Introduction to Polar Coordinates

We are all comfortable using rectangular (i.e., Cartesian) coordinates to describe points on the plane. In Figure 1 let's plot the point $P = \left(\sqrt{3}, 1\right)$ on the rectangular coordinate plane:

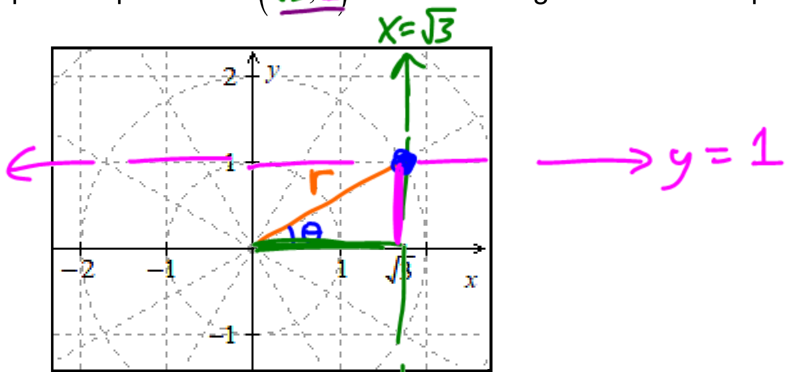
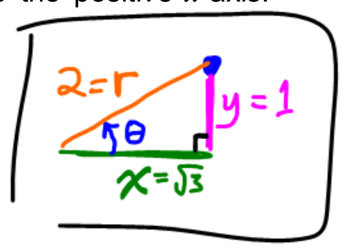


Figure 1

Instead of using these *rectangular* coordinates, we can use a *circular* coordinate system to describe points on the plane, i.e., we can use the **polar coordinate system**. Ordered pairs in polar coordinates have form (r, θ) where r represents the point's distance from the origin and θ represents the angular displacement of the point with respect to the positive x-axis. Let's find the polar coordinates that describe $P = \left(\sqrt{3}, 1\right)$.



Use pythagoras to find r:

$$r = \sqrt{(\sqrt{3})^2 + (1)^2} = \sqrt{3+1} = 2$$

$$\tan(\theta) = \frac{1}{\sqrt{3}}$$

$$\boxed{\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \pi/6}$$

Therefore, polar $\left(2, \frac{\pi}{6}\right)$ is equivalent to rectangular $(\sqrt{3}, 1)$.

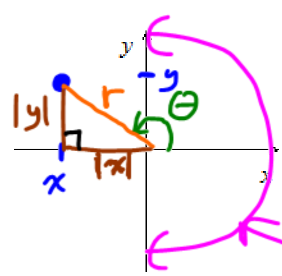
Note: $(\sqrt{3}, 1) \neq \left(2, \frac{\pi}{6}\right)$ $\quad (x, y) = (\sqrt{3}, 1)$

The rectangular coordinates (x, y) are equivalent to the (r, θ) polar coordinates such that:

$$r = \sqrt{x^2 + y^2}$$

&

$$\tan(\theta) = \frac{y}{x} \Rightarrow \cancel{\theta = \tan^{-1}\left(\frac{y}{x}\right)}$$



we don't solve for θ since we may need to adjust for quadrant.

EXAMPLE: Plot the point $A = \left(10, \frac{5\pi}{4}\right)$ on the polar coordinate plane below and determine the rectangular coordinates of point A.

r → 10, θ → $\frac{5\pi}{4}$

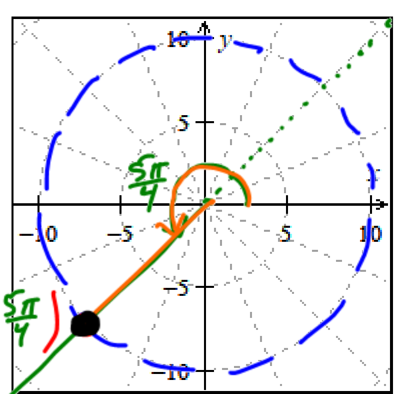
Point is on circumference of radius 10 circle & specified by $\frac{5\pi}{4}$:

$$A = \left(10 \cdot \cos\left(\tfrac{5\pi}{4}\right), 10 \cdot \sin\left(\tfrac{5\pi}{4}\right)\right)$$

$$= \left(10 \cdot \left(-\tfrac{\sqrt{2}}{2}\right), 10 \cdot \left(-\tfrac{\sqrt{2}}{2}\right)\right)$$

$$= \left(-5\sqrt{2}, -5\sqrt{2}\right)$$

$(x, y) = (2, 3)$ $\quad (x, y)_{rect} = (r, \theta)_{polar}$

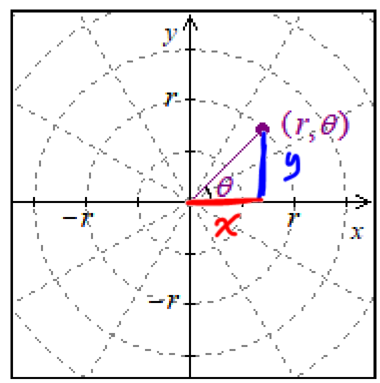


Plot $A = \left(10, \frac{5\pi}{4}\right)$.

The polar coordinates (r, θ) are equivalent to the following rectangular coordinates:

$x = r \cdot \cos(\theta)$ & $y = r \cdot \sin(\theta)$

So polar (r, θ) is equivalent to rectangular $(r \cdot \cos(\theta), r \cdot \sin(\theta))$



What happens if $r < 0$?

$\pi \approx 3.14$

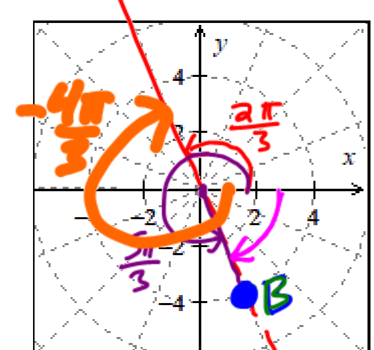
EXAMPLE: Plot the point $B = \left(-4, \frac{2\pi}{3}\right)$ on the polar coordinate plane below and list a few other ordered pairs that are plotted at the same location.

$\left(-4, \frac{2\pi}{3}\right)$, $\left(-4, -\frac{4\pi}{3}\right)$

$\left(4, \frac{5\pi}{3}\right)$

$\left(4, -\frac{\pi}{3}\right)$

coterminal angle makes new points with same location



Plot $B = \left(-4, \frac{2\pi}{3}\right)$.

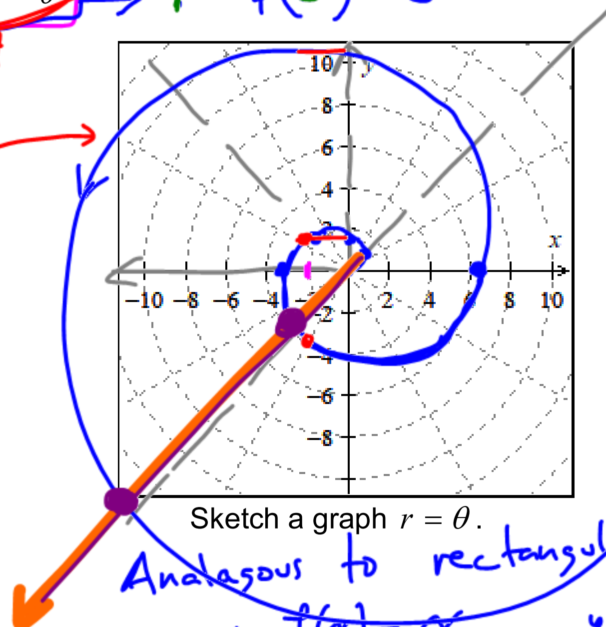
Graphing Polar Functions

Now let's graph polar functions, i.e., functions that involve polar coordinates. These functions will have form $\boxed{r = f(\theta)}$ so the input, θ, is an angle and the output, r, is the distance from the origin. **Notice that in a polar ordered pair, (r, θ), the output variable, r, is the first coordinate and the *input* variable, θ, is the second coordinate** which is a **different order** than rectangular ordered pairs of the form (x, y) in which the *input* variable is the first coordinate and the *output* variable is the second coordinate.

EXAMPLE: Sketch a graph of the polar function $r = \theta$ $\Rightarrow$ $r = f(\theta) = \theta$

θ	r	(r, θ)
0	0	$(0,0)$
$\frac{\pi}{4}$	$\frac{\pi}{4} \approx \frac{3}{4}$	$\left(\frac{3}{4}, \frac{\pi}{4}\right)$
$\frac{\pi}{2}$	$\frac{\pi}{2} \approx 1.5$	$\left(1.5, \frac{\pi}{2}\right)$
$\frac{3\pi}{4}$	$\frac{3\pi}{4} \approx \frac{9}{4}$	$\left(\frac{9}{4}, \frac{3\pi}{4}\right)$
π	$\pi \approx 3$	$(3, \pi)$
$\frac{5\pi}{4}$	$\frac{5\pi}{4} \approx \frac{15}{4}$	$\left(\frac{15}{4}, \frac{5\pi}{4}\right)$
2π	$2\pi \approx 6$	$(6, 2\pi)$

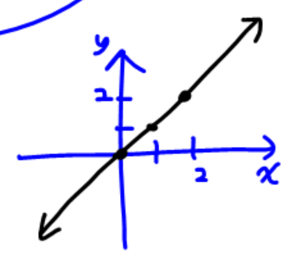
Archimedean spiral



Sketch a graph $r = \theta$.

Analagous to rectangular: $y = f(x) = x$

x	y	(x,y)
0	0	(0,0)
1	1	(1,1)
2	2	(2,2)



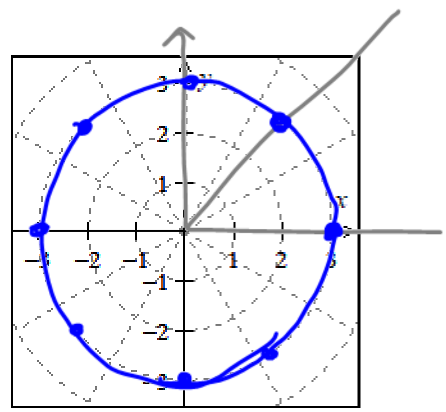
EXAMPLE: Sketch a graph of the polar function $\boxed{r = 3.}$

θ	$r=3$
0	3
$\frac{\pi}{4}$	3
$\frac{\pi}{2}$	3
π	3

Circle of radius 3

rectangular: $x^2 + y^2 = 9$

polar: $r = 3$

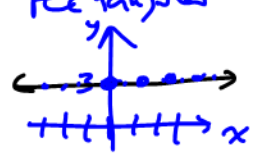


Sketch a graph $r = 3$.

x	y
0	3
1	3
2	3

Analagous to rectangular

$y = 3$



EXAMPLE: Sketch a graph of the $\boxed{r = 2\sin(2\theta)}$ on the polar coordinate plane.

To sketch the graph of $r = 2\sin(2\theta)$, let's find some ordered pairs that satisfy the function.

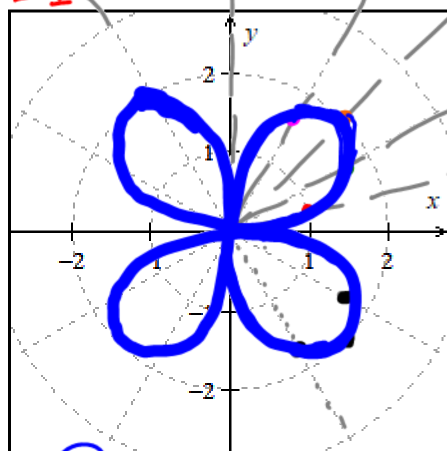
θ	$r = 2\sin(2\theta)$	(r, θ)
0	$\boxed{0}$	$(0,0)$
$\frac{\pi}{12}$	$\boxed{1}$	$(1, \frac{\pi}{12})$
$\frac{\pi}{6}$	$\boxed{\sqrt{3} \approx 1.7}$	$(1.7, \frac{\pi}{6})$
$\frac{\pi}{4}$	$\boxed{2}$	$(2, \frac{\pi}{4})$
$\frac{\pi}{3}$	$\boxed{\sqrt{3} \approx 1.7}$	$(1.7, \frac{\pi}{3})$
$\frac{\pi}{2}$	$\boxed{0}$	$(0, \frac{\pi}{2})$
$\frac{2\pi}{3}$	$-\sqrt{3} \approx -1.7$	$(-1.7, \frac{2\pi}{3})$
$\frac{3\pi}{4}$	-2	
$\frac{5\pi}{6}$	$-\sqrt{3} \approx -1.7$	
π	0	
$\frac{7\pi}{6}$	$\sqrt{3} \approx 1.7$	
$\frac{5\pi}{4}$	2	
$\frac{4\pi}{3}$	$\sqrt{3} \approx 1.7$	
$\frac{3\pi}{2}$	0	
$\frac{5\pi}{3}$	$-\sqrt{3} \approx -1.7$	
$\frac{7\pi}{4}$	-2	
$\frac{11\pi}{6}$	$-\sqrt{3} \approx -1.7$	

$\theta = 0: \quad r = 2 \cdot \sin(2 \cdot 0)$
$= 0$

$\theta = \frac{\pi}{6}: \quad r = 2\sin(2 \cdot \frac{\pi}{6})$
$= \sqrt{3}$

$\theta = \frac{\pi}{12}: \quad r = 2\sin(2 \cdot \frac{\pi}{12})$
$= 2\sin(\frac{\pi}{6})$
$= 2 \cdot \frac{1}{2}$
$= 1$

$\theta = \frac{\pi}{4}: \quad r = 2\sin(2 \cdot \frac{\pi}{4})$
$= 2$

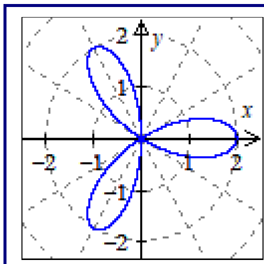


Sketch a graph $r = 2\sin(2\theta)$.

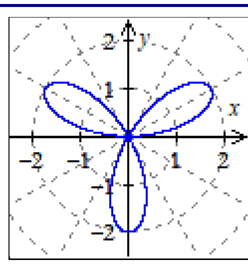
$\theta = \frac{\pi}{3}: \quad r = 2\sin(2 \cdot \frac{\pi}{3})$
$= \sqrt{3}$

$\theta = \frac{\pi}{2}: \quad r = 2\sin(2 \cdot \frac{\pi}{2})$
$= 0$

Below are the graphs of a few other functions defined via polar coordinates.

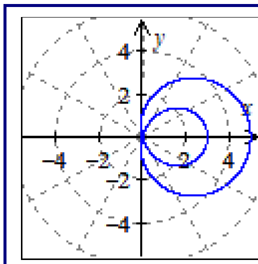


$r = 2\cos(3\theta)$

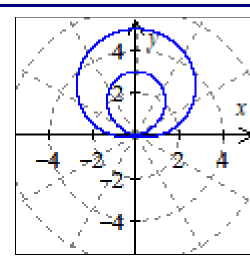


$r = 2\sin(3\theta)$

These graphs are called "**roses**".



$r = 1 + 4\cos(\theta)$



$r = 1 + 4\sin(\theta)$

These graphs are called "**limaçons**".

EXAMPLE: Convert the rectangular equation $\boxed{y = 4x - 3}$ into an equivalent equation in polar coordinates.

$y = 4x - 3$

$r\sin(\theta) = 4r\cos(\theta) - 3$

$r\sin(\theta) - 4r\cos(\theta) = -3$

$r \cdot \left(\sin(\theta) - 4\cos(\theta)\right) = -3$

$$\boxed{r = \frac{-3}{\sin(\theta) - 4\cos(\theta)}}$$

$x = r\cos(\theta)$

$y = r\sin(\theta)$

$r = \sqrt{x^2 + y^2}$

$\tan(\theta) = \frac{y}{x}$

EXAMPLE: Convert the polar equation $\boxed{r = 3\sin(\theta)}$ into an equivalent equation in rectangular coordinates.

$r = 3\sin(\theta)$

$\sqrt{x^2 + y^2} = 3 \cdot \frac{y}{r}$

$\sqrt{x^2 + y^2} = 3 \cdot \frac{y}{\sqrt{x^2 + y^2}}$

$$\boxed{x^2 + y^2 = 3y}$$

"implicit equation"

$x = r\cos(\theta)$

$y = r\sin(\theta)$

$r = \sqrt{x^2 + y^2}$

$\tan(\theta) = \frac{y}{x}$

$y = r\sin(\theta)$

$\Rightarrow \sin(\theta) = \frac{y}{r}$