$(x)^2 = (4)$

The Laws of Sines and Cosines

We've studied right triangle trigonometry and learned how we can use the sine and cosine functions to obtain information about right triangles. Now we'll study how we can use the sine and cosine functions to obtain information about non-right triangles, i.e., *oblique* triangles. (Figures 1 shows a non-right (oblique) triangle since none of its angles measure 90°.)

The *Laws of Sines and Cosines* are **identities** because they apply to *all* triangles (i.e., right triangles and non-right triangles) but we'll only use them when we're working with non-right triangles since we already have lots of tools for right triangles.

In these class-notes, we'll state the Laws and accept/assume that they are true so that we can use them on a few examples. For their derivations, see the videos linked from the corresponding Online Lecture Notes.

Alternative Version: $\dfrac{a}{\sin(A)} = \dfrac{b}{\sin(B)} = \dfrac{c}{\sin(C)}$

THE LAW OF SINES

If a triangle's sides and angles are labeled like the triangle in Figure 1 then…

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}$$

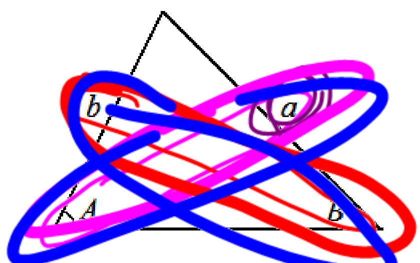


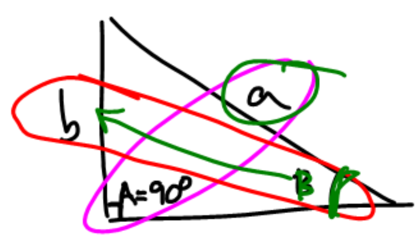
Figure 1

What does the Law of Sines imply when $A = 90^\circ$?

$$\frac{\sin(90^\circ)}{a} = \frac{\sin(B)}{b}$$

$$\Rightarrow b\cdot\frac{1}{a} = \frac{\sin(B)}{\cancel{b}}\cdot \cancel{b} \qquad \sin(B) = \frac{b}{a} = \frac{\text{OPP}}{\text{HYP}}$$

$\boxed{\text{"S.O.H"}}$



THE LAW OF COSINES

If a triangle's sides and angles are labeled like the triangle in Figure 2 then…

$$a^2 = b^2 + c^2 - 2\cdot b\cdot c\cdot \cos(A)$$

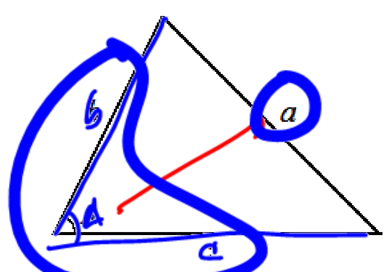


Figure 2

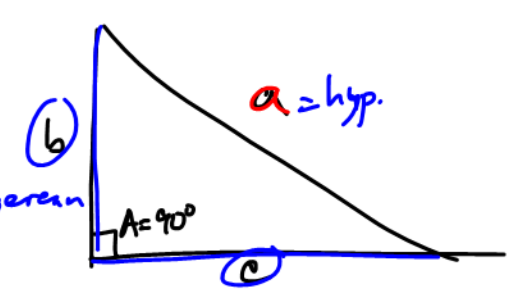
What does the Law of Cosines imply when $A = 90^\circ$?

$$a^2 = b^2 + c^2 - 2\cdot bc\cdot \cos(90^\circ) \qquad (\cos(90^\circ) = 0)$$

$$\Rightarrow a^2 = b^2 + c^2$$

pythagoras.

Moral: Law of Cosines is generalized pythagorean Theorem.



EXAMPLE 1: Find all of the missing angles and side-lengths of the triangle below. (The triangle may not be drawn to scale.)

$$\cancel{\frac{\sin(75^\circ)}{n} = \frac{\sin(55^\circ)}{8}}$$

$$\sin(75^\circ)\cdot\frac{n}{\sin(75^\circ)} = \frac{8}{\sin(55^\circ)}\cdot\sin(75^\circ)$$

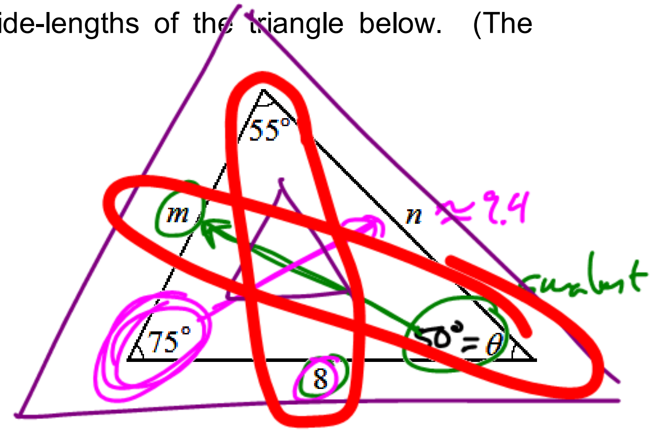
$$n = \frac{8(\sin(75^\circ))}{\sin(55^\circ)}$$

$$n \approx 9.433$$

$$\boxed{\frac{m}{\sin(50^\circ)}} = \boxed{\frac{8}{\sin(55^\circ)}}$$

$$m = \frac{8\sin(50^\circ)}{\sin(55^\circ)}$$

$$\approx 7.48$$



$$\theta + 55^\circ + 75^\circ = 180^\circ$$

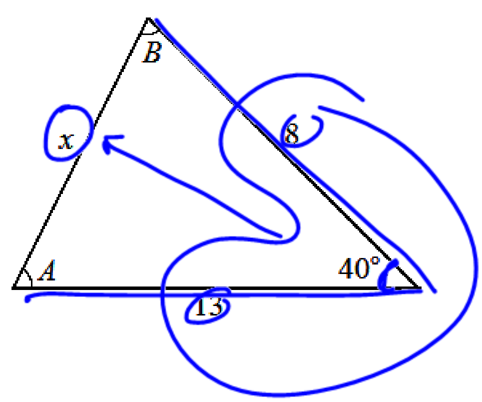
$$\theta = 50^\circ$$

Notice that the Law of Sines involves four parts of a triangle: two angles and the two sides opposite those angles. In order to use the Law of Sines to find a missing part of a triangle, we need to know three of these four parts of a triangle.

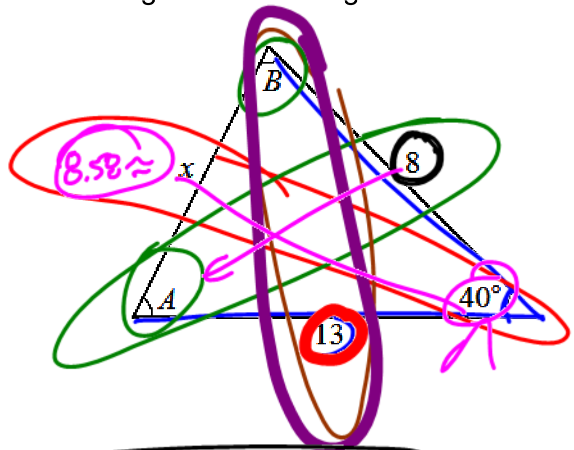
EXAMPLE 2: Find all of the missing angles and side-lengths of the triangle below. (The triangle may not be drawn to scale.)

$$\frac{13}{\sin(B)} = \frac{x}{\sin(40^\circ)}$$

Law of Sines fails



EXAMPLE 2 (CONT): Find all of the missing angles and side-lengths of the triangle below.



Law of Cosines to find x

$$x^2 = 13^2 + 8^2 - 2 \cdot 13 \cdot 8 \cdot \cos(40^\circ)$$

$$x = \sqrt{13^2 + 8 - 2 \cdot 13 \cdot 8 \cos(40^\circ)}$$

$$\approx 8.58$$

$$\frac{\sin(A)}{8} \approx \frac{\sin(40^\circ)}{8.58}$$

$$\sin(A) \approx \frac{8 \sin(40^\circ)}{8.58}$$

$$A \approx \sin^{-1}\left(\frac{8 \sin(40^\circ)}{8.58}\right)$$

$$A \approx 36.8^\circ$$

$$A + B + 40^\circ = 180^\circ$$

$$36.8^\circ + B + 40^\circ \approx 180^\circ$$

$$\boxed{B \approx 103.2^\circ}$$

Dangerous Route

$$\frac{\sin(B_1)}{13} = \frac{\sin(40^\circ)}{x}$$

$$\sin(B_1) \approx \boxed{\frac{13 \sin(40^\circ)}{8.58}}$$

$$B_1 \approx \sin^{-1}\left(\frac{13 \sin(40^\circ)}{8.58}\right)$$

$$\approx 76.89^\circ$$

Recall: $\sin(B) = \sin(180^\circ - B)$

$$B \approx 180^\circ - B_1$$

$$\approx 180^\circ - 76.89^\circ$$

$$= 103.11^\circ$$

When using the Law of Sines to find an angle, always **find the smaller angle first** and then use the "180 degree rule" to find the larger angle.