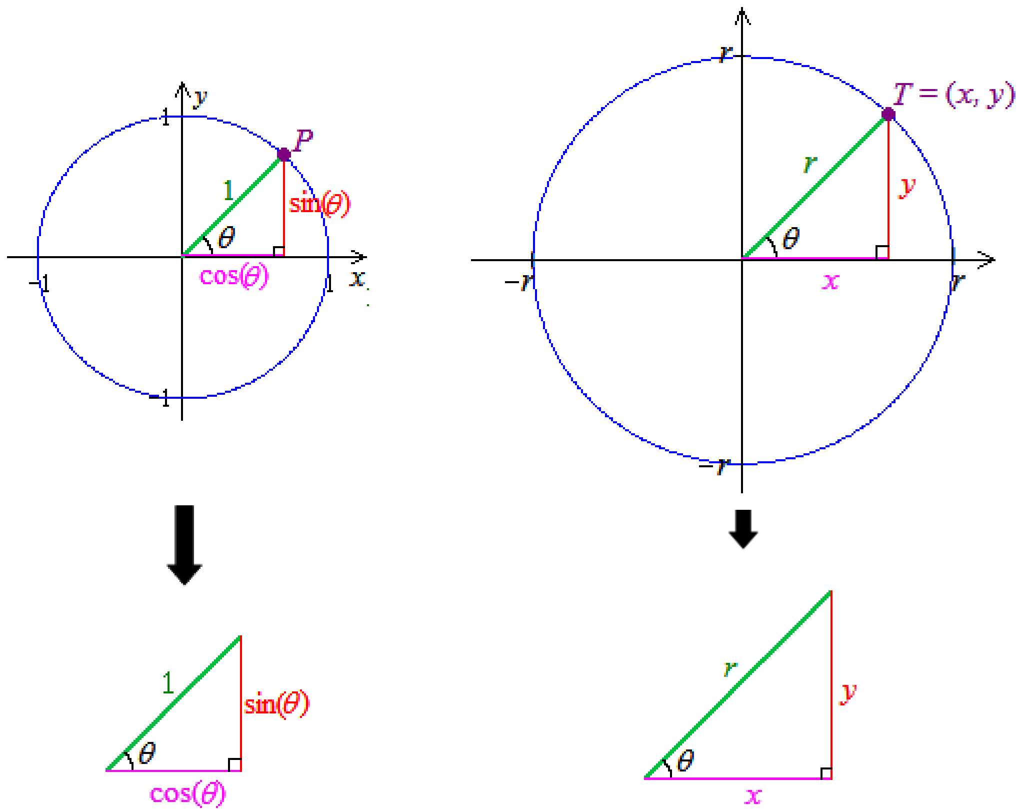


Figure 1: The angle θ in both a unit circle and in a circle of radius r, inducing similar right triangles.

We can use these similar triangles to obtain the following ratios (which we can use to derive expressions for $\sin(\theta)$ and $\cos(\theta)$):

To help remember these ratios, it's best to imagine yourself standing at angle θ looking into the triangle. Then the side labeled y is on the **opposite** side of the triangle while the side labeled x is **adjacent** to you. We use these descriptions (as well as the fact that the side labeled r is the **hypotenuse** of the triangle) to refer to the sides of the triangles in Figure 1.

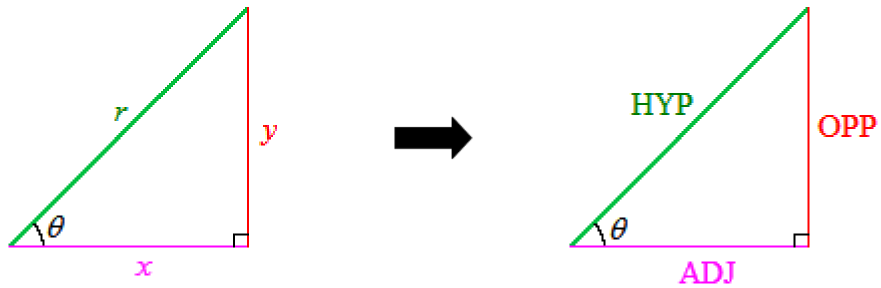


Figure 2: We use the terms **opposite** (or **OPP**), **adjacent** (or **ADJ**), and **hypotenuse** (or **HYP**) to refer to the sides of a right triangle.

DEFINITION: If θ is the angle given in the right triangles in Figure 2, then

$\sin(\theta) =$ $\cos(\theta) =$ $\tan(\theta) =$

Consequently, the other trigonometric functions can be defined as follows:

$\cot(\theta) =$ $\sec(\theta) =$ $\csc(\theta) =$

EXAMPLE 1: Find value for all six trigonometric functions of the angle α given in the right triangle in Figure 3. (The triangle may not be drawn to scale.)

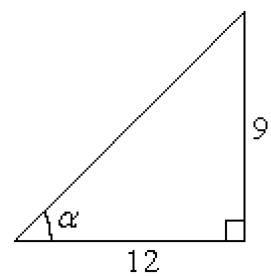


Figure 3

We can use the trigonometric functions, along with the Pythagorean Theorem to "**solve a right triangle**," i.e., find the missing side-lengths and missing angle-measures for a triangle.

EXAMPLE 2: Solve the right triangle given in Figure 4 by finding A, b, and c. (The triangle might not be drawn to scale.)

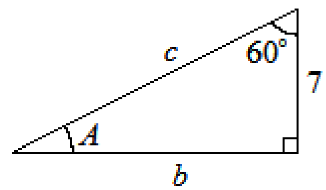


Figure 4

EXAMPLE 3: Solve the right triangle given in Figure 5 by finding c, α, and β. (The triangle might not be drawn to scale.)

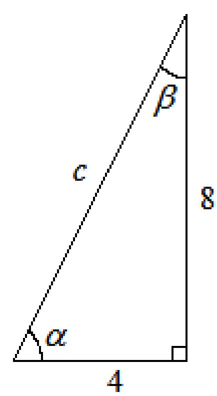


Figure 5

The Laws of Sines and Cosines

We've studied right triangle trigonometry and learned how we can use the sine and cosine functions to obtain information about right triangles. Now we'll study how we can use the sine and cosine functions to obtain information about non-right triangles, i.e., *oblique* triangles. (Figures 1 shows a non-right (oblique) triangle since none of its angles measure 90°.)

The *Laws of Sines and Cosines* are **identities** because they apply to *all* triangles (i.e., right triangles and non-right triangles) but we'll only use them when we're working with non-right triangles since we already have lots of tools for right triangles.

In these class-notes, we'll state the Laws and accept/assume that they are true so that we can use them on a few examples. For their derivations, see the videos linked from the corresponding Online Lecture Notes.

THE LAW OF SINES

If a triangle's sides and angles are labeled like the triangle in Figure 1 then…

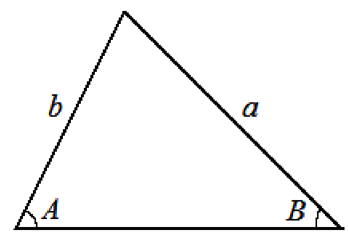


Figure 1

What does the Law of Sines imply when $A = 90^\circ$?

THE LAW OF COSINES

If a triangle's sides and angles are labeled like the triangle in Figure 2 then…

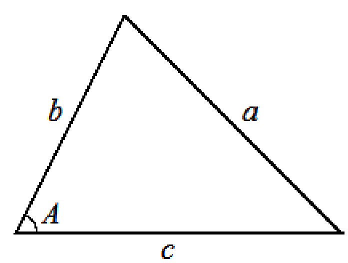
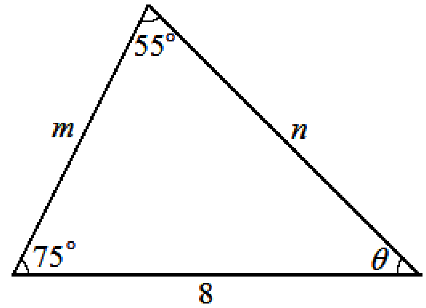


Figure 2

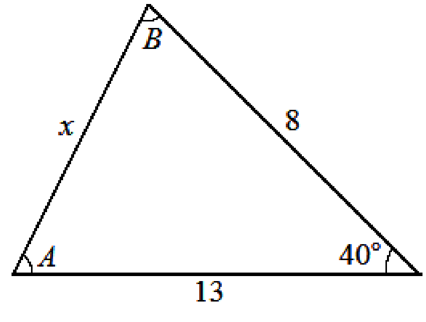
What does the Law of Cosines imply when $A = 90^\circ$?

EXAMPLE 1: Find all of the missing angles and side-lengths of the triangle below. (The triangle may not be drawn to scale.)

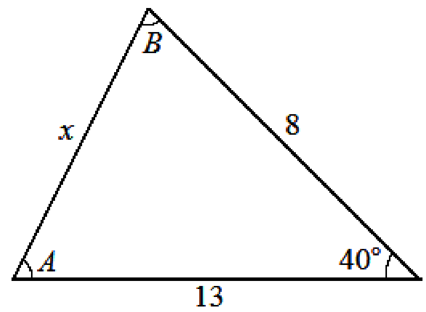


Notice that the Law of Sines involves four parts of a triangle: two angles and the two sides opposite those angles. In order to use the Law of Sines to find a missing part of a triangle, we need to know three of these four parts of a triangle.

EXAMPLE 2: Find all of the missing angles and side-lengths of the triangle below. (The triangle may not be drawn to scale.)



EXAMPLE 2 (CONT): Find all of the missing angles and side-lengths of the triangle below.



When using the Law of Sines to find an angle, always find the smaller angle first and then use the "180 degree rule" to find the larger angle.

Proving Trigonometric Identities

This quarter we've studied many important trigonometric identities. Because these identities are so useful, it is worthwhile to learn (or memorize) many of them (e.g., the Pythagorean Identity). But there are many other identities that aren't particularly important (so they aren't worth memorizing) but they exist and provide us an opportunity to learn another skill: proving mathematical statements. Today we will *prove trig identities*.

Proving statements is a *big* part of math – in a way, it *is* math! There are a great variety of different types of statements, and a correspondingly great variety of "techniques" for proving statements. We'll only focus on proving trig identities using one technique for proving trig identities: we'll focus on a *direct* proof that relies on the ***transitive property of equality***:

> **The Transitive Property of Equality**
>
> Suppose that $a, b, c \in \mathbb{R}$. If $a = b$ and $b = c$, then $a = c$.

This property may seem obvious but it's powerful: it's the property that allows us to "relax our rules" regarding equal signs. "Equal" is, by definition, a ***bi**nary operation*, so it's defined on **two** terms, so "$a = b$" is defined but "$a = b = c$" isn't (initially) defined: the transitive property tells us that a "triple-equation" like this is meaningful" the transitive property allows us to "streamline" the cumbersome argument in **(1)** as showing in the efficient argument in **(2)**.

(1)

(2)

To prove trig identities we'll use the argument-structure in **(2)**. This method works well for trig identities since they consist of two expressions that are always equals so, if we can *literally* show that the two sides of the trig identity are equal, we'll have a rock-solid proof. More importantly, the skill we're practicing (e.g., manipulating an expression "a" into a different-but-equal expression "c") is important in Calculus.

EXAMPLE 1: Prove the identity $\sin(x) = \dfrac{\tan(x)}{\sec(x)}$.

EXAMPLE 2: Prove the identity $\csc(x)\cos(x) = \cot(x)$.

EXAMPLE 3: Prove the identity $\cot(x) + \tan(x) = \csc(x)\sec(x)$.

EXAMPLE 4: Prove the identity $\dfrac{1}{1-\cos(t)}+\dfrac{1}{1+\cos(t)}=2\csc^2(t)$

EXAMPLE 5: Prove the identity $\dfrac{\cos(\theta)}{1-\sin(\theta)} = \dfrac{1+\sin(\theta)}{\cos(\theta)}$.

Other Important Identities

First let's look at identities involving expressions of the form $\sin(A \pm B)$ and $\cos(A \pm B)$. These identities allow us to calculate the sine and cosine of the sum and difference of two angles if we know the sine and cosine of the angles. (There are corresponding identities for tangent but we can use the sine and cosine identities along with the definition of tangent rather than studying another identity for tangent.) We can use these identities to find the trig values of all multiples of $15^\circ = \frac{\pi}{12}$.

THE SUM AND DIFFERENCE IDENTITIES

sine:

$$\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$
$$\sin(A - B) = \sin(A)\cos(B) - \cos(A)\sin(B)$$

cosine:

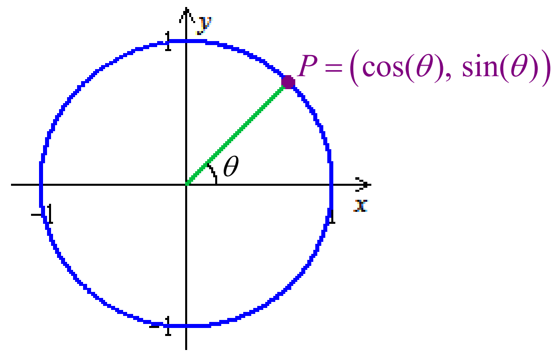
$$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$
$$\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)$$

EXAMPLE 1: Use an appropriate identity to calculate $\cos\left(15^\circ\right)$.

EXAMPLE 2: Use an appropriate identity to calculate $\sin\left(\frac{13\pi}{12}\right)$.

Now we'll familiarize ourselves with the *double-angle identities* and the *half-angle identities*. These identities allow us to find $\sin(2\theta)$ & $\cos(2\theta)$ and $\sin\left(\frac{\theta}{2}\right)$ & $\cos\left(\frac{\theta}{2}\right)$ if we know the values of $\cos(\theta)$ and $\sin(\theta)$.

Let's start by deriving the **double-angle identity for sine**; then we'll derive the **double-angle identity for cosine**.



DOUBLE-ANGLE IDENTITIES

sine :

cosine :

EXAMPLE 1: If $\cos(A) = \frac{1}{3}$, where $\frac{3\pi}{2} < A < 2\pi$, find $\cos(2A)$, $\sin(2A)$, and $\tan(2A)$.

We can use the double-angle identities for cosine to derive **half-angle identities**.

Recall this double-angle identity for cosine: $\cos(2\theta) = 1 - 2\sin^2(\theta)$. We can use this identity to find a half-angle identity for sine:

We can use $\cos(2\theta) = 2\cos^2(\theta) - 1$ to find a half-angle identity for cosine:

HALF-ANGLE IDENTITIES

sine :

cosine :

When using the half-angle identities, you need to decide which sign to use by determining which quadrant $\frac{A}{2}$ falls in.

EXAMPLE 2: If $\cos(A) = \frac{1}{3}$, where $\frac{3\pi}{2} < A < 2\pi$. Find $\cos\left(\frac{A}{2}\right)$ and $\sin\left(\frac{A}{2}\right)$.

EXAMPLE 3: Use a half-angle identity to find $\cos\left(15^\circ\right)$.

Introduction to Polar Coordinates

We are all comfortable using rectangular (i.e., Cartesian) coordinates to describe points on the plane. In Figure 1 let's plot the point $P = \left(\sqrt{3}, 1\right)$ on the rectangular coordinate plane:

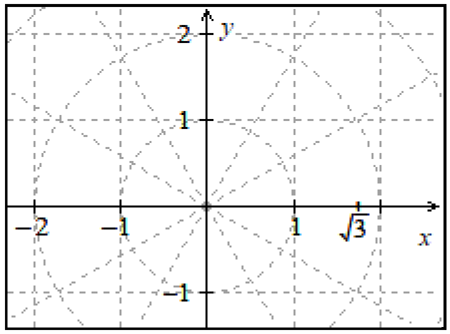
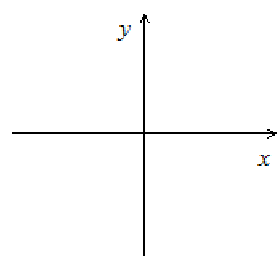


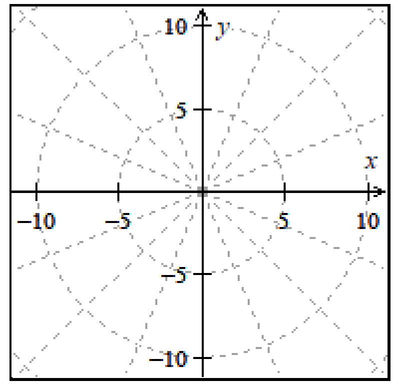
Figure 1

Instead of using these *rectangular* coordinates, we can use a *circular* coordinate system to describe points on the plane, i.e., we can use the **polar coordinate system**. Ordered pairs in polar coordinates have form (r, θ) where r represents the point's distance from the origin and θ represents the angular displacement of the point with respect to the positive x-axis. Let's find the polar coordinates that describe $P = \left(\sqrt{3}, 1\right)$.

The rectangular coordinates (x, y) are equivalent to the (r, θ) polar coordinates such that:

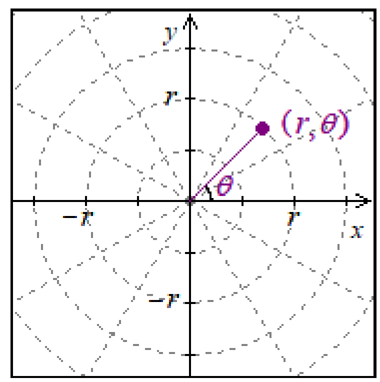


EXAMPLE: Plot the point $A = \left(10, \frac{5\pi}{4}\right)$ on the polar coordinate plane below and determine the rectangular coordinates of point A.



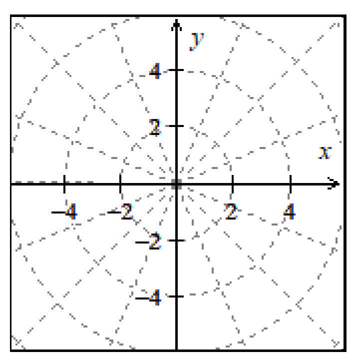
Plot $A = \left(10, \frac{5\pi}{4}\right)$.

The polar coordinates (r, θ) are equivalent to the following rectangular coordinates:



What happens if $r < 0$?

EXAMPLE: Plot the point $B = \left(-4, \frac{2\pi}{3}\right)$ on the polar coordinate plane below and list a few other ordered pairs that are plotted at the same location.

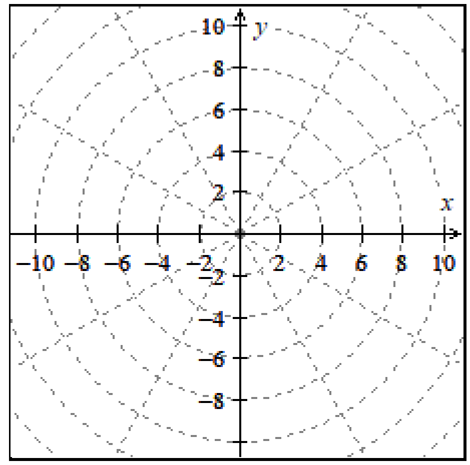


Plot $B = \left(-4, \frac{2\pi}{3}\right)$.

Graphing Polar Functions

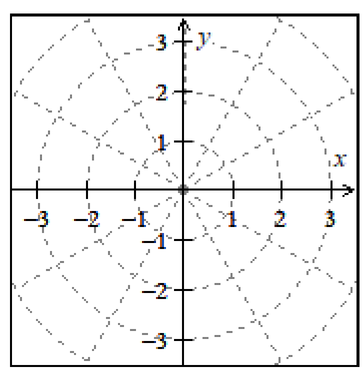
Now let's graph polar functions, i.e., functions that involve polar coordinates. These functions will have form $r = f(\theta)$, so the input, θ, is an angle and the output, r, is the distance from the origin. **Notice that in a polar ordered pair, (r, θ), the *output* variable, *r*, is the first coordinate and the *input* variable, θ, is the second coordinate** which is a **different order** than rectangular ordered pairs of the form (x, y) in which the *input* variable is the first coordinate and the *output* variable is the second coordinate.

EXAMPLE: Sketch a graph of the polar function $r = \theta$.



Sketch a graph $r = \theta$.

EXAMPLE: Sketch a graph of the polar function $r = 3$.

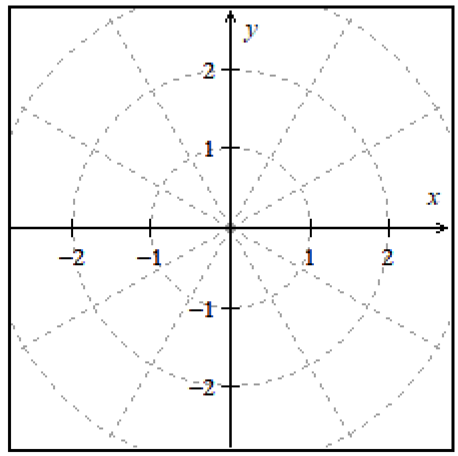


Sketch a graph $r = 3$.

EXAMPLE: Sketch a graph of the $r = 2\sin(2\theta)$ on the polar coordinate plane.

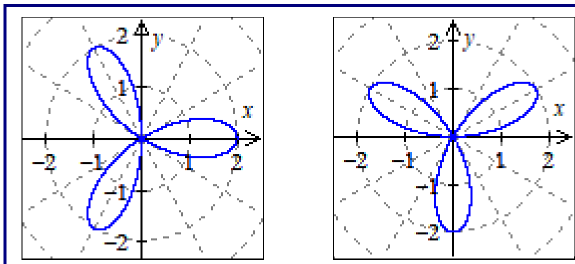
To sketch the graph of $r = 2\sin(2\theta)$, let's find some ordered pairs that satisfy the function.

θ	$r = 2\sin(2\theta)$	$(r,\ \theta)$
0		
$\frac{\pi}{12}$		
$\frac{\pi}{6}$		
$\frac{\pi}{4}$		
$\frac{\pi}{3}$		
$\frac{\pi}{2}$		
$\frac{2\pi}{3}$	$-\sqrt{3} \approx -1.7$	
$\frac{3\pi}{4}$	-2	
$\frac{5\pi}{6}$	$-\sqrt{3} \approx -1.7$	
π	0	
$\frac{7\pi}{6}$	$\sqrt{3} \approx 1.7$	
$\frac{5\pi}{4}$	2	
$\frac{4\pi}{3}$	$\sqrt{3} \approx 1.7$	
$\frac{3\pi}{2}$	0	
$\frac{5\pi}{3}$	$-\sqrt{3} \approx -1.7$	
$\frac{7\pi}{4}$	-2	
$\frac{11\pi}{6}$	$-\sqrt{3} \approx -1.7$	

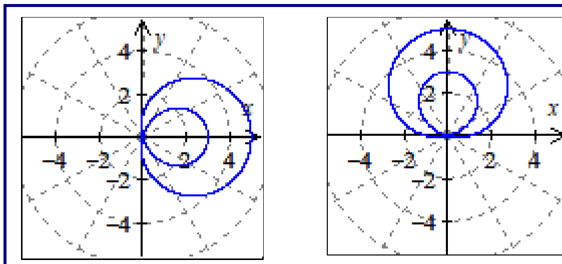


Sketch a graph $r = 2\sin(2\theta)$.

Below are the graphs of a few other functions defined via polar coordinates.


$r = 2\cos(3\theta)$ $r = 2\sin(3\theta)$

These graphs are called "**roses**".


$r = 1 + 4\cos(\theta)$ $r = 1 + 4\sin(\theta)$

These graphs are called "**limaçons**".

EXAMPLE: Convert the rectangular equation $y = 4x - 3$ into an equivalent equation in polar coordinates.

EXAMPLE: Convert the polar equation $r = 3\sin(\theta)$ into an equivalent equation in rectangular coordinates.

Complex Numbers and Polar Coordinates

Recall from Section I: Chapter 0 the definition of the set of complex numbers:

$$\mathbb{C} = \left\{ x \,\middle|\, x = a + bi \text{ and } a,b \in \mathbb{R} \text{ and } i = \sqrt{-1} \right\}.$$

You've probably seen complex numbers in a course like MTH 95 or Algebra 2 when solving quadratic equations like $x^2 - 4x + 13 = 0$:

For a complex number of the form $a + bi$, its *real part* is a and its *imaginary part* is b.

Because a complex number has *two* parts, we can use the *two dimensional* rectangular coordinate plane to plot complex numbers. We use the horizontal axis to represent the real part of the number and the vertical axis to represent the complex part of the number. Thus, the complex number $a + bi$ can be represented by the point (a, b) on the rectangular coordinate plane; see Figure 1.

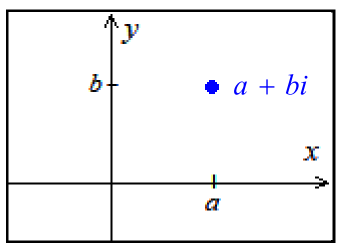


Figure 1

EXAMPLE: Plot the following complex numbers on the coordinate plane in Figure 2.

a. $s = 2 + 5i$

b. $t = \frac{3}{2} - 3i$

c. $u = 3i$

d. $v = -4$

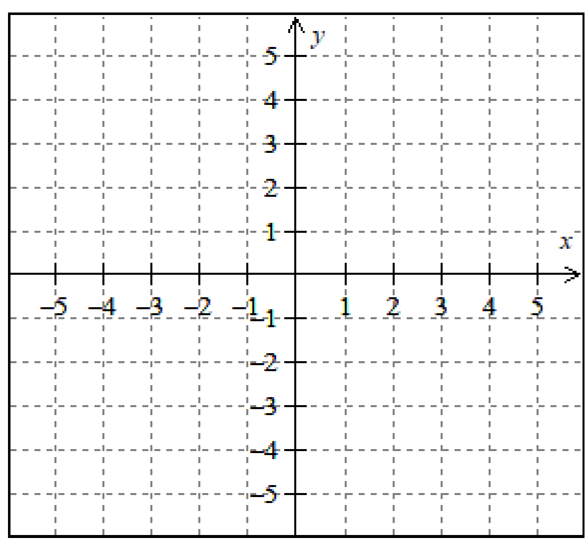


Figure 2

As we've studied, the rectangular ordered pair (a, b) can be represented in polar coordinates (r, θ) where r represents the distance the point is from the origin and θ represents the angle between the positive x-axis and the segment connecting the origin and the point; see Figure 3. Let's recall how we can find r and θ when we know a and b:

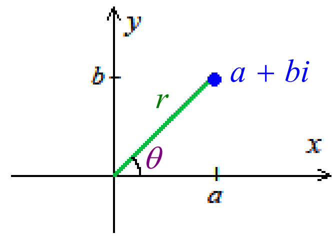


Figure 3

Let's also recall how we can find a and b when we know r and θ:

Using this information, we can write the complex number $a + bi$ in terms of r and θ:

Now we can establish a surprising connection between exponential function $y = e^x$ and complex numbers: Euler's Formula. (If you continue studying mathematics and take a calculus sequence, you have an opportunity to see why this equation is true but, for now, you need to just accept it and learn to work with it.)

EULER'S FORMULA

$$e^{i\theta} = \cos(\theta) + \sin(\theta) \cdot i$$

By multiplying both sides of Euler's formula by r, we obtain the following formula that allows us to write any complex number in **polar form**.

The **polar form** of the complex number $z = a + bi$ is $z = re^{i\theta}$:

EXAMPLE: Express the complex number $z = 6e^{i \cdot \frac{5\pi}{6}}$ in rectangular form $z = a + bi$.

EXAMPLE: Express the complex number $z = 3 - 3i$ in polar form $z = re^{i\theta}$.

Introduction to Vectors

Vectors are mathematical objects used to represent physical quantities like velocity, force, and displacement. Unlike ordinary numbers (or **scalars**), vectors have *both* magnitude and direction. So, for example, we can use a vector to describe the velocity of an object (i.e., the speed *and* direction).

DEFINITION: A **vector** is a mathematical object that has both a *magnitude* (i.e., size) and a *direction*.

In order to distinguish between vectors from scalars (i.e., numbers) we need to use a different notation to denote vectors. In this class, we will use a small arrow above the vector name to denote a vector, so that $\vec{v}$ and $\vec{s}$ represent vectors while v and s represent scalars.

In this class we will focus on **two-dimensional vectors**. A two-dimensional vector can be represented by an **arrow** on the coordinate plane. The **length** of the arrow represents the **magnitude** of the vector and the **direction** of the arrow represents the direction of the vector. (We traditionally use the **angle between the positive x-axis and the arrow** to describe the **direction** of the vector.)

EXAMPLE 1: The vector $\vec{v}$ is depicted as an arrow on the coordinate plane in Figure 1.

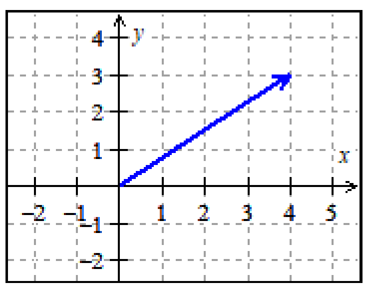


Figure 1: Arrow representing vector $\vec{v}$.

The **tip** of the vector is where the arrow ends and the **tail** of the vector is where the arrow begins. Thus, the tip of $\vec{v}$ is at the point $(4, 3)$ and the tail of the vector is at the origin, $(0, 0)$.

As mentioned above, the **length** of the arrow represents the **magnitude** of the vector. We denote the magnitude of vector $\vec{v}$ by $\|\vec{v}\|$. To find the magnitude of $\vec{v}$, we need to find the length of the arrow; we can do this by thinking of the arrow as being the hypotenuse of a right-triangle with side lengths 4 and 3 and then use the Pythagorean Theorem to find $\|\vec{v}\|$:

We can find the angle between the positive x-axis and the arrow to describe the **direction** of the vector. We've denoted this angle by θ in Figure 2.

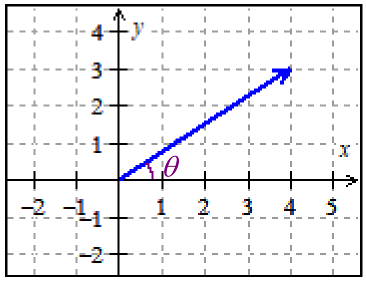


Figure 2

We can use the trigonometry that we studied earlier this quarter to find θ:

Although the magnitude and direction of the vector describe it completely, it is often useful to describe a vector by using its **horizontal and vertical components**. The *horizontal component* of $\vec{v}$ in Figure 2 is 4 units and a *vertical component* of vector $\vec{v}$ is 3 units. Thus, we say that the **component form of vector** $\vec{v}$ is $\langle 4, 3 \rangle$.

It's important to recognize that we could translate this vector anywhere in the coordinate plane and it would still be the same vector. For example, all of the arrows in Figure 3 represent $\vec{v}$ since all of these vectors have a horizontal component of 4 units and a vertical component of 3 units.

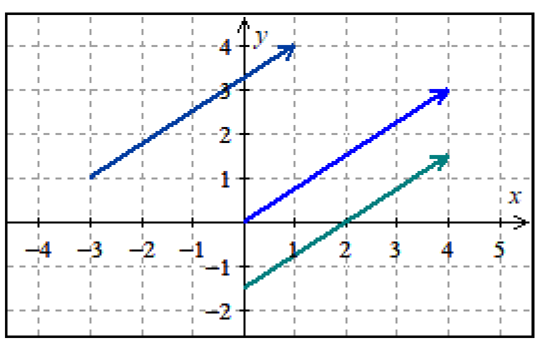


Figure 3: Three copies of $\vec{v}$.

Vector Operations

We can **multiply any vector by a scalar** (i.e., a number) and we can **add or subtract any two vectors**.

When we **multiply a vector by a scalar**, we simply multiply the respective components of the vector by the scalar. Thus, if $\vec{a} = \langle a_1, a_2 \rangle$ and $k \in \mathbb{R}$, then $k\vec{a} = \langle ka_1, ka_2 \rangle$.

EXAMPLE 2: Let $\vec{v} = \langle 4, 3 \rangle$ (from Example 1). Find and draw vectors $\vec{m} = 2\vec{v}$ and $\vec{n} = -2\vec{v}$.

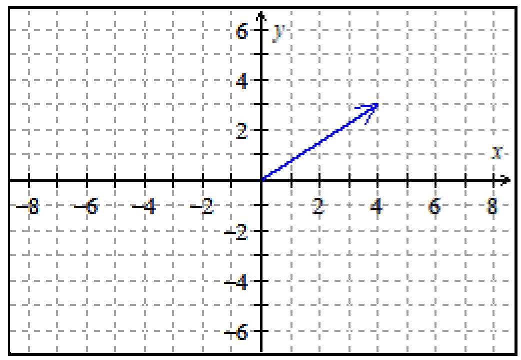


Figure 4: Vector $\vec{v}$.

If $\vec{a} = \langle a_1, a_2 \rangle$ is a vector and $k \in \mathbb{R}$ then $k\vec{a} = \langle ka_1, ka_2 \rangle$ has magnitude $|k| \cdot \|\vec{a}\|$. If $k > 0$ then $k \cdot \vec{a}$ points in the same direction as $\vec{a}$; if $k < 0$ then $k \cdot \vec{a}$ points in the opposite direction as $\vec{a}$.

EXAMPLE 3: Suppose that the vector $\vec{m}$ makes an angle of 37° with respect to the positive x-axis and $\|\vec{m}\| = 20$. Represent $\vec{m}$ in component form.

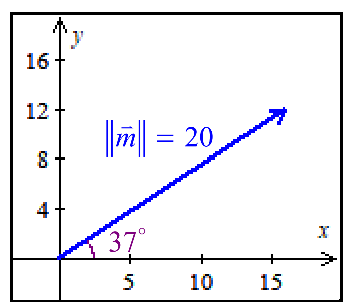


Figure 5: Vector $\vec{m}$

In general, if vector $\vec{v}$ makes an angle θ with the positive x-axis then, in component form, When we **add**, we simply add the respective components of the vectors. Thus, if $\vec{a} = \langle a_1, a_2 \rangle$ and $\vec{b} = \langle b_1, b_2 \rangle$, then $\vec{a} + \vec{b} = \langle a_1 + b_1, a_2 + b_2 \rangle$ and $\vec{a} - \vec{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$

EXAMPLE 4: Let $\vec{v} = \langle 4, 3 \rangle$ (from Example 1) and $\vec{s} = \langle 2, -6 \rangle$. Find $\vec{v} + \vec{s}$.

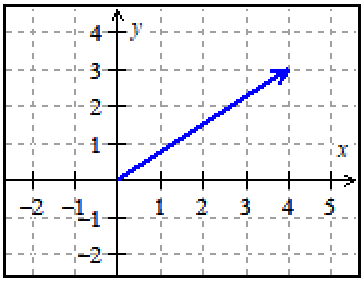


Figure 6: Vector $\vec{v}$.

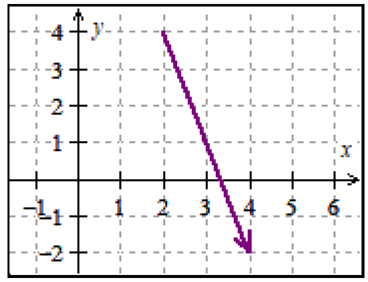


Figure 7: Vector $\vec{s}$.

Let's find $\vec{v} + \vec{s}$:

We can also add vectors by using arrows on a coordinate plane:

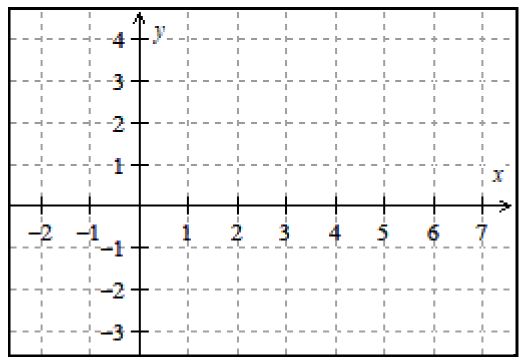


Figure 8: Adding vectors graphically.

Properties of Vector Addition and Scalar Multiplication

If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors and a and b are scalars (i.e., $a, b \in \mathbb{R}$) then the following properties hold true:

1. **Commutativity of Vector Addition:** $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

2. **Associativity of Vector Addition:** $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

3. **Associativity of Scalar Multiplication:** $a(b\vec{v}) = (ab)\vec{v}$

4. **Distributivity:** $(a + b)\vec{v} = a\vec{v} + b\vec{v}$ and $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$

5. **Identities:** $\vec{v} + \vec{0} = \vec{v}$ and $1 \cdot \vec{v} = \vec{v}$

In order to facilitate the communication and manipulation of vectors, it is useful to consider **unit vectors**.

DEFINITION: A **unit vector** is a vector whose magnitude is 1 unit. So if $\vec{a}$ is a unit vector then $\|\vec{a}\| = 1$.

The **standard unit vectors** are the unit vectors that point in the horizontal and vertical directions.

DEFINITION:

The vector $\vec{i}$ is the unit vector that points in the **positive horizontal direction**. Since its horizontal component is 1 and its vertical component is 0, we see that $\vec{i} = \langle 1, 0 \rangle$.

The vector $\vec{j}$ is the unit vector that points in the **positive vertical direction**. Since its horizontal component is 0 and its vertical component is 1, we see that $\vec{j} = \langle 0, 1 \rangle$.

Note that since they are *unit vectors*, $\|\vec{i}\| = 1$ and $\|\vec{j}\| = 1$.

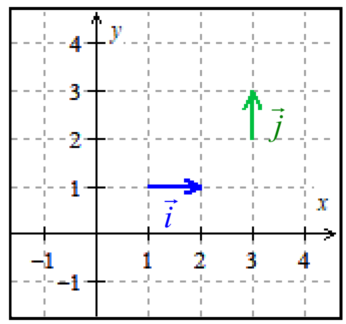


Figure 9: Unit vectors $\vec{i}$ and $\vec{j}$.

We can use vectors $\vec{i}$ and $\vec{j}$ to describe all other two-dimensional vectors. For example, we can describe $\vec{v} = \langle 4, 3 \rangle$ (from Example 1) using vectors $\vec{i}$ and $\vec{j}$: along with scalar multiplication and vector addition:

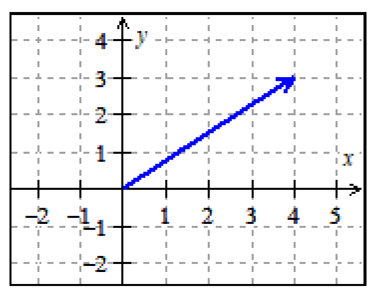


Figure 10: Vector $\vec{v}$.

In general, if $\vec{a} = \langle a_1, a_2 \rangle$ is a vector, then _________________.

The Dot Product

We've studied how to add and subtract vectors and how to multiply vectors by scalars. Now we'll study how to multiply one vector by another. This type of multiplication is called the **dot product**. Since we are focusing on two-dimensional vectors in this class, we will define the dot product in terms of two-dimensional vectors:

DEFINITION: If $\vec{u} = \langle u_1, u_2 \rangle$ and $\vec{v} = \langle v_1, v_2 \rangle$, then **the dot product of $\vec{u}$ and $\vec{v}$**, denoted $\vec{u} \cdot \vec{v}$, is defined as follows:

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2$$

Thus, to compute the dot product of two vectors, we simply multiply the horizontal components of the two vectors and the vertical components of the two vectors and then add the results. It is important to note that the dot product produces a **scalar**.

EXAMPLE 1: If $\vec{a} = \langle 3, -9 \rangle$ and $\vec{b} = \langle 6, -1 \rangle$, find $\vec{a} \cdot \vec{b}$.

Properties of the Dot Product

If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors then the following properties hold true:

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$ (commutative property)

2. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$ (distributive property)

3. $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$

4. $\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cos(\theta)$ where θ is the angle between $\vec{u}$ and $\vec{v}$.

The dot product can be used to find the angle between two vectors.

EXAMPLE 2: Find the angle between vectors $\vec{a} = \langle 3, -9 \rangle$ and $\vec{b} = \langle 6, -1 \rangle$ from Example 1.

EXAMPLE 3: If the angle between $\vec{u}$ and $\vec{v}$ is $\theta = 90^\circ$ (i.e., if $\vec{u}$ and $\vec{v}$ are perpendicular), find $\vec{u} \cdot \vec{v}$.

EXAMPLE 4: If $\vec{u}$ and $\vec{v}$ are non-zero vectors and $\vec{u} \cdot \vec{v} > 0$, what can you say about the angle θ between vectors $\vec{u}$ and $\vec{v}$. What if $\vec{u} \cdot \vec{v} < 0$?

Now we'll prepare to derive $\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cos(\theta)$. We'll need to use the **difference of two vectors** so first let's explore that:

EXAMPLE 5: Let $\vec{v} = \langle 4, 3 \rangle$ and $\vec{s} = \langle 2, -6 \rangle$; find $\vec{v} - \vec{s}$.

We can also subtract vectors by using arrows on the coordinate plane. Notice that $\vec{v} - \vec{s} = \vec{v} + (-\vec{s})$: so in Figure 1 we can subtract $\vec{s}$ from $\vec{v}$ by adding $-\vec{s}$ to $\vec{v}$:

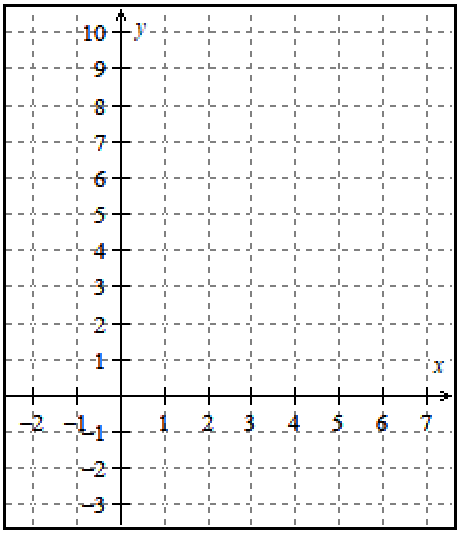


Figure 1

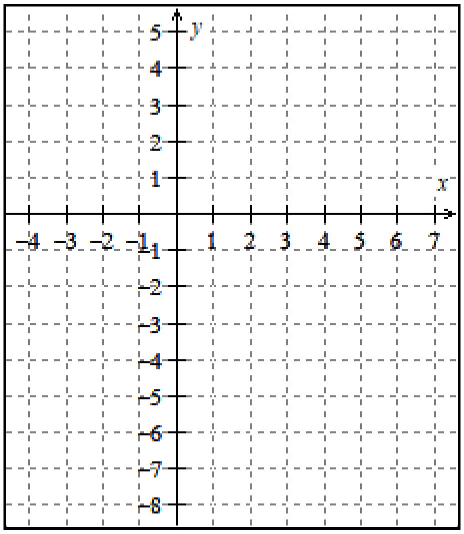


Figure 2

Interestingly, $\vec{v} - \vec{s}$ is the vector that we need to add to $\vec{s}$ in order to create $\vec{v}$: we can see this symbolically and then use it obtain another way to draw $\vec{v} - \vec{s}$ in Figure 2.

We'll also need to use the identity $\vec{v}\cdot\vec{v}=\|\vec{v}\|^2$ so let's prove it by computing $\vec{v}\cdot\vec{v}$ for a generic vector $\vec{v}=\langle v_1, v_2\rangle$ and showing that the result is equal to $\|\vec{v}\|^2$ (this proof will have the same structure as our proofs of trig identities).

Now we're ready to derive $\vec{u}\cdot\vec{v}=\|\vec{u}\|\cdot\|\vec{v}\|\cos(\theta)$ where θ is the angle between $\vec{u}$ and $\vec{v}$:

First let's draw two vectors $\vec{u}$ and $\vec{v}$ so that θ is the angle between the vectors:

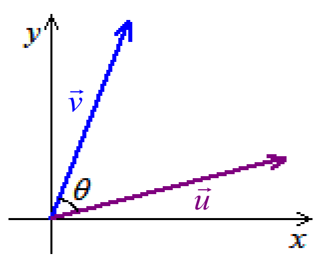


Figure 3

Now let's construct the vector $\vec{w}=\vec{v}-\vec{u}$. Recall that we can obtain the vector $\vec{v}-\vec{u}$ by drawing an arrow that starts at the tip of $\vec{u}$ and ends at the tip of $\vec{v}$.

If we think of the arrows as being line segments instead of arrows, we have a triangle with side lengths $\|\vec{u}\|$, $\|\vec{v}\|$, and $\|\vec{w}\|$ and angle θ opposite the side $\|\vec{w}\|$.

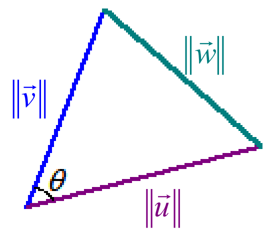


Figure 4

Now we can use the Law of Cosines to obtain an equation that relates the magnitudes of the vectors and the angle θ.

We can use this equation to obtain the statement given in fourth property in the table above. First, let's find $\|\vec{w}\|^2$. Recall that $\vec{w} = \vec{v} - \vec{u}$. So...

We can now substitute this expression for $\|\vec{w}\|^2$ in the equation we found above, and simplify to obtain $\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cos(\theta)$:

A Summary via Examples

EXAMPLE 1a: Find B and c in the right triangle given in Figure 1. (The triangle might not be drawn to scale.)

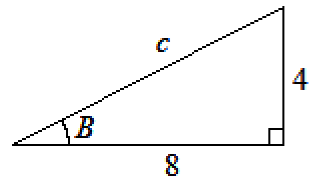


Figure 1

EXAMPLE 1b: Find a and b in the right triangle given in Figure 2. (The triangle might not be drawn to scale.)

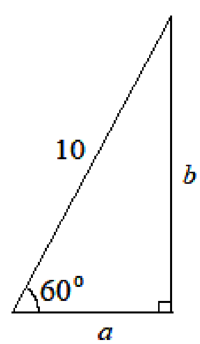
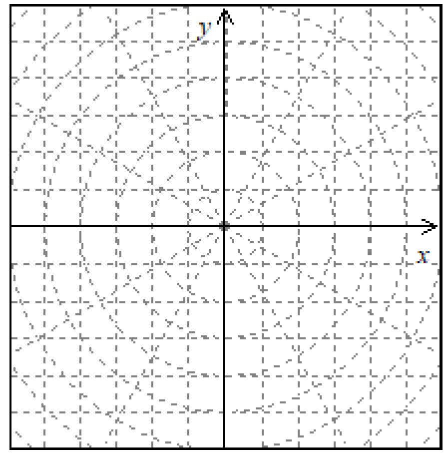
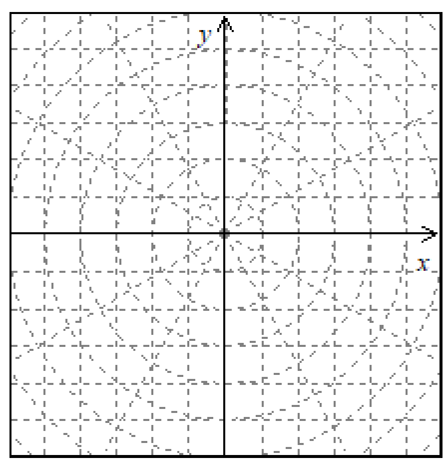


Figure 2

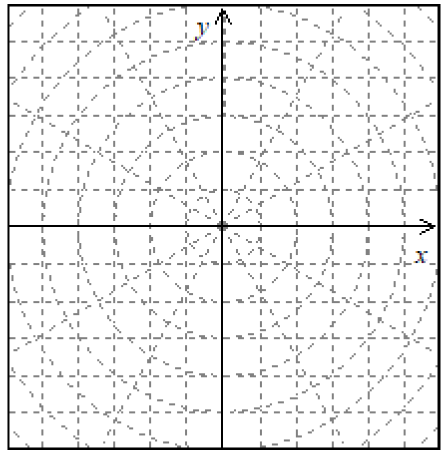
EXAMPLE 2a: Translate the rectangular ordered pair $(-8, -4)$ into polar coordinates (r, θ).



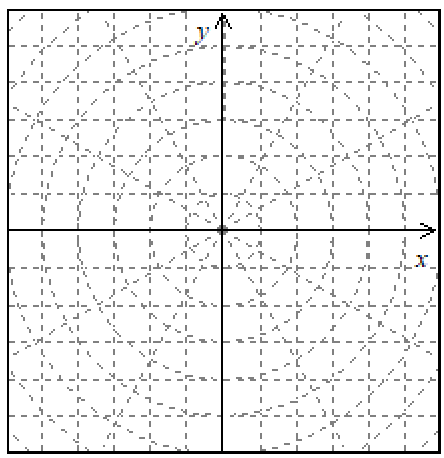
EXAMPLE 2b: Translate the polar ordered pair $\left(10, \frac{4\pi}{3}\right)$ into rectangular coordinates (x, y).



EXAMPLE 3a: Find the polar form, $z = r \cdot e^{i \cdot \theta}$, of the complex number $z = -8 + 4i$.



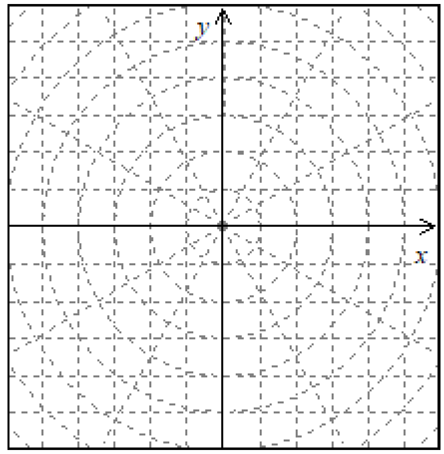
EXAMPLE 3b: Find the rectangular form, $z = a + bi$, of the complex number $z = 10 \cdot e^{i \cdot \frac{2\pi}{3}}$.



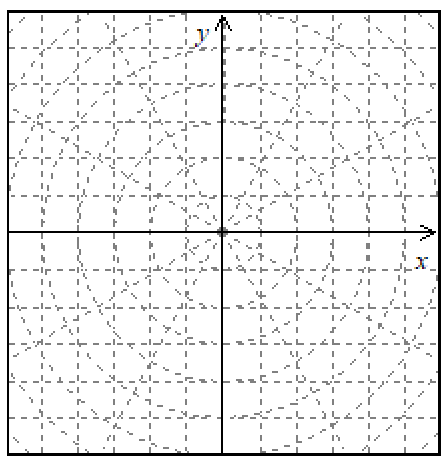
EXAMPLE 4a: If $\vec{v} = 8\vec{i} - 4\vec{j}$, find

(1) the direction of $\vec{v}$ (measured in degrees with respect to the positive x-axis)

(2) $\|\vec{v}\|$



EXAMPLE 4b: Find the components of the vector $\vec{w}$ if $\|\vec{w}\| = 10$ and $\vec{w}$ has a direction of 300° with respect to the positive x-axis.



"bonus video"

Introduction

In this Class Notes Video, we're going start by acknowledging the existence of the Sum-to-Product Identities so that, later in the video, we can use one of these identities to explain a common technique for tuning musical instruments. In the middle of the video we'll discuss how sinusoidal functions -- and manipulations of sinusoidal functions -- can be used to model real-word behavior.

Sum-to-Product Identities

Now let's look at identities involving expressions of the form $\sin(A) \pm \sin(B)$ and $\cos(A) \pm \cos(B)$.

THE SUM (AND DIFFERENCE) TO PRODUCT IDENTITIES

$$\sin(A) + \sin(B) = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\sin(A) - \sin(B) = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\cos(A) + \cos(B) = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\cos(A) - \cos(B) = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

These identities have a variety of real-word applications. As mentioned above, later we'll investigate how one of these identities (namely, the first one on the list) can be used to explain a common technique for tuning musical instruments.

Modeling with Sinusoidal Functions

Since many real-world scenarios are more complicated than the simple rotation around a unit circle, we often need to modify the sine and cosine functions to use them to model the real-world. As we've discussed, we can use simple graph transformations to warp the graphs of $y = \sin(t)$ and $y = \cos(t)$ into sinusoidal functions with any period, midline, and amplitude: this can allow us to model real-world behavior like the rotation of a Ferris wheel and the oscillation of the rabbit population in a national park. The simple graph transformations work well to allow us to model many real-world scenarios but we need to use other tools to modify a sinusoidal function when we want to model real-world behavior that isn't truly periodic (e.g., an oscillating spring whose amplitude decays over time) and behavior that is more complicated than a simple sine wave (e.g., the combination of multiple sound waves).

Before we get started, let's first make an important observation about the graphs of sinusoidal functions:

EXAMPLE 1: Let's graph $f(t) = 4\cos(\pi t)$:

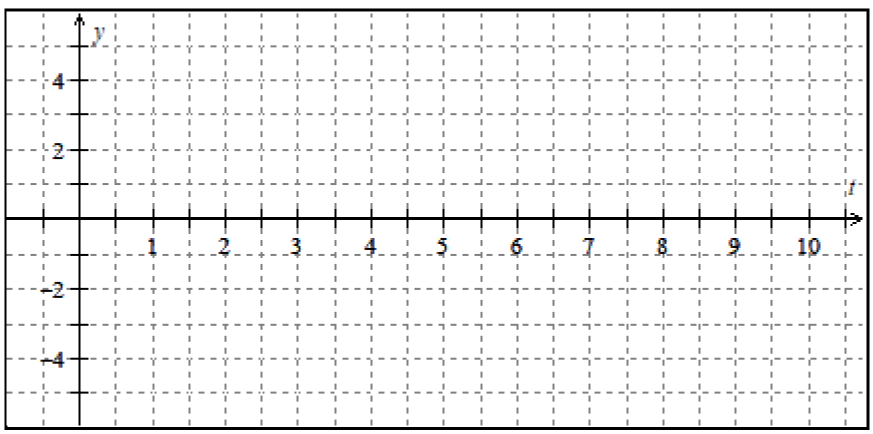


Figure 1: Graph of $f(t) = 4\cos(\pi t)$.

Notice in Figure 1 that the graph of $f(t) = 4\cos(\pi t)$ appears to "bounce" up and down between the lines $y = 4$ and $y = -4$; since the amplitude is 4, it's reasonable to think of the amplitude determining a pair of lines that guide the graph of the wave. I like to think of these lines as forming "railroad tracks" that the function "bounces" between. Mathematicians sometimes say that these lines form the *envelope for the function* since the function never escapes the region between the two lines.

EXAMPLE 2a: A weight is suspended from a spring. Suppose that the weight is pulled 4 inches below its resting position and released, and that it bounces up and down once every 2 seconds without any dampening, i.e., the weight continuously bounces 4 inches above the resting position and 4 inches below the resting positing without losing any energy. Find a sinusoidal function f that models the weight's displacement below its resting position t seconds after it was released.

SOLUTION:

First we need to decide of we want to use sine or cosine to construct our function. Since the weight was at its maximum displacement below the equilibrium when it was released, it might be easiest to use the cosine function since cosine is at its maximum output when the input is 0. So our function will have form $f(t) = A\cos(wt)$.

- The weight bounces up and down once every 2 seconds so the period is 2. Thus,

$$2 = \frac{2\pi}{w} \quad \Rightarrow \quad w = \pi .$$

- Since the weight bounces 4 inches above the resting position and 4 inches below, the amplitude $A = 4$.

Thus, weight's displacement below its resting position t seconds after it was released is given by $f(t) = 4\cos(\pi t)$, the same function we graphed in Example 1.

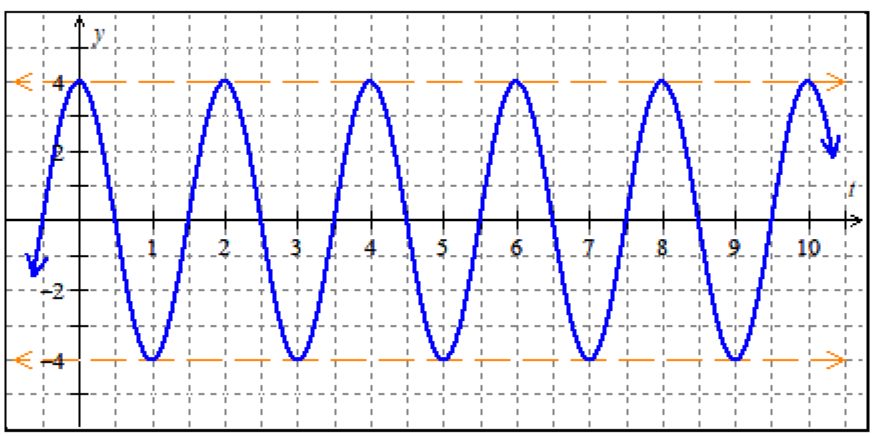


Figure 2: The graph of $f(t) = 4\cos(\pi t)$ (from Example 1).

EXAMPLE 2b: A weight is suspended from a spring. Suppose that the weight is pulled 4 inches below its resting position and released, and that it bounces up and down once every 2 seconds. Suppose further that there are "**damping forces**" that cause the displacement of the weight to decrease exponentially at the rate of 10% per second. Find a sinusoidal function g that models the weight's displacement below its resting position t seconds after it was released.

SOLUTION:

First we need to decide of we want to use sine or cosine to construct our function. Since the situation is similar to the situation in Example 2a, we can again use a function with the form $g(t) = A\cos(wt)$ with $w = \pi$.

Unlike in Example 2a, in this example there are damping forces that cause the displacement of the weight to decrease exponentially at the rate of 10% per second. In order to represent these damping forces in the algebraic rule for the function, we can replace the amplitude $A = 4$ with a function $A(t)$ that has initial value 4 and decreases exponentially at the rate of 10% per second. Recall from MTH 111 that

$$\begin{aligned} A(t) &= 4\cdot(1+r)^t \\ &= 4\cdot(1+(-0.10))^t \\ &= 4(0.90)^t \end{aligned}$$

Thus, the weight's displacement below its resting position t seconds after it was released is given by $g(t) = 4(0.90)^t \cdot \cos(\pi t)$.

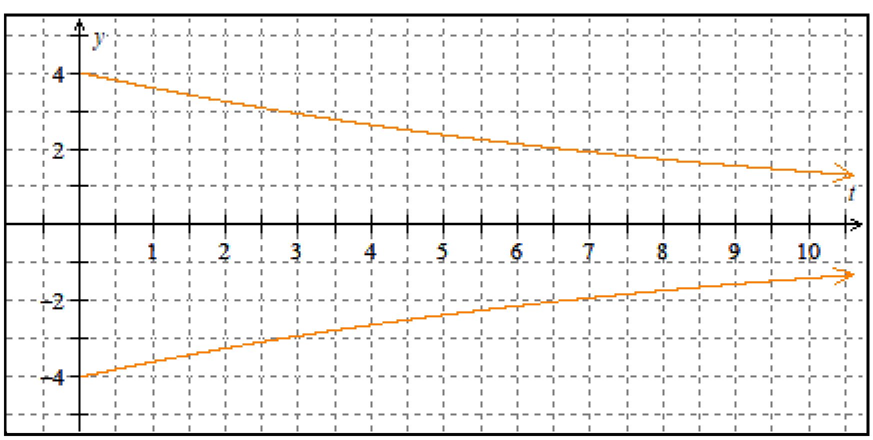
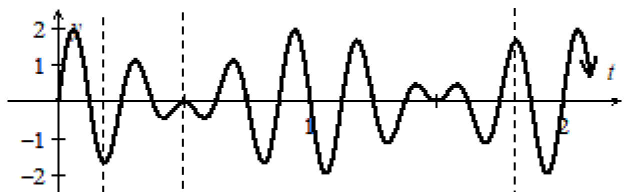


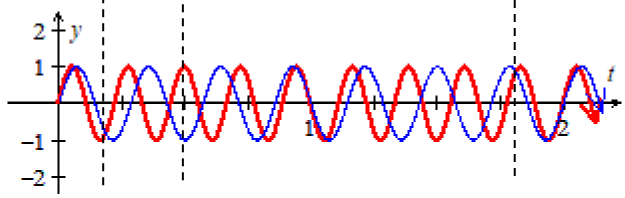
Figure 3: Graph of $g(t) = 4(0.90)^t \cdot \cos(\pi t)$.

Modeling Sound Waves with Sinusoidal Functions

Suppose that the functions $y = \sin(9\pi t)$ and $y = \sin(7\pi t)$ represent two sound waves. (In reality, the frequencies of these waves are too low for human ears to hear but using these functions will make it easier for use to draw graphs by hand.) If both of these "sounds" are made at the same time, the function that describes the combined sound (or the *chord* in music lingo) is the sum of the two respective sound waves: $f(t) = \sin(9\pi t) + \sin(7\pi t)$. Below is a graph of $f(t) = \sin(9\pi t) + \sin(7\pi t)$ followed by graphs of $y = \sin(9\pi t)$ and $y = \sin(7\pi t)$.



$f(t) = \sin(9\pi t) + \sin(7\pi t)$



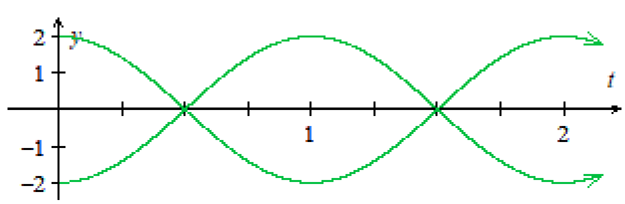
$y = \sin(9\pi t)$ and $y = \sin(7\pi t)$

To understand "why" the graph of $y = f(t)$ looks as it does, we could do point by-point-calculations (suggested by the dotted lines). Instead, we can use one of our new identities to obtain insight on the shape of graphs like this, and we can use this insight to enhance our understanding of acoustics. Recall (from page 1):

$$\sin(A) + \sin(B) = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right).$$

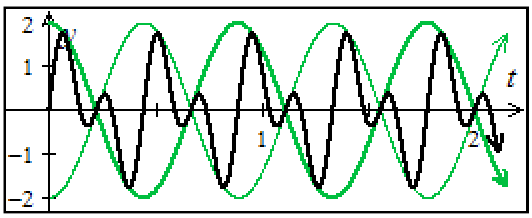
By letting $A = 9\pi t$ and $B = 7\pi t$, we can re-write $f(t)$:

So when we consider the graph of $y = f(t)$, we can think of $A(t) = 2\cos(\pi t)$ as the "amplitude-like" function that defines the envelope for $y = \sin(8\pi t)$ to bounce inside. (As we learned in the previous examples, we need to graph both $y = 2\cos(\pi t)$ and $y = -2\cos(\pi t)$ in order to obtain our envelope.) Then we need to draw $y = \sin(8\pi t)$ so that stays inside the envelope. This function has period $\frac{2\pi}{8\pi} = \frac{1}{4}$, so there are four periods in 1 unit. Graphs of $y = 2\cos(\pi t)$ and $y = -2\cos(\pi t)$ are drawn; sketch $f(t) = \sin(9\pi t) + \sin(7\pi t)$.



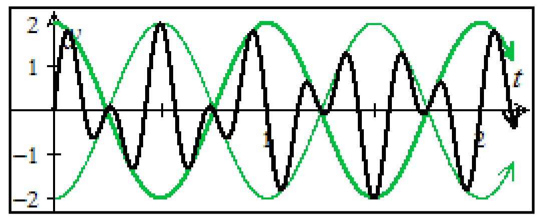
On the previous page we learned that we can analyze the sum of two sound waves (i.e., a *musical chord*) using “envelopes” that we can find using trig identities. Interestingly, the graphs of consonant chords (which tend to be pleasant sounding) look nice and stable while the graphs of dissonant chords (which tend to be unpleasant sounding) look messy and unstable. The chords that are considered to have *perfect consonance* are **octaves**, **fifths**, and **fourths**. There are many other chords that are less consonant, and many more that are dissonant. In the graphs below, notice how consonance vs. dissonance is represented in the stable vs. un-stable nature of the waves.

An **octave** is a cord that consists of two sounds whose frequencies have a ratio of 2:1. This means that the period of one the sound-waves is twice that of the other wave. e.g., $y = \sin(9\pi t)$ and $y = \sin(4.5\pi t)$ are separated by an octave.



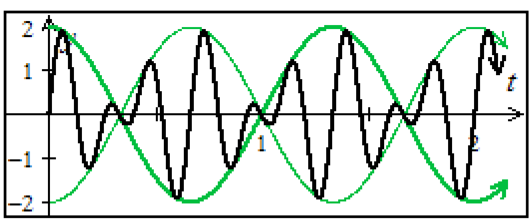
$oct(t) = \sin(9\pi t) + \sin(4.5\pi t)$

A **minor seventh** is a cord that consists of two sounds whose frequencies have a ratio of 9:5, so the period of one the sound-waves is $9/5$ times larger than the other. e.g., $y = \sin(9\pi t)$ and $y = \sin(5\pi t)$ are separated by a minor seventh.



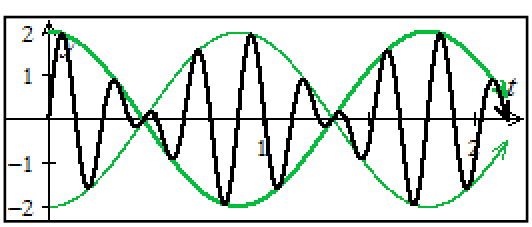
$svn(t) = \sin(9\pi t) + \sin(5\pi t)$

A **fifth** is a cord that consists of two sounds whose frequencies have a ratio of 3:2, so the period of one of the sound-waves is $3/2$ times larger than the other wave. e.g., $y = \sin(9\pi t)$ and $y = \sin(6\pi t)$ are separated by a fifth.



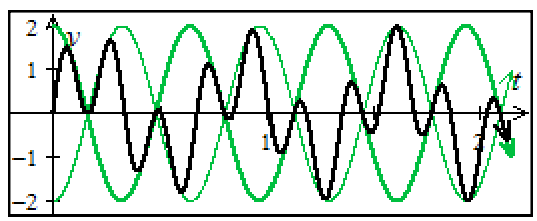
$fif(t) = \sin(9\pi t) + \sin(6\pi t)$

A **fourth** is a cord that consists of two sounds whose frequencies have a ratio of 4:3, so the period of one of the sound-waves is $4/3$ times larger than the other wave. e.g., $y = \sin(9\pi t)$ and $y = \sin(6.75\pi t)$ are separated by a fourth.

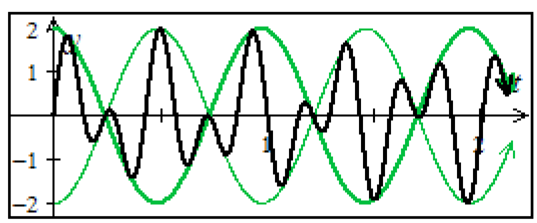


$fr(t) = \sin(9\pi t) + \sin(6.75\pi t)$

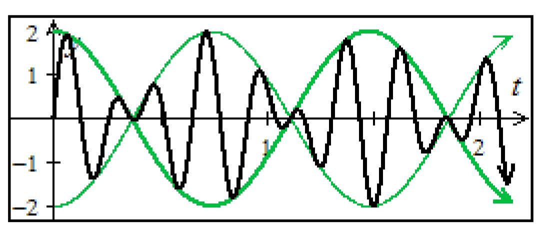
The graphs below represent highly dissonant chords:



$a(t) = \sin(9\pi t) + \sin(2.8\pi t)$ (ratio 9:2.8)

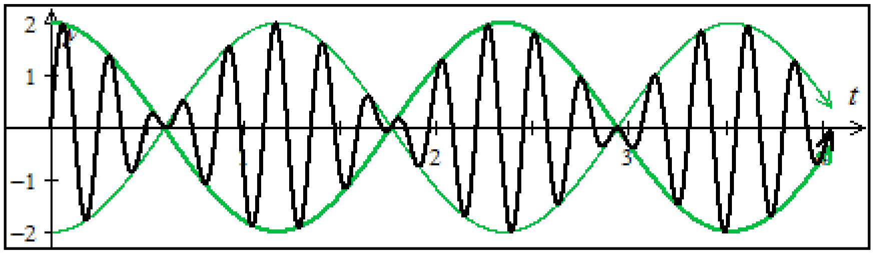


$b(t) = \sin(9\pi t) + \sin(4.9\pi t)$ (ratio: 9:4.9)

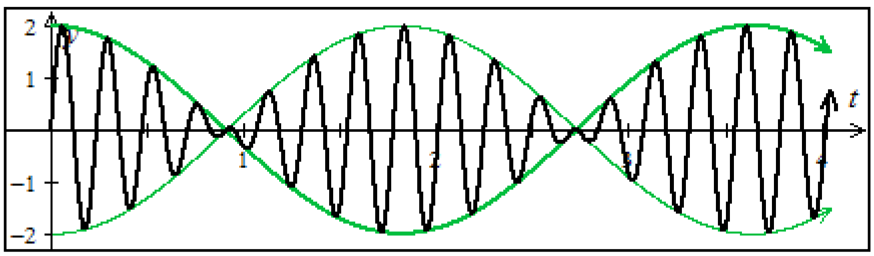


$c(t) = \sin(9\pi t) + \sin(6.3\pi t)$ (ratio: 9:6.3)

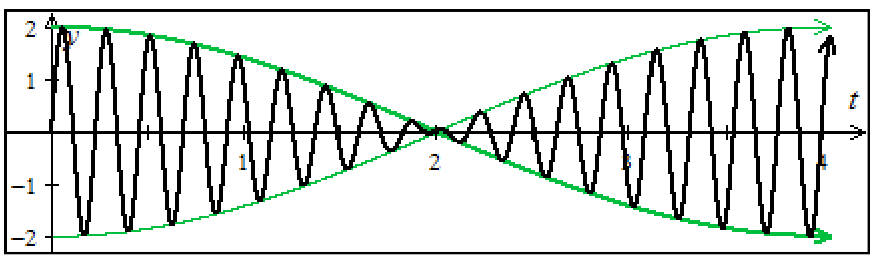
Finally, let's investigate what happens when two sounds have *almost* the same frequency. The graphs below show $F(t) = \sin(9\pi t) + \sin(B\pi t)$ with value of B getting closer and closer to 9.



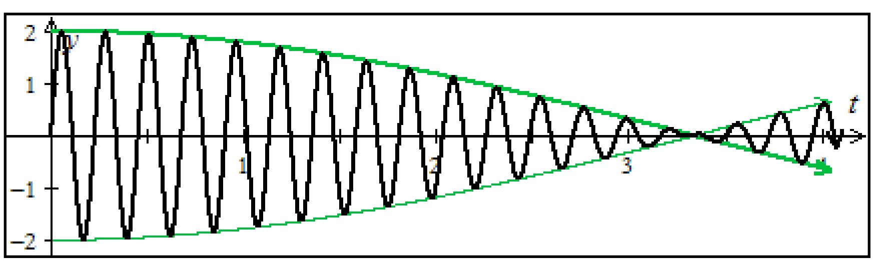
$F(t) = \sin(9\pi t) + \sin(7.3\pi t)$



$F(t) = \sin(9\pi t) + \sin(7.9\pi t)$



$F(t) = \sin(9\pi t) + \sin(8.5\pi t)$



$F(t) = \sin(9\pi t) + \sin(8.7\pi t)$

As the frequencies of the waves get more similar, the length of time between the low-volume instances increases. When these are actual sound waves, the human ear can discern this change in volume and musicians can use the "beats" that they hear to tune their instruments.

We can listen to acoustic beats here:
https://academo.org/demos/wave-interference-beat-frequency/.