Inverse Trig Functions

As we studied in MTH 111, the inverse of a function reverses the roles of the inputs and the outputs. (For more info on inverse functions, check out my MTH 111 Online Lecture Notes.)

Suppose that f and f^{-1} are inverses. If $f(a)=b$, then ______________.

Inverse functions are extremely valuable since they "undo" one another and allow us to solve equations. For example, we can solve the equation $x^3=10$ by using the inverse of the cubing function, the cube-root function, to "undo" the cubing involved in the equation:

The cubing function has an inverse function because it's **one-to-one**, which means that each output value corresponds to exactly one input value (e.g., the only number with a cube of 8 is 2) – this will allow us to reverse the roles of the inputs and outputs and still have a function. Let's use the graphs below of $y=x^2$ and $y=x^3$ to review one-to-one functions vs. not one-to-one functions.

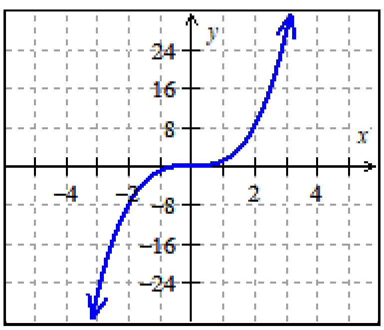


Figure 1: $y=x^3$

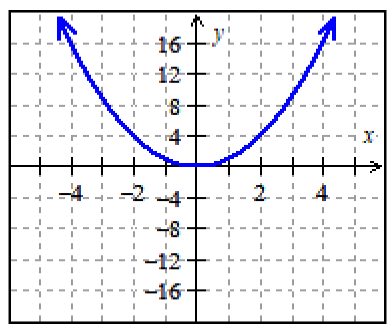


Figure 2: $y=x^2$

Unfortunately, the trig functions aren't one-to-one so, in their natural form, they don't have inverse functions. For example, consider the output $\frac{1}{2}$ for the cosine function: this output corresponds to the inputs $-\frac{\pi}{3}$, $\frac{\pi}{3}$, $\frac{5\pi}{3}$, $\frac{7\pi}{3}$, etc.; see Figure 3.

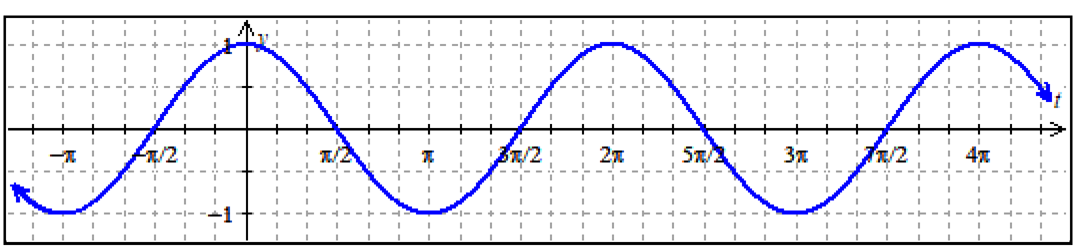
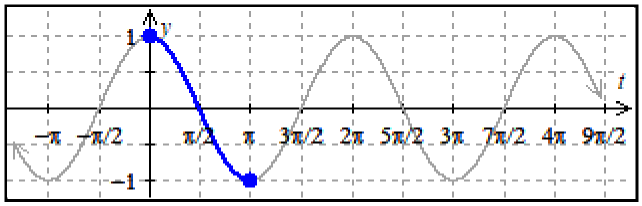


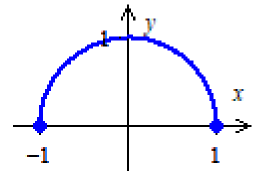
Figure 3: The graph of $y=\cos(t)$.

Since inverse functions are so valuable, we *really* want inverse trig functions, so we need to **restrict the domains** of the functions to intervals on which they are one-to-one, and then we can construct inverse functions.

Let's start by constructing the inverse of the cosine function.



A graph of $y = \cos(t)$.



DEFINITION: The **inverse cosine function**, $y = \cos^{-1}(t)$, is defined by:

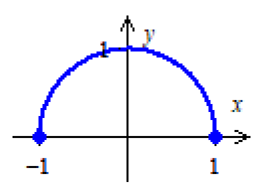
$$\text{If } 0 \le y \le \pi \text{ and } \cos(y) = t \text{, then } y = \cos^{-1}(t).$$

The *domain* of $y = \cos^{-1}(t)$ is ____________________ (which is the range of the cosine function) and the *range* of $y = \cos^{-1}(t)$ is ____________________.

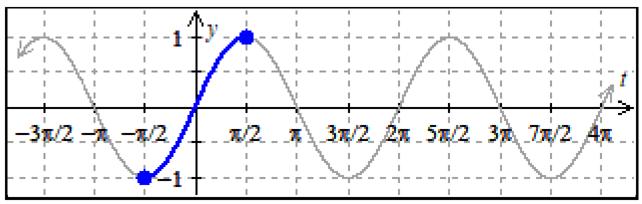
This function is often called **arccosine** and is expressed as $y = \arccos(t)$.

Key Point: Inverse Notation vs. Exponent Notation:

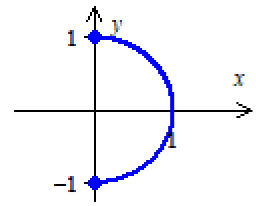
EXAMPLE: Evaluate $\cos^{-1}\left(\frac{1}{2}\right)$.



Now we'll construct the inverse of the sine function.



A graph of $y = \sin(t)$.



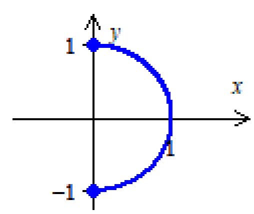
DEFINITION: The **inverse sine function**, $y = \sin^{-1}(t)$, is defined by the following:

$$\text{If } -\tfrac{\pi}{2} \le y \le \tfrac{\pi}{2} \text{ and } \sin(y) = t\text{, then } y = \sin^{-1}(t).$$

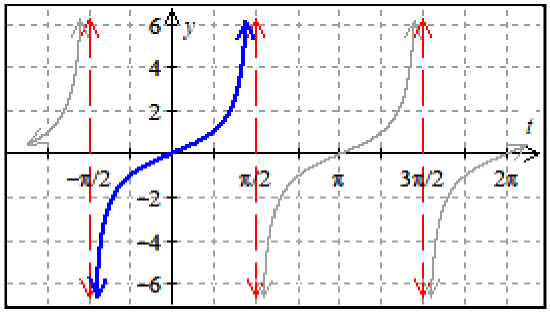
The *domain* of $y = \sin^{-1}(t)$ is ____________________ (which is the range of the sine function) and the *range* of $y = \sin^{-1}(t)$ is ____________________.

This function is often called the **arcsine** and is expressed as $y = \arcsin(t)$.

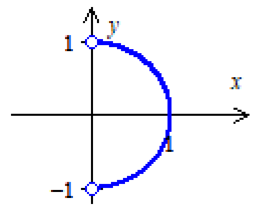
EXAMPLE: Evaluate $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$.



Now let's define the inverse tangent function.



A graph of $y = \tan(t)$.

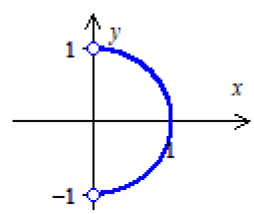


DEFINITION: The **inverse tangent function**, $y = \tan^{-1}(t)$, is defined by:

$$\text{If } -\frac{\pi}{2} < y < \frac{\pi}{2} \text{ and } \tan(y) = t \text{, then } y = \tan^{-1}(t).$$

The *domain* of $y = \tan^{-1}(t)$ is _____________________ (which is the range of the tangent function) and the *range* of $y = \tan^{-1}(t)$ is ______________. This function is often called the **arctangent** and is expressed as $y = \arctan(t)$.

EXAMPLE: Evaluate $\tan^{-1}\left(-\sqrt{3}\right)$.

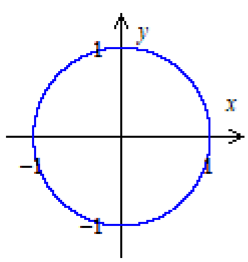


EXAMPLE: Evaluate the following expressions.

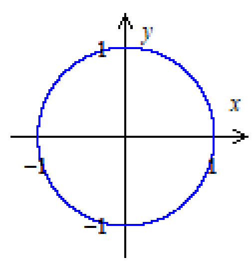
a. $\sin\left(\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)$.

b. $\sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)$.

c. $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$.



d. $\cos^{-1}\left(\cos\left(\frac{7\pi}{4}\right)\right)$.



e. $\sin^{-1}\left(\sin\left(\frac{10\pi}{11}\right)\right)$.