Introduction

In this Class Notes Video, we're going start by acknowledging the existence of the Sum-to-Product Identities so that, later in the video, we can use one of these identities to explain a common technique for tuning musical instruments. In the middle of the video we'll discuss how sinusoidal functions -- and manipulations of sinusoidal functions -- can be used to model real-word behavior.

Sum-to-Product Identities

Now let's look at identities involving expressions of the form $\sin(A) \pm \sin(B)$ and $\cos(A) \pm \cos(B)$.

THE SUM (AND DIFFERENCE) TO PRODUCT IDENTITIES

$$\sin(A) + \sin(B) = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\sin(A) - \sin(B) = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\cos(A) + \cos(B) = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\cos(A) - \cos(B) = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

These identities have a variety of real-word applications. As mentioned above, later we'll investigate how one of these identities (namely, the first one on the list) can be used to explain a common technique for tuning musical instruments.

Modeling with Sinusoidal Functions

Since many real-world scenarios are more complicated than the simple rotation around a unit circle, we often need to modify the sine and cosine functions to use them to model the real-world. As we've discussed, we can use simple graph transformations to warp the graphs of $y = \sin(t)$ and $y = \cos(t)$ into sinusoidal functions with any period, midline, and amplitude: this can allow us to model real-world behavior like the rotation of a Ferris wheel and the oscillation of the rabbit population in a national park. The simple graph transformations work well to allow us to model many real-world scenarios but we need to use other tools to modify a sinusoidal function when we want to model real-world behavior that isn't truly periodic (e.g., an oscillating spring whose amplitude decays over time) and behavior that is more complicated than a simple sine wave (e.g., the combination of multiple sound waves).

Before we get started, let's first make an important observation about the graphs of sinusoidal functions:

EXAMPLE 1: Let's graph $f(t) = 4\cos(\pi t)$:

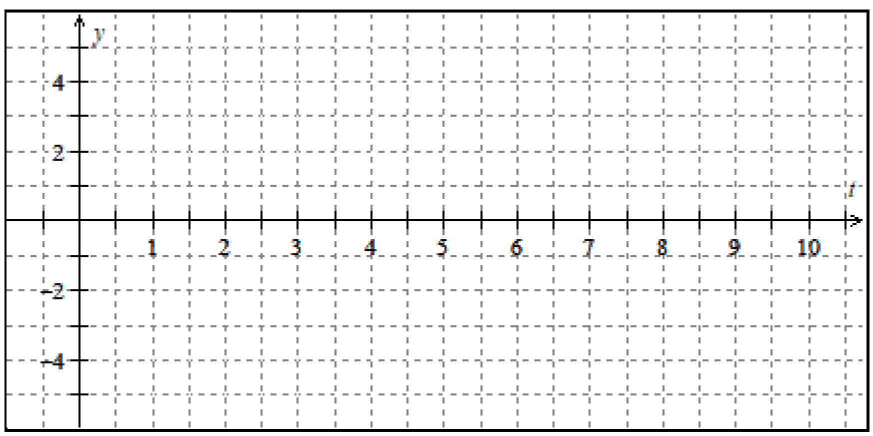


Figure 1: Graph of $f(t) = 4\cos(\pi t)$.

Notice in Figure 1 that the graph of $f(t) = 4\cos(\pi t)$ appears to "bounce" up and down between the lines $y = 4$ and $y = -4$; since the amplitude is 4, it's reasonable to think of the amplitude determining a pair of lines that guide the graph of the wave. I like to think of these lines as forming "railroad tracks" that the function "bounces" between. Mathematicians sometimes say that these lines form the *envelope for the function* since the function never escapes the region between the two lines.

EXAMPLE 2a: A weight is suspended from a spring. Suppose that the weight is pulled 4 inches below its resting position and released, and that it bounces up and down once every 2 seconds without any dampening, i.e., the weight continuously bounces 4 inches above the resting position and 4 inches below the resting positing without losing any energy. Find a sinusoidal function f that models the weight's displacement below its resting position t seconds after it was released.

SOLUTION:

First we need to decide of we want to use sine or cosine to construct our function. Since the weight was at its maximum displacement below the equilibrium when it was released, it might be easiest to use the cosine function since cosine is at its maximum output when the input is 0. So our function will have form $f(t) = A\cos(wt)$.

- The weight bounces up and down once every 2 seconds so the period is 2. Thus,

$$2 = \frac{2\pi}{w} \quad \Rightarrow \quad w = \pi .$$

- Since the weight bounces 4 inches above the resting position and 4 inches below, the amplitude $A = 4$.

Thus, weight's displacement below its resting position t seconds after it was released is given by $f(t) = 4\cos(\pi t)$, the same function we graphed in Example 1.

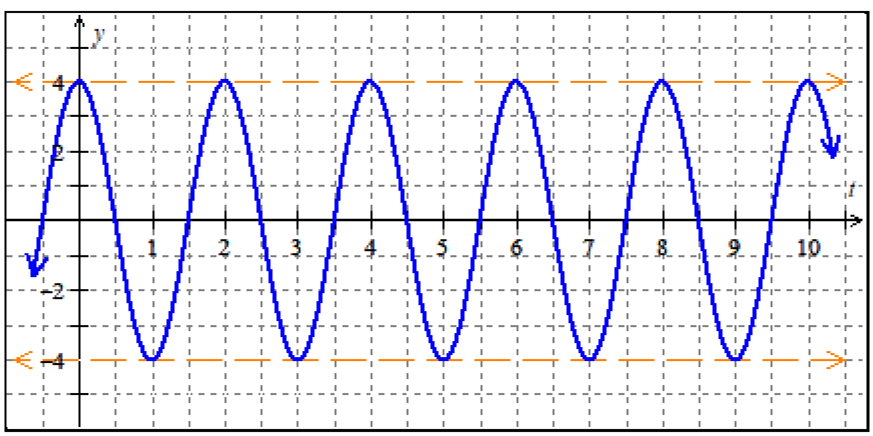


Figure 2: The graph of $f(t) = 4\cos(\pi t)$ (from Example 1).

EXAMPLE 2b: A weight is suspended from a spring. Suppose that the weight is pulled 4 inches below its resting position and released, and that it bounces up and down once every 2 seconds. Suppose further that there are "**damping forces**" that cause the displacement of the weight to decrease exponentially at the rate of 10% per second. Find a sinusoidal function g that models the weight's displacement below its resting position t seconds after it was released.

SOLUTION:

First we need to decide of we want to use sine or cosine to construct our function. Since the situation is similar to the situation in Example 2a, we can again use a function with the form $g(t) = A\cos(wt)$ with $w = \pi$.

Unlike in Example 2a, in this example there are damping forces that cause the displacement of the weight to decrease exponentially at the rate of 10% per second. In order to represent these damping forces in the algebraic rule for the function, we can replace the amplitude $A = 4$ with a function $A(t)$ that has initial value 4 and decreases exponentially at the rate of 10% per second. Recall from MTH 111 that

$$\begin{aligned} A(t) &= 4\cdot\left(1 + r\right)^t \\ &= 4\cdot\left(1 + (-0.10)\right)^t \\ &= 4(0.90)^t \end{aligned}$$

Thus, the weight's displacement below its resting position t seconds after it was released is given by $g(t) = 4(0.90)^t \cdot \cos(\pi t)$.

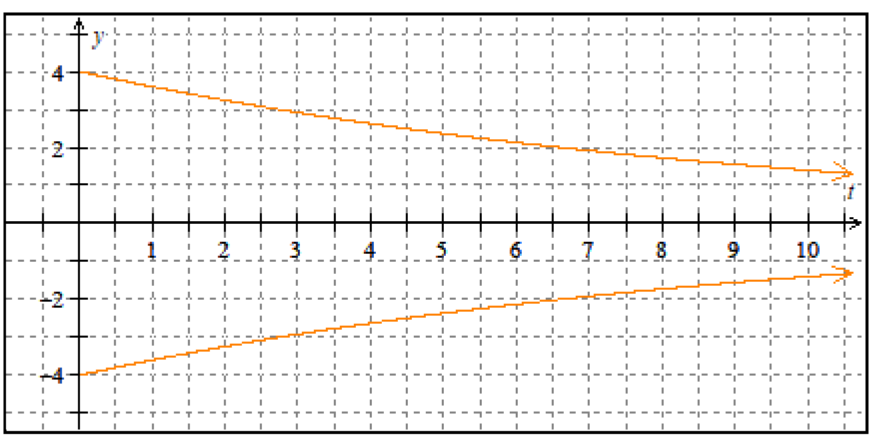
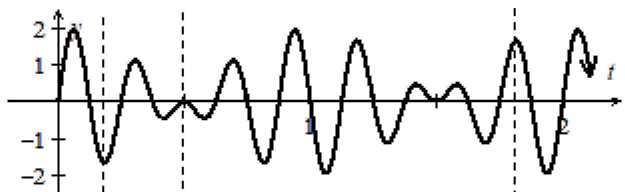


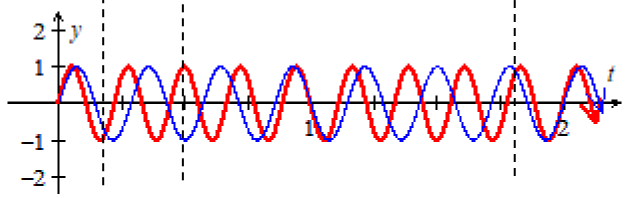
Figure 3: Graph of $g(t) = 4(0.90)^t \cdot \cos(\pi t)$.

Modeling Sound Waves with Sinusoidal Functions

Suppose that the functions $y = \sin(9\pi t)$ and $y = \sin(7\pi t)$ represent two sound waves. (In reality, the frequencies of these waves are too low for human ears to hear but using these functions will make it easier for use to draw graphs by hand.) If both of these "sounds" are made at the same time, the function that describes the combined sound (or the *chord* in music lingo) is the sum of the two respective sound waves: $f(t) = \sin(9\pi t) + \sin(7\pi t)$. Below is a graph of $f(t) = \sin(9\pi t) + \sin(7\pi t)$ followed by graphs of $y = \sin(9\pi t)$ and $y = \sin(7\pi t)$.



$f(t) = \sin(9\pi t) + \sin(7\pi t)$



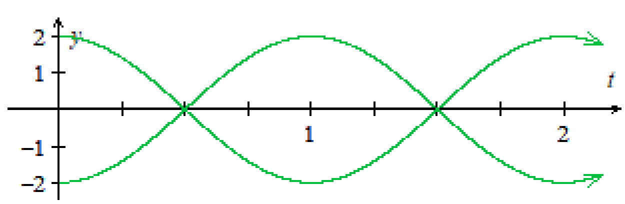
$y = \sin(9\pi t)$ and $y = \sin(7\pi t)$

To understand "why" the graph of $y = f(t)$ looks as it does, we could do point by-point-calculations (suggested by the dotted lines). Instead, we can use one of our new identities to obtain insight on the shape of graphs like this, and we can use this insight to enhance our understanding of acoustics. Recall (from page 1):

$$\sin(A) + \sin(B) = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right).$$

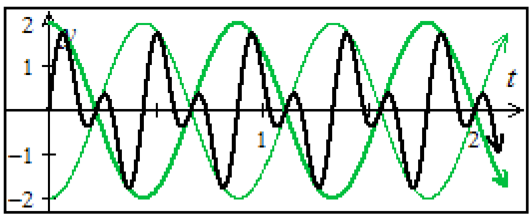
By letting $A = 9\pi t$ and $B = 7\pi t$, we can re-write $f(t)$:

So when we consider the graph of $y = f(t)$, we can think of $A(t) = 2\cos(\pi t)$ as the "amplitude-like" function that defines the envelope for $y = \sin(8\pi t)$ to bounce inside. (As we learned in the previous examples, we need to graph both $y = 2\cos(\pi t)$ and $y = -2\cos(\pi t)$ in order to obtain our envelope.) Then we need to draw $y = \sin(8\pi t)$ so that stays inside the envelope. This function has period $\frac{2\pi}{8\pi} = \frac{1}{4}$, so there are four periods in 1 unit. Graphs of $y = 2\cos(\pi t)$ and $y = -2\cos(\pi t)$ are drawn; sketch $f(t) = \sin(9\pi t) + \sin(7\pi t)$.



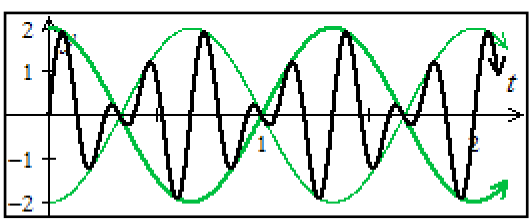
On the previous page we learned that we can analyze the sum of two sound waves (i.e., a *musical chord*) using "envelopes" that we can find using trig identities. Interestingly, the graphs of consonant chords (which tend to be pleasant sounding) look nice and stable while the graphs of dissonant chords (which tend to be unpleasant sounding) look messy and unstable. The chords that are considered to have *perfect consonance* are **octaves**, **fifths**, and **fourths**. There are many other chords that are less consonant, and many more that are dissonant. In the graphs below, notice how consonance vs. dissonance is represented in the stable vs. un-stable nature of the waves.

An **octave** is a cord that consists of two sounds whose frequencies have a ratio of 2:1. This means that the period of one the sound-waves is twice that of the other wave. e.g., $y = \sin(9\pi t)$ and $y = \sin(4.5\pi t)$ are separated by an octave.



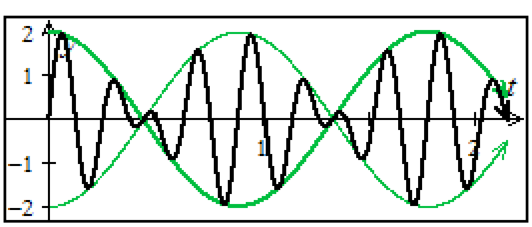
$oct(t) = \sin(9\pi t) + \sin(4.5\pi t)$

A **fifth** is a cord that consists of two sounds whose frequencies have a ratio of 3:2, so the period of one of the sound-waves is $3/2$ times larger than the other wave. e.g., $y = \sin(9\pi t)$ and $y = \sin(6\pi t)$ are separated by a fifth.



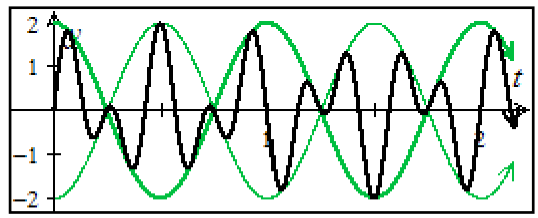
$fif(t) = \sin(9\pi t) + \sin(6\pi t)$

A **fourth** is a cord that consists of two sounds whose frequencies have a ratio of 4:3, so the period of one of the sound-waves is $4/3$ times larger than the other wave. e.g., $y = \sin(9\pi t)$ and $y = \sin(6.75\pi t)$ are separated by a fourth.



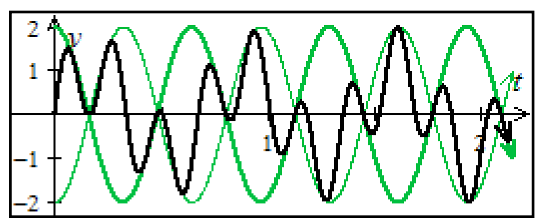
$fr(t) = \sin(9\pi t) + \sin(6.75\pi t)$

A **minor seventh** is a cord that consists of two sounds whose frequencies have a ratio of 9:5, so the period of one the sound-waves is $9/5$ times larger than the other. e.g., $y = \sin(9\pi t)$ and $y = \sin(5\pi t)$ are separated by a minor seventh.

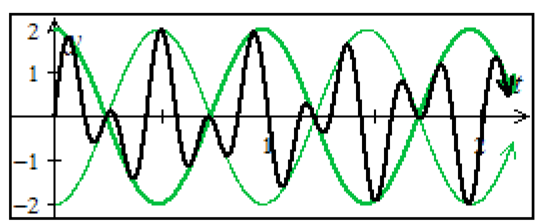


$svn(t) = \sin(9\pi t) + \sin(5\pi t)$

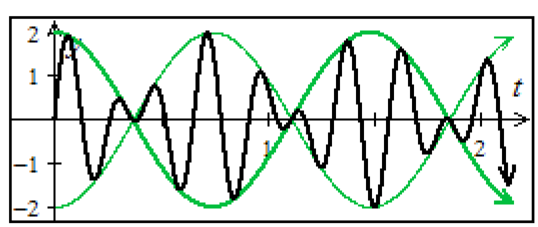
The graphs below represent highly dissonant chords:



$a(t) = \sin(9\pi t) + \sin(2.8\pi t)$ (ratio 9:2.8)

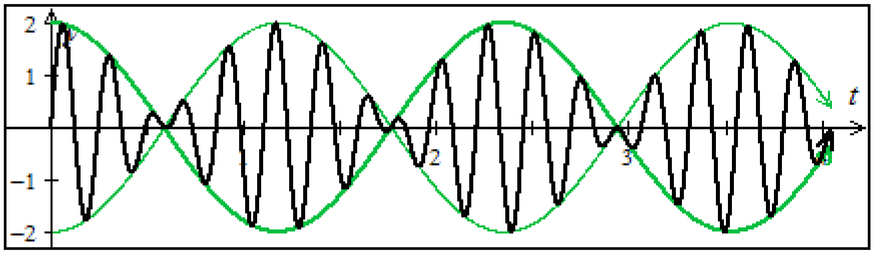


$b(t) = \sin(9\pi t) + \sin(4.9\pi t)$ (ratio: 9:4.9)

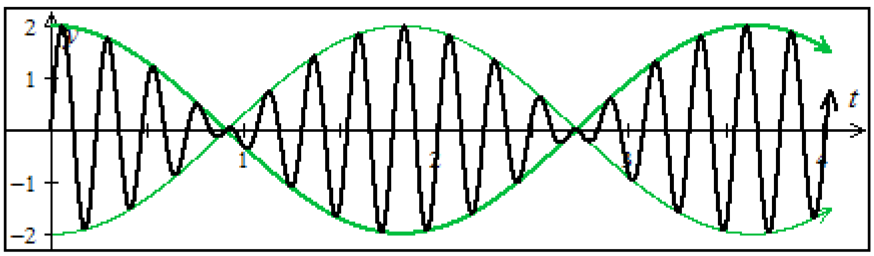


$c(t) = \sin(9\pi t) + \sin(6.3\pi t)$ (ratio: 9:6.3)

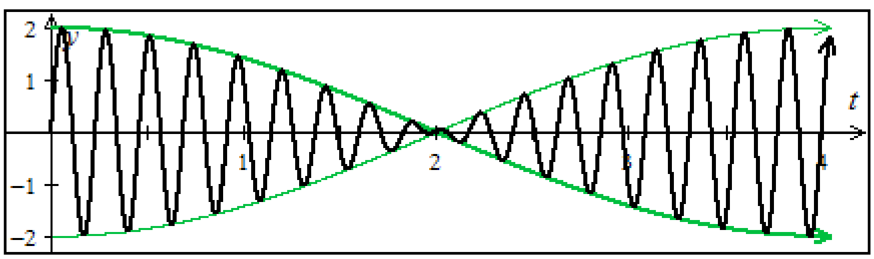
Finally, let's investigate what happens when two sounds have *almost* the same frequency. The graphs below show $F(t) = \sin(9\pi t) + \sin(B\pi t)$ with value of B getting closer and closer to 9.



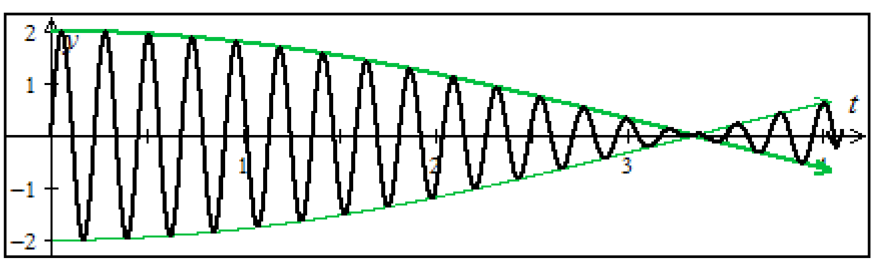
$F(t) = \sin(9\pi t) + \sin(7.3\pi t)$



$F(t) = \sin(9\pi t) + \sin(7.9\pi t)$



$F(t) = \sin(9\pi t) + \sin(8.5\pi t)$



$F(t) = \sin(9\pi t) + \sin(8.7\pi t)$

As the frequencies of the waves get more similar, the length of time between the low-volume instances increases. When these are actual sound waves, the human ear can discern this change in volume and musicians can use the "beats" that they hear to tune their instruments.

We can listen to acoustic beats here:
https://academo.org/demos/wave-interference-beat-frequency/.