Other Important Identities

First let's look at identities involving expressions of the form $\sin(A \pm B)$ and $\cos(A \pm B)$. These identities allow us to calculate the sine and cosine of the sum and difference of two angles if we know the sine and cosine of the angles. (There are corresponding identities for tangent but we can use the sine and cosine identities along with the definition of tangent rather than studying another identity for tangent.) We can use these identities to find the trig values of all multiples of $15^\circ = \frac{\pi}{12}$.

$\frac{30^\circ}{2} = \frac{\pi/6}{2} \Rightarrow 15^\circ = \frac{\pi}{12}$

THE SUM AND DIFFERENCE IDENTITIES

sine :

$$\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$
$$\sin(A - B) = \sin(A)\cos(B) - \cos(A)\sin(B)$$

cosine :

$$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$
$$\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)$$

EXAMPLE 1: Use an appropriate identity to calculate $\cos(15^\circ)$.

$15^\circ = 45^\circ - 30^\circ$

$$\begin{aligned}
\cos(15^\circ) &= \cos(45^\circ - 30^\circ) \\
&= \cos(45^\circ)\cos(30^\circ) + \sin(45^\circ)\sin(30^\circ) \\
&= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) \\
&= \frac{\sqrt{6} + \sqrt{2}}{4}
\end{aligned}$$

EXAMPLE 2: Use an appropriate identity to calculate $\sin\left(\frac{13\pi}{12}\right)$.

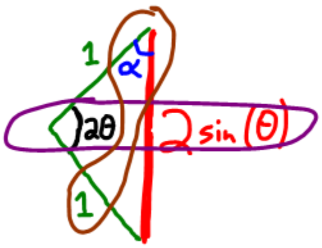
$\frac{13\pi}{12} = \frac{10\pi}{12} + \frac{3\pi}{12} = \frac{5\pi}{6} + \frac{\pi}{4}$

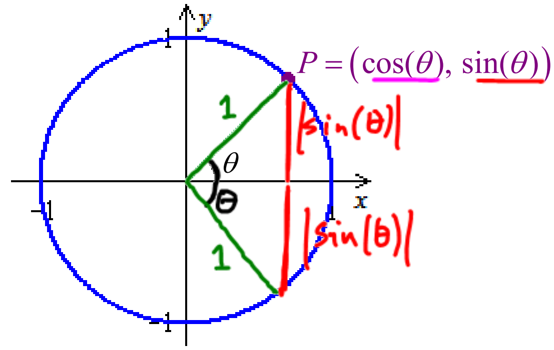
$$\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$

$$\begin{aligned}
\sin\left(\frac{13\pi}{12}\right) &= \sin\left(\frac{5\pi}{6} + \frac{\pi}{4}\right) \\
&= \sin\left(\frac{5\pi}{6}\right)\cos\left(\frac{\pi}{4}\right) + \cos\left(\frac{5\pi}{6}\right)\sin\left(\frac{\pi}{4}\right) \\
&= \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) \\
&= \frac{\sqrt{2} - \sqrt{6}}{4}
\end{aligned}$$

Now we'll familiarize ourselves with the *double-angle identities* and the *half-angle identities*. These identities allow us to find $\sin(2\theta)$ & $\cos(2\theta)$ and $\sin\left(\frac{\theta}{2}\right)$ & $\cos\left(\frac{\theta}{2}\right)$ if we know the values of $\cos(\theta)$ and $\sin(\theta)$.

Let's start by deriving the **double-angle identity for sine**; then we'll derive the **double-angle identity for cosine**.



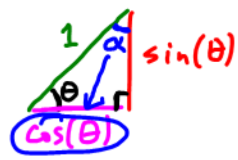


Law of Sines

$$\frac{\sin(2\theta)}{2\sin(\theta)} = \frac{\sin(\alpha)}{1}$$

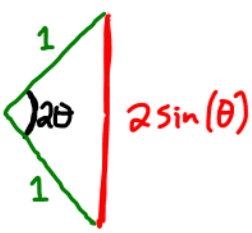
$$\sin(2\theta) = 2\sin(\theta)\sin(\alpha)$$

$$\Rightarrow \boxed{\sin(2\theta) = 2\sin(\theta)\cos(\theta)}$$



$$\sin(\alpha) = \frac{\text{opp}}{\text{hyp}} = \frac{\cos(\theta)}{1}$$

$$\boxed{\sin(\alpha) = \cos(\theta)}$$



Law of Cosines

$$(2\sin(\theta))^2 = 1^2 + 1^2 - 2\cdot 1\cdot 1\cos(2\theta)$$

$$\Rightarrow 4\sin^2(\theta) = 2 - 2\cos(2\theta)$$

$$\Rightarrow 2\cos(2\theta) = 2 - 4\sin^2(\theta)$$

$$\Rightarrow \boxed{\cos(2\theta) = 1 - 2\sin^2(\theta)}$$

$$\cos(2\theta) = 1 - 2\sin^2(\theta)$$

$$= 1 - 2(1 - \cos^2(\theta))$$

$$= 1 - 2 + 2\cos^2(\theta)$$

$$\boxed{\cos(2\theta) = 2\cos^2(\theta) - 1}$$

$$\cos(2\theta) = 1 - 2\sin^2(\theta)$$

$$= \sin^2(\theta) + \cos^2(\theta) - 2\sin^2(\theta)$$

$$\boxed{\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)}$$

DOUBLE-ANGLE IDENTITIES

sine: $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$

cosine: $\cos(2\theta) = 1 - 2\sin^2(\theta)$

$\cos(2\theta) = 2\cos^2(\theta) - 1$

$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$

Q4

EXAMPLE 1: If $\cos(A) = \frac{1}{3}$, where $\frac{3\pi}{2} < A < 2\pi$, find $\cos(2A)$, $\sin(2A)$, and $\tan(2A)$.

Find $\sin(A)$:

$\sin^2(A) + \cos^2(A) = 1$

$\sin^2(A) + \left(\frac{1}{3}\right)^2 = 1$

$\sin^2(A) = 1 - \frac{1}{9} = \frac{8}{9}$

$\sin(A) = -\frac{2\sqrt{2}}{3}$ ✓

$\cos(2A) = 2\cos^2(A) - 1$

$= 2\left(\frac{1}{3}\right)^2 - 1$

$= 2 \cdot \frac{1}{9} - 1$

$= -\frac{7}{9}$

$\sin(2A) = 2\sin(A)\cos(A)$

$= 2\left(-\frac{2\sqrt{2}}{3}\right)\left(\frac{1}{3}\right)$

$= -\frac{4\sqrt{2}}{9}$

Q3

$\tan(2A) = \frac{\sin(2A)}{\cos(2A)}$

$= \frac{+4\sqrt{2}/9}{+7/9}$

$= \frac{4\sqrt{2}}{7}$

We can use the double-angle identities for cosine to derive **half-angle identities**.

Recall this double-angle identity for cosine: $\cos(2\theta) = 1 - 2\sin^2(\theta)$. We can use this identity to find a half-angle identity for sine:

$$\cos(2\theta) = 1 - 2\sin^2(\theta)$$

$$2\sin^2(\theta) = 1 - \cos(2\theta)$$

$$\sin^2(\theta) = \frac{1 - \cos(2\theta)}{2}$$

$$\sin(\theta) = \pm\sqrt{\frac{1 - \cos(2\theta)}{2}}$$

Let $A = 2\theta$
Then $\frac{A}{2} = \theta$

$$\sin\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1 - \cos(A)}{2}}$$

We can use $\cos(2\theta) = 2\cos^2(\theta) - 1$ to find a half-angle identity for cosine:

$$\cos(2\theta) = 2\cos^2(\theta) - 1$$

$$1 + \cos(2\theta) = 2\cos^2(\theta)$$

$$\frac{1 + \cos(2\theta)}{2} = \cos^2(\theta)$$

$$\cos(\theta) = \pm\sqrt{\frac{1 + \cos(2\theta)}{2}}$$

Let $A = 2\theta$
Then $\frac{A}{2} = \theta$

$$\cos\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1 + \cos(A)}{2}}$$

HALF-ANGLE IDENTITIES

sine: $\sin\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1 - \cos(A)}{2}}$

cosine: $\cos\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1 + \cos(A)}{2}}$

When using the half-angle identities, you need to decide which sign to use by determining which quadrant $\frac{A}{2}$ falls in.

EXAMPLE 2: If $\cos(A) = \frac{1}{3}$, where $\frac{3\pi}{2} < A < 2\pi$. Find $\cos\left(\frac{A}{2}\right)$ and $\sin\left(\frac{A}{2}\right)$.

$$\frac{3\pi/2}{2} < \frac{A}{2} < \frac{2\pi}{2}$$

$$\Rightarrow \frac{3\pi}{4} < \frac{A}{2} < \pi$$

$$\cos\left(\frac{A}{2}\right) = -\sqrt{\frac{1+\cos(A)}{2}} = -\sqrt{\frac{1+\frac{1}{3}}{2}} = -\sqrt{\frac{4/3}{2}} = -\sqrt{\frac{4}{3}\cdot\frac{1}{2}} = -\sqrt{\frac{2}{3}}$$

$$\sin\left(\frac{A}{2}\right) = +\sqrt{\frac{1-\cos(A)}{2}} = \sqrt{\frac{1-1/3}{2}} = \sqrt{\frac{2/3}{2}} = \sqrt{\frac{1}{3}}$$

EXAMPLE 3: Use a half-angle identity to find $\cos\left(15^\circ\right)$.

$A = 30^\circ$

$$\cos(15^\circ) = \cos\left(\frac{30^\circ}{2}\right) = +\sqrt{\frac{1+\cos(30^\circ)}{2}} = \sqrt{\frac{1+\sqrt{3}/2}{2}\cdot\frac{2}{2}} = \sqrt{\frac{2+\sqrt{3}}{4}} = \frac{\sqrt{2+\sqrt{3}}}{2}$$

From page 1

$$\cos(15^\circ) = \frac{\sqrt{6}+\sqrt{2}}{4}$$