$2^{-1} = \frac{1}{2}$

Inverse Trig Functions

As we studied in MTH 111, the inverse of a function reverses the roles of the inputs and the outputs. (For more info on inverse functions, check out my MTH 111 Online Lecture Notes.)

Suppose that f and f^{-1} are inverses. If $f(a) = b$, then $f^{-1}(b) = a$.

Inverse functions are extremely valuable since they "undo" one another and allow us to solve equations. For example, we can solve the equation $x^3 = 10$ by using the inverse of the cubing function, the cube-root function, to "undo" the cubing involved in the equation:

$$\sqrt[3]{x^3} = \sqrt[3]{10}$$

$$x = \sqrt[3]{10}$$

The cubing function has an inverse function because it's **one-to-one**, which means that each output value corresponds to exactly one input value (e.g., the only number with a cube of 8 is 2) – this will allow us to reverse the roles of the inputs and outputs and still have a function. Let's use the graphs below of $y = x^2$ and $y = x^3$ to review one-to-one functions vs. not one-to-one functions.

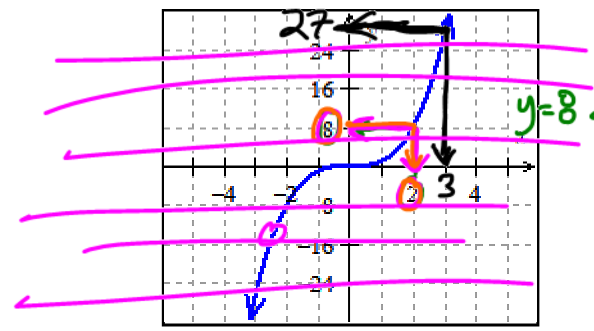


Figure 1: $y = x^3$

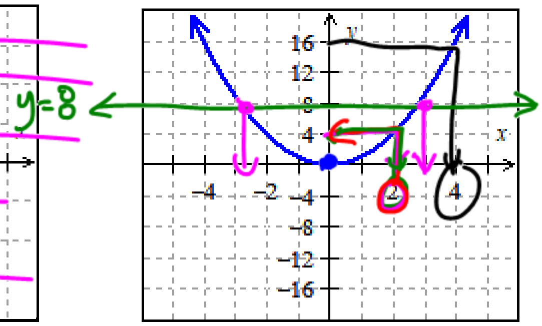


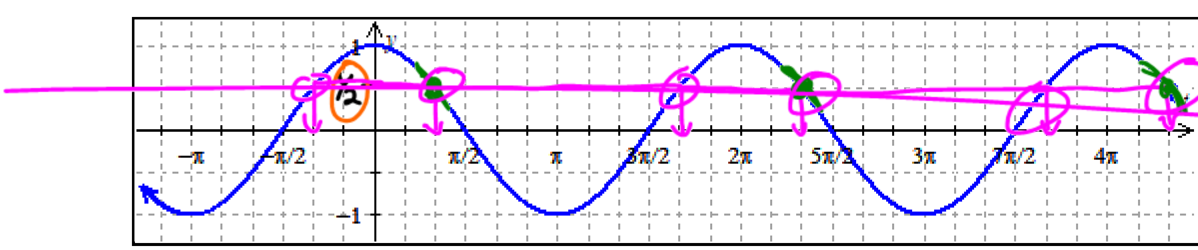
Figure 2: $y = x^2$

$y = x^2$ is "two to one"

$\sqrt{x^2} = \pm\sqrt{16}$

$x = 4$ or $x = -4$

Unfortunately, the trig functions aren't one-to-one so, in their natural form, they don't have inverse functions. For example, consider the output $\frac{1}{2}$ for the cosine function: this output corresponds to the inputs $-\frac{\pi}{3}$, $\frac{\pi}{3}$, $\frac{5\pi}{3}$, $\frac{7\pi}{3}$, etc.; see Figure 3.



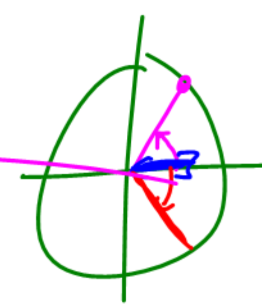
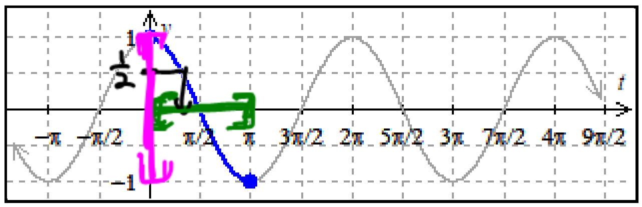


Figure 3: The graph of $y = \cos(t)$.

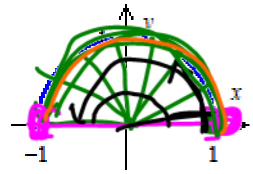
"infinitely many to one"

Since inverse functions are so valuable, we *really* want inverse trig functions, so we need to **restrict the domains** of the functions to intervals on which they are one-to-one, and then we can construct inverse functions.

Let's start by constructing the inverse of the cosine function.



A graph of $y = \cos(t)$.



DEFINITION: The **inverse cosine function**, $y = \cos^{-1}(t)$, is defined by:

If $0 \le y \le \pi$ and $\cos(y) = t$, then $y = \cos^{-1}(t)$.

The *domain* of $y = \cos^{-1}(t)$ is $[-1, 1]$ (which is the range of the cosine function) and the *range* of $y = \cos^{-1}(t)$ is $[0, \pi]$.

This function is often called **arccosine** and is expressed as $y = \arccos(t) = \cos^{-1}(t)$

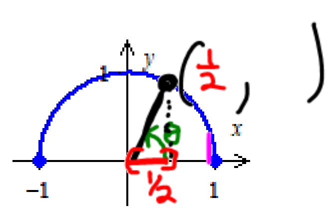
Key Point: Inverse Notation vs. Exponent Notation:

$\boxed{\cos^{-1}(x)}$ not an exponent

$(\cos(x))^{-1} = \dfrac{1}{\cos(t)}$

$= \sec(t)$

EXAMPLE: Evaluate $\cos^{-1}\left(\frac{1}{2}\right)$



Suppose $\cos^{-1}\left(\frac{1}{2}\right) = \theta$

$\Rightarrow$ $\boxed{\cos(\theta) = \frac{1}{2}}$

$\Rightarrow$ $\theta = \frac{\pi}{3}$

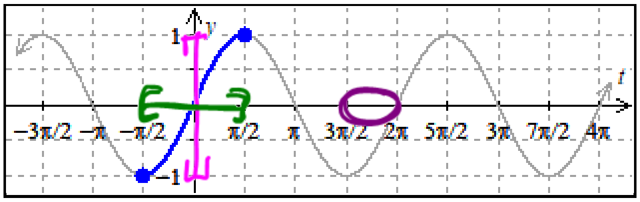
$\boxed{\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}}$ because $\boxed{\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}}$

AND $0 \le \frac{\pi}{3} \le \pi$

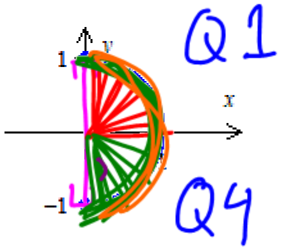
$\sqrt{x^2} = \pm\sqrt{4}$

$x = \pm 2$

Now we'll construct the inverse of the sine function.



A graph of $y = \sin(t)$.



DEFINITION: The **inverse sine function**, $y = \sin^{-1}(t)$, is defined by the following:

$$\text{If } -\frac{\pi}{2} \le y \le \frac{\pi}{2} \text{ and } \boxed{\sin(y) = t}, \text{ then } y = \sin^{-1}(t).$$

The *domain* of $y = \sin^{-1}(t)$ is $[-1, 1]$ (which is the range of the sine function) and the *range* of $y = \sin^{-1}(t)$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

This function is often called the **arcsine** and is expressed as $y = \arcsin(t)$.

EXAMPLE. Evaluate $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$.

Let $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = \theta$

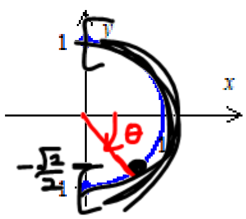
Then $\sin(\theta) = -\frac{\sqrt{2}}{2}$

$\theta = -\frac{\pi}{4}$

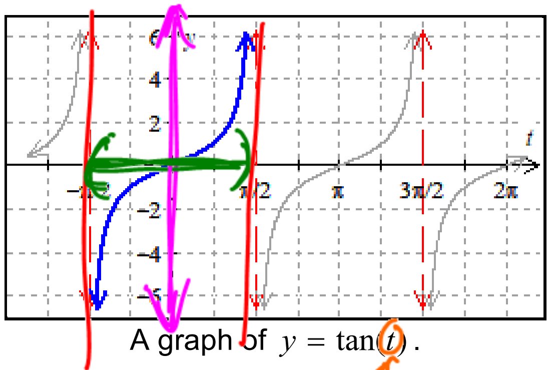
$\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$ since $\sin\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

& $-\frac{\pi}{2} \le -\frac{\pi}{4} \le \frac{\pi}{2}$

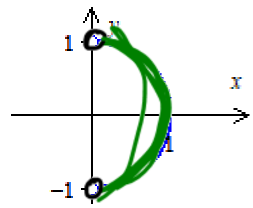
$\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$



Now let's define the inverse tangent function.



A graph of $y = \tan(t)$.



DEFINITION: The **inverse tangent function**, $y = \tan^{-1}(t)$, is defined by:

$$\text{If } -\frac{\pi}{2} < y < \frac{\pi}{2} \text{ and } \tan(y) = t, \text{ then } y = \tan^{-1}(t).$$

The *domain* of $y = \tan^{-1}(t)$ is $\mathbb{R} = (-\infty, \infty)$ (which is the range of the tangent function) and the *range* of $y = \tan^{-1}(t)$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. This function is often called the **arctangent** and is expressed as $y = \arctan(t)$.

EXAMPLE: Evaluate $\boxed{\tan^{-1}\left(-\sqrt{3}\right)}$.

Let $\tan^{-1}(-\sqrt{3}) = \theta$

Then $\tan(\theta) = -\sqrt{3}$

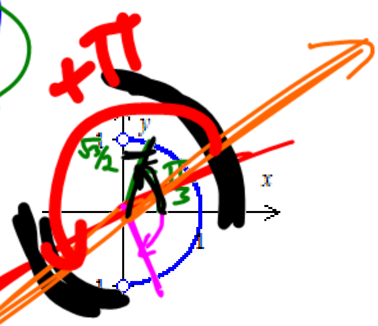
$$\sqrt{3} = \frac{-\sqrt{3}/2}{1/2} = \frac{\sin(\theta)}{\cos(\theta)} = -\sqrt{3}$$

$\sin(\theta) = -\frac{\sqrt{3}}{2}$

$\theta = -\frac{\pi}{3}$

$\frac{\sqrt{3}}{2} > \frac{1}{2}$

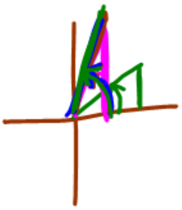
$\frac{\pi}{6}$ or $\frac{\pi}{3}$



$$\tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$$

Check:

$$\tan\left(-\frac{\pi}{3}\right) = \frac{\sin(-\frac{\pi}{3})}{\cos(-\frac{\pi}{3})} = \frac{-\sqrt{3}/2}{+1/2} = -\sqrt{3} \checkmark$$



EXAMPLE: Evaluate the following expressions.

a. $\sin\left(\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)$.

$$\sin\left(\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right) = \sin\left(\frac{\pi}{3}\right)$$
$$= \frac{\sqrt{3}}{2}$$

$\frac{\sqrt{3}}{2} > \frac{1}{2}$

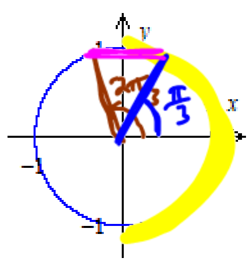
$\frac{\pi}{6}$ & $\frac{\pi}{3}$

b. $\sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)$.

$$\sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right) = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$
$$= \frac{\pi}{3}$$

c. $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$.

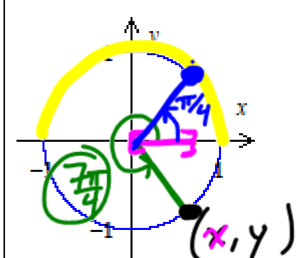
$$\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right) = \sin^{-1}\left(+\frac{\sqrt{3}}{2}\right)$$
$$= \frac{\pi}{3}$$



d. $\cos^{-1}\left(\cos\left(\frac{7\pi}{4}\right)\right)$.

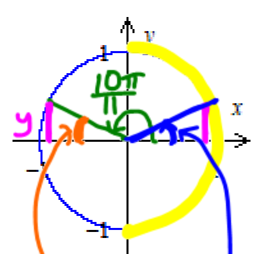
$$\cos^{-1}\left(\cos\left(\frac{7\pi}{4}\right)\right) = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$$
$$= \frac{\pi}{4}$$

$$\cos^{-1}\left(\cos\left(\frac{7\pi}{4}\right)\right) = \frac{\pi}{4}$$



e. $\sin^{-1}\left(\sin\left(\frac{10\pi}{11}\right)\right)$.

$$\sin^{-1}\left(\underbrace{\sin\left(\frac{10\pi}{11}\right)}_{y}\right) = \frac{\pi}{11}$$



$$\frac{10\pi}{11} = \frac{11\pi}{11} - \frac{1\pi}{11}$$
$$= \pi - \frac{\pi}{11}$$