Introduction to Vectors

Vectors are mathematical objects used to represent physical quantities like velocity, force, and displacement. Unlike ordinary numbers (or **scalars**), vectors have *both* magnitude and direction. So, for example, we can use a vector to describe the velocity of an object (i.e., the speed *and* direction).

> **DEFINITION:** A **vector** is a mathematical object that has both a *magnitude* (i.e., size) and a *direction*.

In order to distinguish between vectors from scalars (i.e., numbers) we need to use a different notation to denote vectors. In this class, we will use a small arrow above the vector name to denote a vector, so that $\vec{v}$ and $\vec{s}$ represent vectors while v and s represent scalars.

In this class we will focus on **two-dimensional vectors**. A two-dimensional vector can be represented by an **arrow** on the coordinate plane. The **length** of the arrow represents the **magnitude** of the vector and the **direction** of the arrow represents the direction of the vector. (We traditionally use the **angle between the positive x-axis and the arrow** to describe the **direction** of the vector.)

EXAMPLE 1: The vector $\vec{v}$ is depicted as an arrow on the coordinate plane in Figure 1.

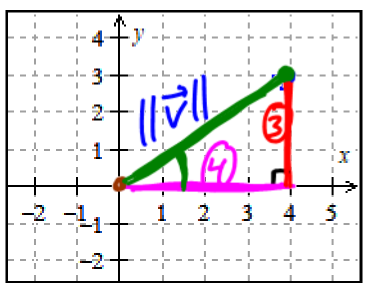

Figure 1: Arrow representing vector $\vec{v}$.

The **tip** of the vector is where the arrow ends and the **tail** of the vector is where the arrow begins. Thus, the tip of $\vec{v}$ is at the point $(4, 3)$ and the tail of the vector is at the origin, $(0, 0)$.

As mentioned above, the **length** of the arrow represents the **magnitude** of the vector. We denote the magnitude of vector $\vec{v}$ by $\|\vec{v}\|$. To find the magnitude of $\vec{v}$, we need to find the length of the arrow; we can do this by thinking of the arrow as being the hypotenuse of a right-triangle with side lengths 4 and 3 and then use the Pythagorean Theorem to find $\|\vec{v}\|$:

$$\|\vec{v}\| = \sqrt{4^2 + 3^2} = \sqrt{16+9} = 5$$

We can find the angle between the positive x-axis and the arrow to describe the **direction** of the vector. We've denoted this angle by θ in Figure 2.

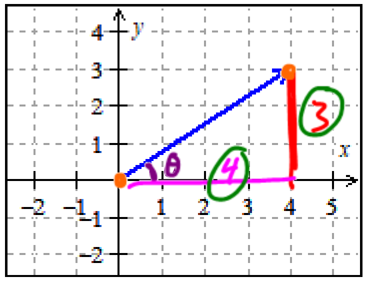

Figure 2

We can use the trigonometry that we studied earlier this quarter to find θ:

$\tan(\theta) = \frac{3}{4}$

$\theta = \tan^{-1}\left(\frac{3}{4}\right)$

$\approx 36.9°$

Although the magnitude and direction of the vector describe it completely, it is often useful to describe a vector by using its **horizontal and vertical components**. The *horizontal component* of $\vec{v}$ in Figure 2 is 4 units and a *vertical component* of vector $\vec{v}$ is 3 units. Thus, we say that the **component form of vector** $\vec{v}$ is $\langle 4, 3\rangle$.

$\vec{v} = \langle 4, 3 \rangle$

It's important to recognize that we could translate this vector anywhere in the coordinate plane and it would still be the same vector. For example, all of the arrows in Figure 3 represent $\vec{v}$ since all of these vectors have a horizontal component of 4 units and a vertical component of 3 units.

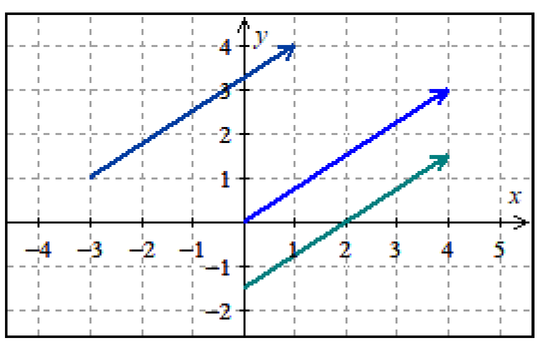

Figure 3: Three copies of $\vec{v}$.

Vector Operations

We can **multiply any vector by a scalar** (i.e., a number) and we can **add or subtract any two vectors**.

When we **multiply a vector by a scalar**, we simply multiply the respective components of the vector by the scalar. Thus, if $\vec{a} = \langle a_1, a_2 \rangle$ and $k \in \mathbb{R}$, then $k\vec{a} = \langle ka_1, ka_2 \rangle$.

EXAMPLE 2: Let $\vec{v} = \langle 4, 3 \rangle$ (from Example 1). Find and draw vectors $\vec{m} = 2\vec{v}$ and $\vec{n} = -2\vec{v}$.

$\vec{m} = 2 \cdot \vec{v}$
$= 2 \cdot \langle 4, 3 \rangle$
$= \langle 2 \cdot 4, 2 \cdot 3 \rangle$
$= \langle 8, 6 \rangle$

$\vec{n} = -2 \cdot \vec{v}$
$= -2 \cdot \langle 4, 3 \rangle$
$= \langle -2 \cdot 4, -2 \cdot 3 \rangle$
$= \langle -8, -6 \rangle$

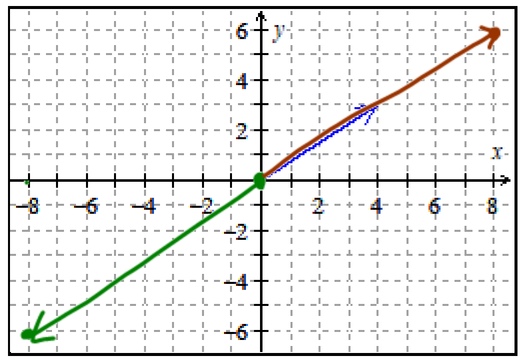

Figure 4: Vector $\vec{v}$.

> If $\vec{a} = \langle a_1, a_2 \rangle$ is a vector and $k \in \mathbb{R}$ then $k\vec{a} = \langle ka_1, ka_2 \rangle$ has magnitude $|k| \cdot \|\vec{a}\|$. If $k > 0$ then $k \cdot \vec{a}$ points in the same direction as $\vec{a}$; if $k < 0$ then $k \cdot \vec{a}$ points in the opposite direction as $\vec{a}$.

EXAMPLE 3: Suppose that the vector $\vec{m}$ makes an angle of 37° with respect to the positive x-axis and $\|\vec{m}\| = 20$. Represent $\vec{m}$ in component form.

$\vec{m} = \langle x, y \rangle$
$= \langle 20 \cdot \cos(37^\circ), 20 \cdot \sin(37^\circ) \rangle$
$\approx \langle 15.97, 12.04 \rangle$
$\approx 15.97\vec{i} + 12.04\vec{j}$

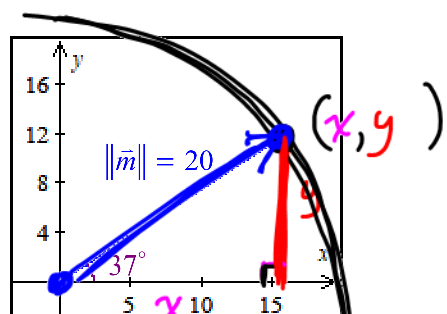

Figure 5: Vector $\vec{m}$

> In general, if vector $\vec{v}$ makes an angle θ with the positive x-axis then, in component form,
> $\vec{v} = \langle \|\vec{v}\| \cos(\theta), \|\vec{v}\| \cdot \sin(\theta) \rangle$

When we **add**, we simply add the respective components of the vectors. Thus, if $\vec{a} = \langle a_1, a_2 \rangle$ and $\vec{b} = \langle b_1, b_2 \rangle$, then $\vec{a} + \vec{b} = \langle a_1 + b_1, a_2 + b_2 \rangle$ and $\vec{a} - \vec{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$

EXAMPLE 4: Let $\vec{v} = \langle 4, 3 \rangle$ (from Example 1) and $\vec{s} = \langle 2, -6 \rangle$. Find $\vec{v} + \vec{s}$.

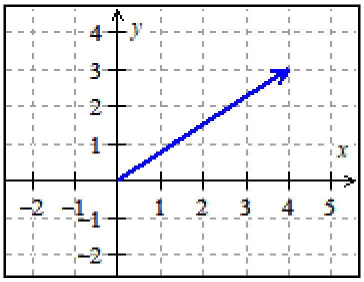

Figure 6: Vector $\vec{v}$.

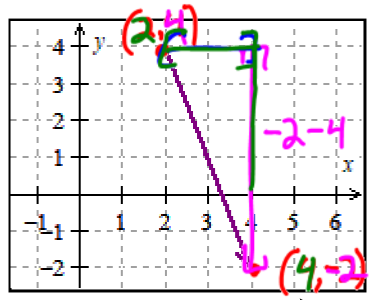

Figure 7: Vector $\vec{s}$.

$\vec{s} = \langle 4-2, -2-4 \rangle$
$= \langle +2, -6 \rangle$

Let's find $\vec{v} + \vec{s}$:

$\vec{v} + \vec{s} = \langle 4, 3 \rangle + \langle 2, -6 \rangle$
$= \langle 4+2, 3+(-6) \rangle$
$= \langle 6, -3 \rangle$

We can also add vectors by using arrows on a coordinate plane:

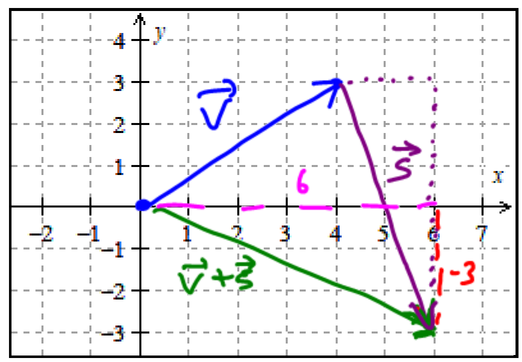

Figure 8: Adding vectors graphically.

Add $\vec{s}$ to $\vec{v}$ by attaching tail of $\vec{s}$ to tip of $\vec{v}$ & connect tail of $\vec{v}$ to tip of $\vec{s}$

Properties of Vector Addition and Scalar Multiplication

If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors and a and b are scalars (i.e., $a, b \in \mathbb{R}$) then the following properties hold true:

1. **Commutativity of Vector Addition:** $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
2. **Associativity of Vector Addition:** $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
3. **Associativity of Scalar Multiplication:** $a(b\vec{v}) = (ab)\vec{v}$
4. **Distributivity:** $(a + b)\vec{v} = a\vec{v} + b\vec{v}$ and $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$
5. **Identities:** $\vec{v} + \vec{0} = \vec{v}$ and $1 \cdot \vec{v} = \vec{v}$

In order to facilitate the communication and manipulation of vectors, it is useful to consider **unit vectors**.

> **DEFINITION:** A **unit vector** is a vector whose magnitude is 1 unit. So if $\vec{a}$ is a unit vector then $\|\vec{a}\| = 1$.

The **standard unit vectors** are the unit vectors that point in the horizontal and vertical directions.

($\vec{k}$ is usually the third famous unit vector)

> **DEFINITION:**
>
> The vector $\vec{i}$ is the unit vector that points in the **positive horizontal direction**. Since its horizontal component is 1 and its vertical component is 0, we see that $\vec{i} = \langle 1, 0 \rangle$.
>
> The vector $\vec{j}$ is the unit vector that points in the **positive vertical direction**. Since its horizontal component is 0 and its vertical component is 1, we see that $\vec{j} = \langle 0, 1 \rangle$.
>
> Note that since they are *unit vectors*, $\|\vec{i}\| = 1$ and $\|\vec{j}\| = 1$.

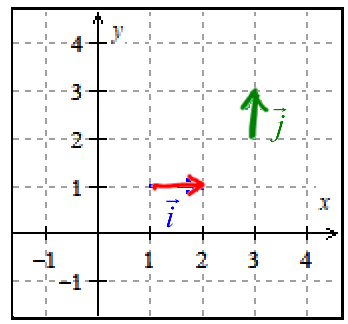

Figure 9: Unit vectors $\vec{i}$ and $\vec{j}$.

We can use vectors $\vec{i}$ and $\vec{j}$ to describe all other two-dimensional vectors. For example, we can describe $\vec{v} = \langle 4, 3 \rangle$ (from Example 1) using vectors $\vec{i}$ and $\vec{j}$: along with scalar multiplication and vector addition:

$$\vec{v} = \langle 4, 3 \rangle$$
$$= \langle 4, 0 \rangle + \langle 0, 3 \rangle$$
$$= 4 \cdot \langle 1, 0 \rangle + 3 \cdot \langle 0, 1 \rangle$$
$$= 4 \cdot \vec{i} + 3\vec{j}$$

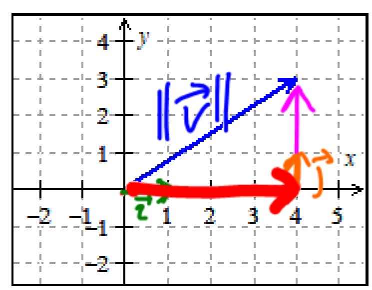

Figure 10: Vector $\vec{v}$.

> In general, if $\vec{a} = \langle a_1, a_2 \rangle$ is a vector, then $a_1\vec{i} + a_2\vec{j}$.