Intro to the Trigonometric Functions: Part 1

(these notes correspond to Parts 1 & 2 from my MTH 112 Online Lecture Notes)

As we noticed in the Ferris wheel example, a circle rotating about its center lends itself naturally to the study of periodic functions. In fact, the two most important *trigonometric functions* are defined in terms of a unit circle: the **sine** and **cosine** functions. (Note that a **unit circle** is a circle with a radius of 1 unit; see Figure 1.)

DEFINITIONS:

The **sine function**, denoted $\sin(\theta)$, associates each angle θ with the vertical coordinate (i.e.. the y-coordinate) of the point P specified by the angle θ on the circumference of a unit circle.

The **cosine function**, denoted $\cos(\theta)$, associates each angle θ with the horizontal coordinate (i.e., the x-coordinate) of the point P specified by the angle θ on the circumference of a unit circle.

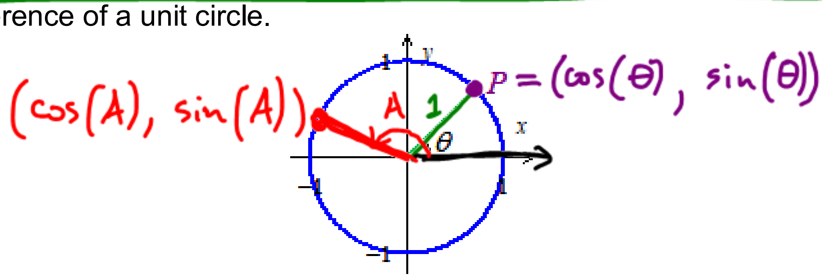


Figure 1: A Unit Circle

There are four other trigonometric functions. These four functions are defined in terms of the sine and cosine functions so first let's get familiar with sine and cosine. In order to enhance our understanding of the sine and cosine functions, we should determine some particular values for the functions and sketch their graphs. The easiest values for us to find are represented by the points where the unit circle intersects the coordinate axes. Use the graph in Figure 2 to complete Table 1. (Keep in mind that cosine represents the horizontal coordinate and sine represents the vertical coordinate.)

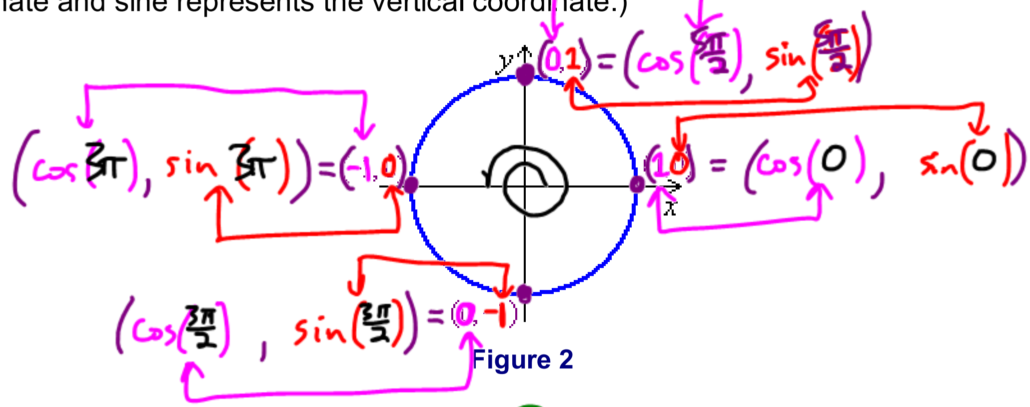
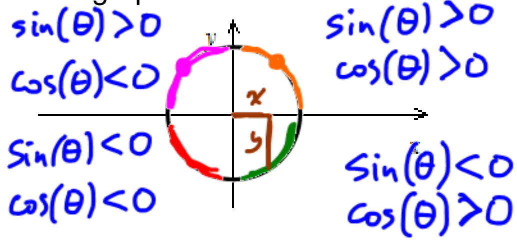


Figure 2

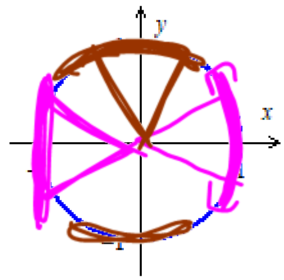
θ (degrees)	0°	90°	180°	270°	360°	450°	540°
θ (radians)	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π
$\cos(\theta)$	1	0	-1	0	1	0	-1
$\sin(\theta)$	0	1	0	-1	0	1	0

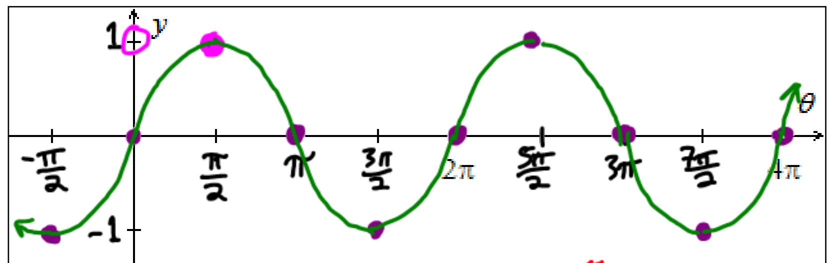
Table 1

Let's determine the signs (positive or negative) of the sine and cosine functions in the four different quadrants of the coordinate plane. Notice that our analysis of the signs of the sine and cosine functions agree with the graphs of the sine and cosine functions.

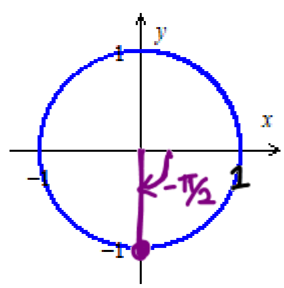


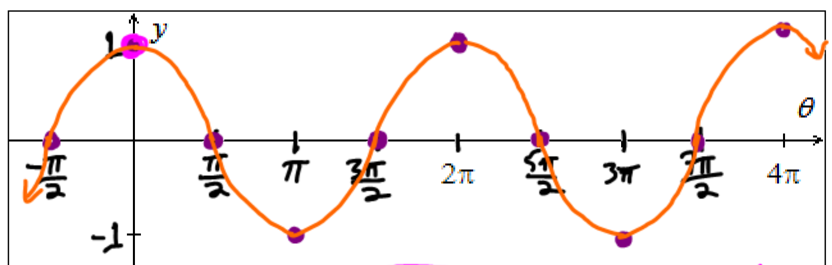
Now, let's plot on coordinate planes the info from Table 1 and sketch graphs of the sine and cosine functions. Be sure to add **scale** to the y-axis and label *all* of the "tics" on the θ-axis.





The graph of $y = \sin(\theta)$. ← "y-coords on unit circle"





The graph of $y = \cos(\theta)$. ← x-coords on unit circle.

Notice that angles of measure 2π radians (i.e., $\theta = 360^\circ$) and 0 radians specify the same point on the unit circle: $(1, 0)$. Thus, the sine and cosine values for 2π radians and 0 radians are the same. In general, θ and $\theta + 2\pi$ specify the *same* point on the unit circle, so the sine and cosine values of θ and $\theta + 2\pi$ are the same. Thus, **the period of the sine and cosine functions is 2π radians**.

For all θ, $\sin(\theta) = \sin(\theta + 2\pi)$ and $\cos(\theta) = \cos(\theta + 2\pi)$.

[Let's use Desmos to confirm that we have correct graphs and to discuss what happens when we graph these functions with θ in degrees instead of radians.

$x^2 = 4$

$x = 2$ OR $x = -2$

Notice that the graphs of $y = \sin(\theta)$ and $y = \cos(\theta)$ are very similar. In fact, if we shift the graph of $y = \sin(\theta)$ .... left $\frac{\pi}{2}$ units it becomes $y = \cos(\theta)$.

$$\boxed{\sin\left(\theta + \frac{\pi}{2}\right) = \cos(\theta)}$$

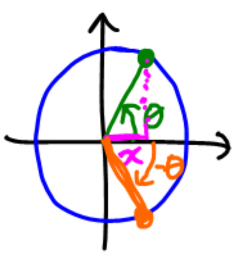
This is equation is known as an **identity** since the left and right sides of the equation are *always* identical, no matter what value of θ is used.

DEFINITION: An **identity** is an equation that is true for all values in the domains of the involved expressions.

Earlier in these notes we observed a couple of identities but didn't call them identities. The equations

$$\boxed{\sin(\theta) = \sin(\theta + 2\pi) \quad \text{and} \quad \cos(\theta) = \cos(\theta + 2\pi)}$$

are identities since they are true for *all* values of θ. We can use the definitions and graphs of sine and cosine to determine a few other important identities. (You should spend some time using what you learned in MTH 111 about graph transformations and symmetry to make sense of WHY these identities are true.)

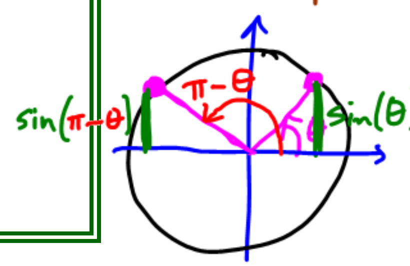


$y = \cos(-\theta)$

reflect about y-axis

SOME IMPORTANT TRIG IDENTITIES

- $\cos(\theta) = \cos(\theta + 2\pi)$
- $\sin(\theta) = \sin(\theta + 2\pi)$
- $\cos(\theta) = \sin\left(\theta + \frac{\pi}{2}\right)$
- $\sin(\theta) = \cos\left(\theta - \frac{\pi}{2}\right)$
- $\boxed{\cos(-\theta) = \cos(\theta)}$
- $\sin(-\theta) = -\sin(\theta)$
- $\boxed{\cos(\theta) = \cos(2\pi - \theta)}$
- $\sin(\theta) = \sin(\pi - \theta)$



We'll study more identities in these lecture notes, and even more over the next few weeks.

Recall the Pythagorean Theorem from your previous math course-work:

THE PYTHAGOREAN THEOREM:

If the sides of a right triangle (i.e., a triangle with a 90° angle) are labeled like the one given in Figure 3, then $a^2 + b^2 = c^2$.

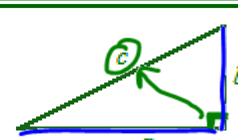


Figure 3

We can use the Pythagorean Theorem along with the definitions of sine and cosine to derive another important identity. In Figure 4, notice how the definitions of sine and cosine naturally lead us to a right triangle with side-lengths $\sin(\theta)$, $\cos(\theta)$, and 1.

$a + b = b + c$

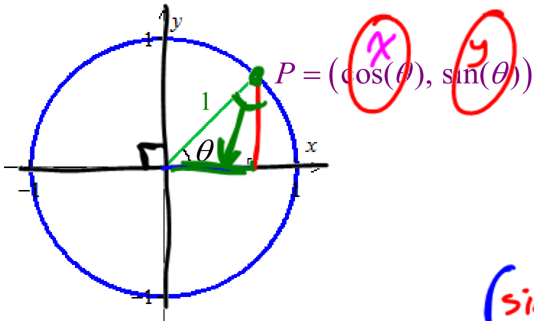
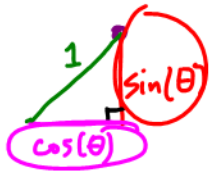


Figure 4



$$(\sin(\theta))^2 + (\cos(\theta))^2 = 1^2$$

The Pythagorean Identity: $\sin^2(\theta) + \cos^2(\theta) = 1$

We can generalize the definitions of the sine and cosine functions so that they are applicable to circles of any size, rather than only for unit circles.

DEFINITION: If the point $T = (x, y)$ is specified by the angle θ on the circumference of a circle of radius r as shown in Figure 5 below, then

$\cos(\theta) = \frac{x}{r}$ $\sin(\theta) = \frac{y}{r}$

Notice that if $r = 1$ then this definition $\cos(\theta)$ and $\sin(\theta)$ is equivalent we saw at the beginning of this chapter: $\cos(\theta) = \frac{x}{1}$ $\sin(\theta) = \frac{y}{1}$

$\Rightarrow \cos(\theta) = x$ $\sin(\theta) = y$

If we solve the equations $\cos(\theta) = \dfrac{x}{r}$ and $\sin(\theta) = \dfrac{y}{r}$ for x and y, respectively, we can obtain the coordinates of a point on the circumference of a circle of any r:

If the point $T = (x, y)$ is specified by the angle θ on the circumference of a circle of radius, r, then

$x = r\cos(\theta)$ & $y = r\sin(\theta)$,

Let's add this information to Figure 5.

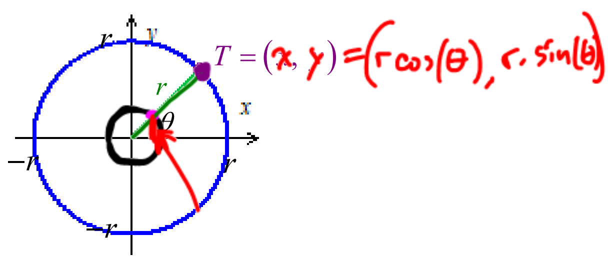


Figure 5

As mentioned at the beginning of these notes, there are four other trigonometric functions. These four functions are defined in terms of the sine and cosine functions:

DEFINITIONS: The **tangent function**, denoted $\tan(\theta)$, is defined by $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$

reciprocal

The **cotangent function**, denoted $\cot(\theta)$, is defined by $\cot(\theta) = \frac{1}{\tan(\theta)}$.

Consequently: $\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$

The **secant function**, denoted $\sec(\theta)$, is defined by $\sec(\theta) = \frac{1}{\cos(\theta)}$.

The **cosecant function**, denoted $\csc(\theta)$, is defined by $\csc(\theta) = \frac{1}{\sin(\theta)}$

We can use the Pythagorean Identity to derive two identities that involve these so-called "other trig functions":

$$\boxed{\sin^2(\theta) + \cos^2(\theta) = 1}$$

$$\frac{\sin^2(\theta)}{\cos^2(\theta)} + \frac{\cos^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)}$$

$$\Rightarrow \boxed{\tan^2(\theta) + 1 = \sec^2(\theta)}$$

$$\frac{\sin^2(\theta)}{\sin^2(\theta)} + \frac{\cos^2(\theta)}{\sin^2(\theta)} = \frac{1}{\sin^2(\theta)}$$

$$\Rightarrow \boxed{1 + \cot^2(\theta) = \csc^2(\theta)}$$

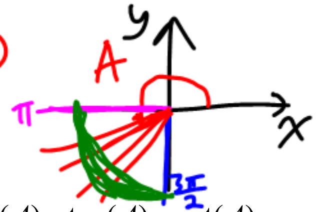
The Pythagorean Identities

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

$$\tan^2(\theta) + 1 = \sec^2(\theta)$$

$$1 + \cot^2(\theta) = \csc^2(\theta)$$

A is in Quad 3



EXAMPLE: If $\cos(A) = -\frac{5}{7}$ and $\pi < A < \frac{3\pi}{2}$, find $\sin(A)$, $\tan(A)$, $\cot(A)$, $\sec(A)$, and $\csc(A)$.

Use pythagoras to find sin(A):

$$\boxed{\sin^2(A) + \cos^2(A) = 1}$$

$$\sin^2(A) + \left(-\frac{5}{7}\right)^2 = 1$$

$$\sin^2(A) + \frac{25}{49} = 1$$

$$\sin^2(A) = \frac{49}{49} - \frac{25}{49}$$

$$\sqrt{\sin^2(A)} = -\sqrt{\frac{24}{49}}$$

$$\sin(A) = -\frac{\sqrt{24}}{7} = \frac{-2\sqrt{6}}{7}$$

$$\tan(A) = \frac{\sin(A)}{\cos(A)} = \frac{-2\sqrt{6}/\cancel{7}}{-5/\cancel{7}} = -\frac{2\sqrt{6}}{\cancel{7}} \cdot \left(-\frac{\cancel{7}}{5}\right) = \frac{2\sqrt{6}}{5}$$

$$\cot(A) = \frac{1}{\tan(A)} = \frac{5}{2\sqrt{6}}$$

$$\sec(A) = \frac{1}{\cos(A)} = \frac{1}{-5/7} = -\frac{7}{5}$$

$$\csc(A) = \frac{1}{\sin(A)} = \frac{1}{-2\sqrt{6}/7} = -\frac{7}{2\sqrt{6}}$$