$2^{-1} = \frac{1}{2}$

Inverse Trig Functions and Solving Trig Equations

As we studied in MTH 111, the inverse of a function reverses the roles of the inputs and the outputs. (For more info on inverse functions, check out my MTH 111 Online Lecture Notes.)

Suppose that f and f^{-1} are inverses. If $f(a) = b$, then $f^{-1}(b) = a$.

Inverse functions are extremely valuable since they "undo" one another and allow us to solve equations. For example, we can solve the equation $x^3 = 10$ by using the inverse of the cubing function, the cube-root function, to "undo" the cubing involved in the equation:

$$\sqrt[3]{x^3} = \sqrt[3]{10}$$

$$x = \sqrt[3]{10}$$

The cubing function has an inverse function because it's **one-to-one**, which means that each output value corresponds to exactly one input value (e.g., the only number with a cube of 8 is 2) – this will allow us to reverse the roles of the inputs and outputs and still have a function. Let's use the graphs below of $y = x^2$ and $y = x^3$ to review one-to-one functions vs. not one-to-one functions.

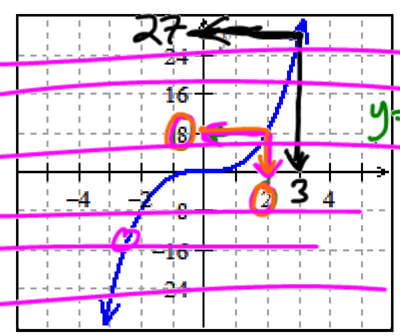


Figure 1: $y = x^3$

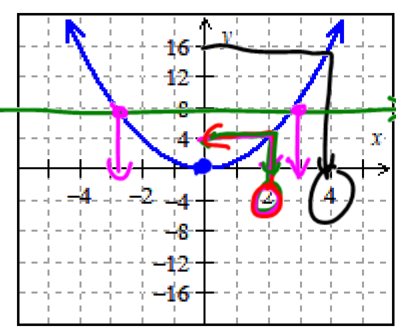


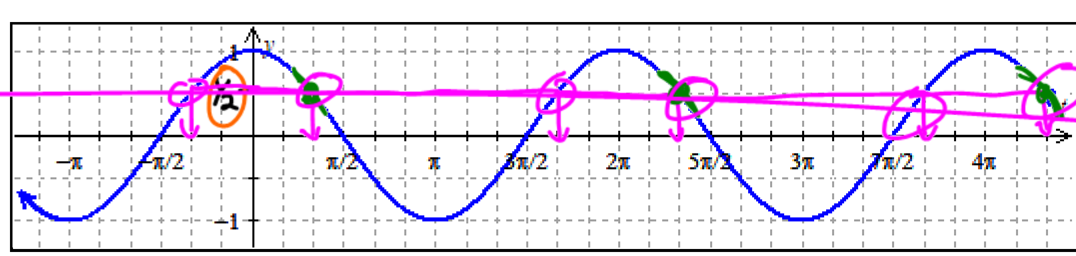
Figure 2: $y = x^2$

$y = x^2$ is "two to one"

$\sqrt{x^2} = \pm\sqrt{16}$

$x = 4$ OR $x = -4$

Unfortunately, the trig functions aren't one-to-one so, in their natural form, they don't have inverse functions. For example, consider the output $\frac{1}{2}$ for the cosine function: this output corresponds to the inputs $-\frac{\pi}{3}$, $\frac{\pi}{3}$, $\frac{5\pi}{3}$, $\frac{7\pi}{3}$, etc.; see Figure 3.



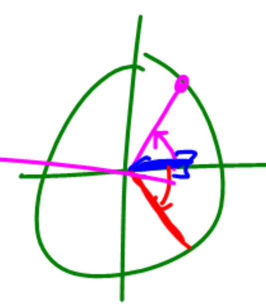
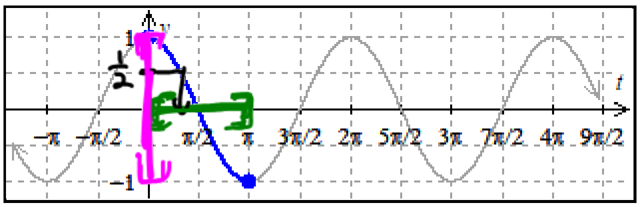


Figure 3: The graph of $y = \cos(t)$.

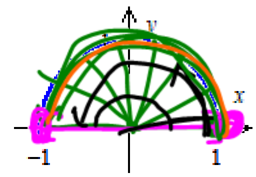
"infinitely many to one"

Since inverse functions are so valuable, we *really* want inverse trig functions, so we need to **restrict the domains** of the functions to intervals on which they are one-to-one, and then we can construct inverse functions.

Let's start by constructing the inverse of the cosine function.



A graph of $y = \cos(t)$.



DEFINITION: The **inverse cosine function**, $y = \cos^{-1}(t)$, is defined by:

$$\text{If } 0 \le y \le \pi \text{ and } \cos(y) = t, \text{ then } y = \cos^{-1}(t).$$

The *domain* of $y = \cos^{-1}(t)$ is $[-1, 1]$ (which is the range of the cosine function) and the *range* of $y = \cos^{-1}(t)$ is $[0, \pi)$.

This function is often called **arccosine** and is expressed as $y = \arccos(t) = \cos^{-1}(t)$

Key Point: Inverse Notation vs. Exponent Notation:

$\boxed{\cos^{-1}(x)}$ not an exponent

$(\cos(x))^{-1} = \dfrac{1}{\cos(t)}$

$= \sec(t)$

EXAMPLE: Evaluate $\boxed{\cos^{-1}\left(\frac{1}{2}\right)}$

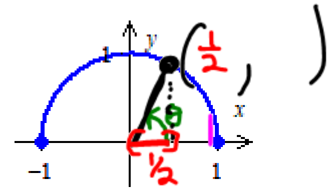
Suppose $\cos^{-1}\left(\frac{1}{2}\right) = \theta$

$\Rightarrow$ $\boxed{\cos(\theta) = \frac{1}{2}}$

$\Rightarrow$ $\theta = \frac{\pi}{3}$

$\boxed{\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}}$ because $\boxed{\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}}$

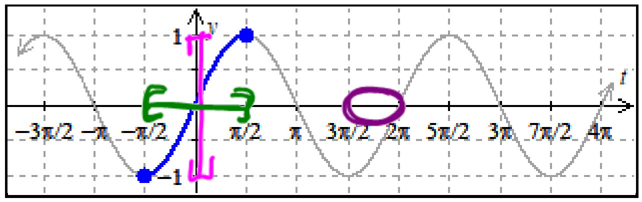
AND $0 \le \frac{\pi}{3} \le \pi$



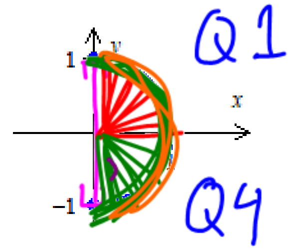
$\sqrt{x^2} = \pm\sqrt{4}$

$x = \pm 2$

Now we'll construct the inverse of the sine function.



A graph of $y = \sin(t)$.



DEFINITION: The **inverse sine function**, $y = \sin^{-1}(t)$, is defined by the following:

$$\text{If } -\frac{\pi}{2} \le y \le \frac{\pi}{2} \text{ and } \boxed{\sin(y) = t}, \text{ then } y = \sin^{-1}(t).$$

The *domain* of $y = \sin^{-1}(t)$ is $[-1, 1]$ (which is the range of the sine function) and the *range* of $y = \sin^{-1}(t)$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

This function is often called the **arcsine** and is expressed as $y = \arcsin(t)$.

EXAMPLE. Evaluate $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$.

Let $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = \theta$

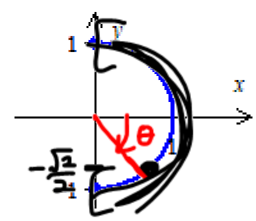
Then $\sin(\theta) = -\frac{\sqrt{2}}{2}$

$\theta = -\frac{\pi}{4}$

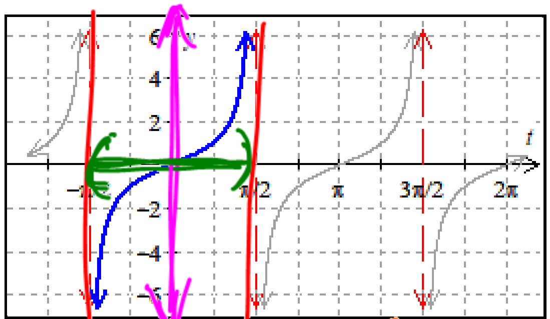
$\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$ since $\sin\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

& $-\frac{\pi}{2} \le -\frac{\pi}{4} \le \frac{\pi}{2}$

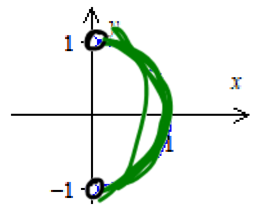
$\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$



Now let's define the inverse tangent function.



A graph of $y = \tan(t)$.



DEFINITION: The **inverse tangent function**, $y = \tan^{-1}(t)$, is defined by:

$$\text{If } -\frac{\pi}{2} < y < \frac{\pi}{2} \text{ and } \tan(y) = t \text{ then } y = \tan^{-1}(t).$$

The *domain* of $y = \tan^{-1}(t)$ is $\mathbb{R} = (-\infty, \infty)$ (which is the range of the tangent function) and the *range* of $y = \tan^{-1}(t)$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. This function is often called the **arctangent** and is expressed as $y = \arctan(t)$.

EXAMPLE: Evaluate $\boxed{\tan^{-1}\left(-\sqrt{3}\right)}$.

Let $\tan^{-1}(-\sqrt{3}) = \theta$

Then $\tan(\theta) = -\sqrt{3}$

$$\sqrt{3} = \frac{-\sqrt{3}/2}{1/2} = \frac{\sin(\theta)}{\cos(\theta)} = -\sqrt{3}$$

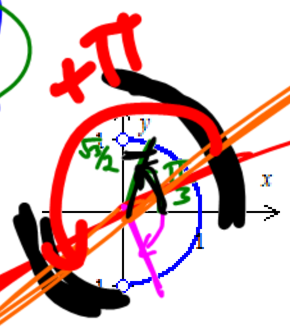
$\sin(\theta) = \frac{-\sqrt{3}}{2}$

$\theta = \frac{-\pi}{3}$

$\frac{\sqrt{3}}{2} > \frac{1}{2}$

$\boxed{\frac{\pi}{6} \text{ or } \frac{\pi}{3}}$

$+\pi$



$$\tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$$

Check: $\tan\left(-\frac{\pi}{3}\right) = \frac{\sin(-\frac{\pi}{3})}{\cos(-\frac{\pi}{3})}$

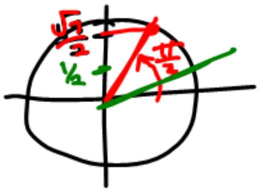
$= \frac{-\sqrt{3}/\cancel{2}}{+1/\cancel{2}}$

$= -\sqrt{3}$ ✓

EXAMPLE: Evaluate the following expressions.

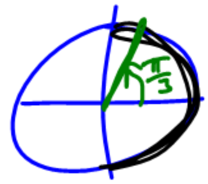
a. $\sin\left(\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)$.

$$\sin\left(\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right) = \sin\left(\frac{\pi}{3}\right)$$
$$= \frac{\sqrt{3}}{2}$$



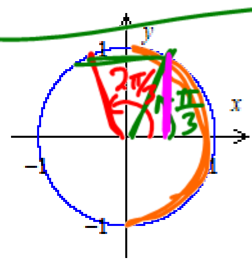
b. $\sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)$.

$$\sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right) = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$
$$= \frac{\pi}{3}$$



c. $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$.

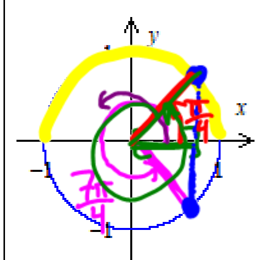
$$\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right) = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$
$$= \frac{\pi}{3}$$



d. $\cos^{-1}\left(\cos\left(\frac{7\pi}{4}\right)\right)$.

$$\cos^{-1}\left(\cos\left(\frac{7\pi}{4}\right)\right) = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$$
$$= \frac{\pi}{4}$$

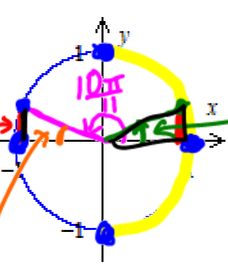
$$\cos^{-1}\left(\cos\left(\frac{7\pi}{4}\right)\right) = \boxed{\frac{\pi}{4}}$$



e. $\sin^{-1}\left(\sin\left(\frac{10\pi}{11}\right)\right)$.

$$\sin^{-1}\left(\sin\left(\frac{10\pi}{11}\right)\right) = \frac{\pi}{11}$$

$\sin\left(\frac{10\pi}{11}\right)$



$\frac{\pi}{11}$

$$\boxed{\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}}$$

Friendly Angles

$0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$, & multiples

$\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$ & multiples

Friendly Sine/cosine Values

$0, 1, -1$

$\pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2}, \pm\frac{\sqrt{2}}{2}$

$\sqrt[3]{x^3} = \sqrt[3]{10}$

$x = \sqrt[3]{10}$

vs

$x^3 = 8$

$\Rightarrow x = 2$

Solving Trig Equations

EXAMPLE: Find all of the solutions to the equation $8\cos(t) + 3 = -1$.

$8\cos(t) + 3 = -1$ — isolate trig function

$8\cos(t) = -4$

$\cos(t) = \frac{-4}{8} = -\frac{1}{2}$

$\cos^{-1}(\cos(t)) = \cos^{-1}\left(-\frac{1}{2}\right)$

$t = \frac{2\pi}{3} + 2k\pi$ OR $t = -\frac{2\pi}{3} + 2k\pi$

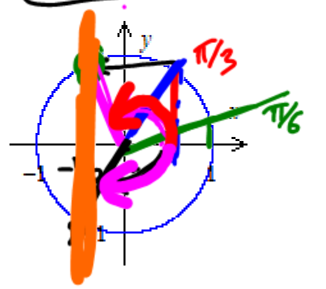
for $k \in \mathbb{Z}$

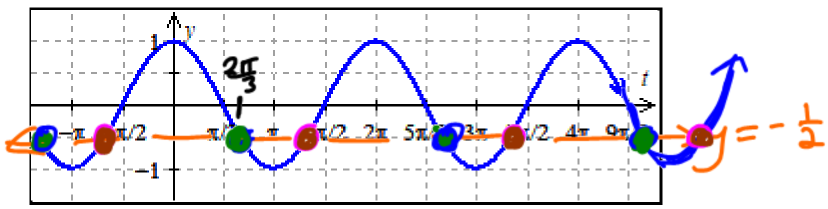
set of integers (whole numbers)

"is an element of"

Identity

$\cos(\theta) = \cos(-\theta)$





A graph of $y = \cos(t)$.

EXAMPLE: Find all of the solutions to the equation $4\sin(t) + 2\sqrt{3} = 0$.

$4\sin(t) + 2\sqrt{3} = 0$

$4\sin(t) = -2\sqrt{3}$

$\sin(t) = -\frac{2\sqrt{3}}{4} = -\frac{\sqrt{3}}{2}$

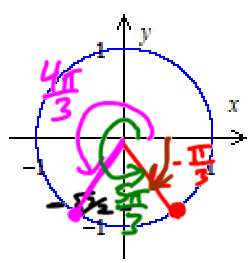
$\sin^{-1}(\sin(t)) = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

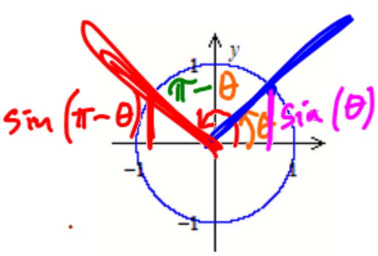
$t = -\frac{\pi}{3} + 2k\pi$ OR $t = \pi - \left(-\frac{\pi}{3}\right) + 2k\pi$

$= \frac{4\pi}{3} + 2k\pi$

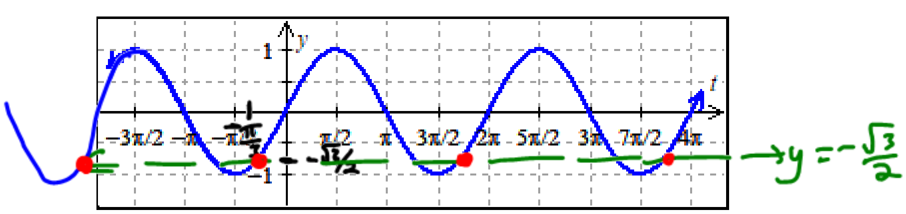
for $k \in \mathbb{Z}$

$2\pi \cdot k = 2k\pi$





$\sin(\theta) = \sin(\pi - \theta)$



A graph of $y = \sin(t)$.

$y = \cos(3t)$ has period: $P = 2\pi \cdot \frac{1}{3} = \frac{2\pi}{3}$

EXAMPLE: First find *all* of the solutions to the equation $2\sqrt{2}\cos(3t) = -2$. Then find the particular solutions on the interval $[0, 2\pi)$.

$[0, \pi)$

$2\sqrt{2}\cos(3t) = -2$

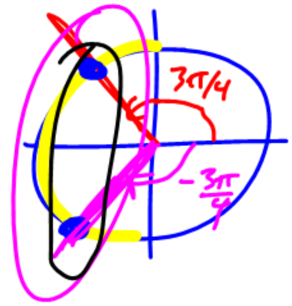
$\cos(3t) = -\frac{2}{2\sqrt{2}} = -\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$

$\cos(3t) = -\frac{\sqrt{2}}{2}$

$\cos^{-1}(\cos(3t)) = \cos^{-1}\left(-\sqrt{2}/2\right)$

$\frac{3t}{3} = \frac{3\pi/4}{3} + \frac{2k\pi}{3}$ OR $\frac{3t}{3} = \frac{-3\pi/4}{3} + \frac{2k\pi}{3}$, $k \in \mathbb{Z}$

$\boxed{t = \frac{\pi}{4} + \frac{2\pi}{3} \cdot k \quad \text{OR} \quad t = -\frac{\pi}{4} + \frac{2k\pi}{3}, \; k \in \mathbb{Z}}$



Find solutions on $[0, 2\pi) = [0, \frac{24\pi}{12})$

$k=0$: $t = \frac{\pi}{4} + \frac{2 \cdot 0 \cdot \pi}{3} = \frac{\pi}{4}$ ✓ OR $t = -\frac{\pi}{4} + \frac{2 \cdot 0 \cdot \pi}{3} = -\frac{\pi}{4}$ X

$k=1$: $t = \frac{\pi}{4} + \frac{2 \cdot 1 \pi}{3} = \frac{3\pi}{12} + \frac{8\pi}{12} = \frac{11\pi}{12}$ ✓ ; $t = -\frac{\pi}{4} + \frac{2 \cdot 1 \pi}{3} = -\frac{3\pi}{12} + \frac{8\pi}{12} = \frac{5\pi}{12}$ ✓

$k=2$: $t = \frac{\pi}{4} + \boxed{\frac{2 \cdot 2\pi}{3}} = \frac{3\pi}{12} + \frac{16\pi}{12} = \frac{19\pi}{12}$ ✓ ; $t = -\frac{\pi}{4} + \boxed{\frac{2 \cdot 2\pi}{3}} = -\frac{3\pi}{12} + \frac{16\pi}{12} = \frac{13\pi}{12}$ ✓

$k=3$: $t = \frac{\pi}{4} + \frac{2 \cdot 3\pi}{3} = \frac{3\pi}{12} + \frac{24\pi}{12} = \frac{27\pi}{12} > \frac{24\pi}{12} = 2\pi$ X ; $t = -\frac{\pi}{4} + \frac{2 \cdot \cancel{3}\pi}{\cancel{3}} = \frac{7\pi}{4}$ ✓

$k=4$: NOT WORTH TRYING ; $t = -\frac{\pi}{4} + \frac{2 \cdot 4\pi}{3} = -\frac{3\pi}{12} + \frac{32\pi}{12} = \frac{35\pi}{12} > 2\pi$ X

Solution set is: $\left\{\frac{\pi}{4}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{7\pi}{4}\right\}$

$k = -1$: $t = \frac{\pi}{4} + \frac{2(-1)\pi}{3}$ $\qquad t < 0$ ×

$= \frac{3\pi}{12} + \frac{-8\pi}{12}$

$= \frac{-5\pi}{12} < 0$ ×