Graphing Sinusoidal Functions

Below is a summary of what is studied in MTH 111 about graph transformations; see Section I, Units 6–8 from my online notes for MTH 111 to review graph transformations.

SUMMARY OF GRAPH TRANSFORMATIONS

Suppose that f and g are functions such that $g(t) = A \cdot f\big(\omega(t-h)\big) + k$ and $A, \omega, h, k \in \mathbb{R}$. In order to transform the graph of the function f into the graph of g:

1st: horizontally stretch/compress the graph of f by a factor of $\frac{1}{|\omega|}$ and, if $\omega < 0$, reflect it about the y-axis. (Stretch if $|\omega| < 1$; compress if $|\omega| > 1$.)

2nd: shift the graph horizontally h units (shift right if $h > 0$; shift left if $h < 0$).

3rd: vertically stretch/compress the graph by a factor of $|A|$ and, if $A < 0$, reflect it about the t-axis. (Stretch if $|A| > 1$; compress if $|A| < 1$.)

4th: shift the graph vertically k units (shift up if k is positive and down if k is negative).

(The *order* in which these transformations are performed *matters*.)

When we apply these graph transformations to the graphs of $y = \sin(t)$ and $y = \cos(t)$ we obtain *sinusoidal functions*:

DEFINITION: A **sinusoidal function** is function f of the form

$$\left\{ f(t) = A\sin\big(\omega(t-h)\big) + k \quad \text{or} \quad f(t) = A\cos\big(\omega(t-h)\big) + k \right\}$$

where $A, \omega, h, k \in \mathbb{R}$, $A \neq 0$, and $\omega \neq 0$.

A sinusoidal function of this form has the following properties:

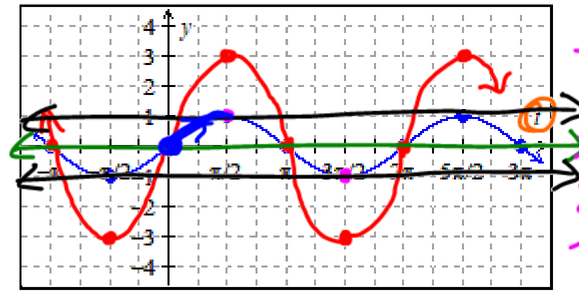
• Amplitude of $\|A\|$ units	• If the input for sine or cosine is factored as shown in the template then we can "start" our sine or cosine wave at $t = h$.
• Midline: $y = k$	
• Period: $2\pi \cdot \frac{1}{\omega}$	

(We'll use the examples below to determine the properties for the box above.)

EXAMPLE: The graph of $f(t) = \sin(t)$ is given below. Sketch a graph of $y = 3\sin(t)$.

Compared with $f(t) = \sin(t)$, $y = 3 \cdot \sin(t)$ is stretch vertically by a factor of 3

Amp is 3 units.



Sketch a graph of $y = 3\sin(t)$.

Use [Desmos](#) to graph $f(t) = \sin(t)$ & $y = A\sin(t)$ and $g(t) = \cos(t)$ & $y = A\cos(t)$ for various values of $A > 0$; then complete the following sentence:

- The graphs of $y = A\sin(t)$ and $y = A\cos(t)$ have have amplitude A units.

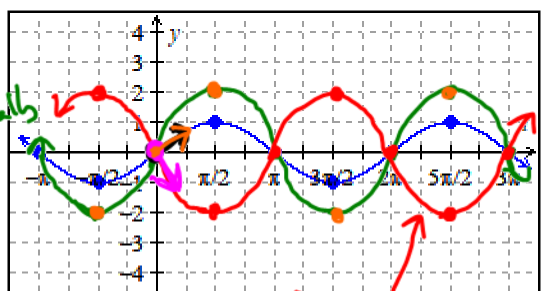
EXAMPLE: The graph of $f(t) = \sin(t)$ is given below; sketch a graph of $y = -2\sin(t)$; also the graph of $g(t) = \cos(t)$ is given below; sketch a graph of $y = -4\cos(t)$.

$y = -2\sin(t)$
$= -1 \cdot 2\sin(t)$

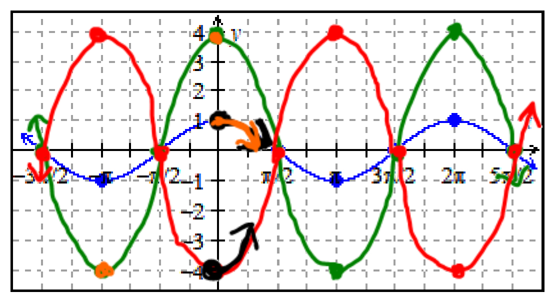
stretch vertically by factor 2

reflect the about t-axis

The amp is $|-2| = 2$



Sketch a graph of $y = -2\sin(t)$.



Sketch a graph of $y = -4\cos(t)$.

$y = -4\cos(t)$
$= -1 \cdot 4\cos(t)$

stretching vertically by factor 4

reflect about t-axis

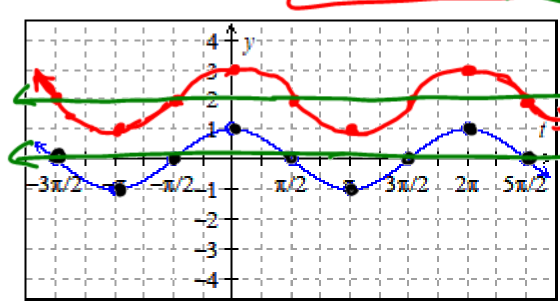
Observation: The reflected sine wave "starts" at midline & travels DOWN

while the reflected cosine wave "starts" minimum & travels UP.

EXAMPLE: The graph of $g(t) = \cos(t)$ is given below. Sketch a graph of $y = \cos(t) + 2$.

Compared with $g(t) = \cos(t)$, The graph of $y = \cos(t) + 2$ is shifted Up 2 units.

The midline $y = 2$



Sketch a graph of $y = \cos(t) + 2$.

Use [Desmos](#) to graph $g(t) = \cos(t)$ & $y = \cos(t) + k$ and $f(t) = \sin(t)$ & $y = \sin(t) + k$ for various values of k; then complete the following sentence:

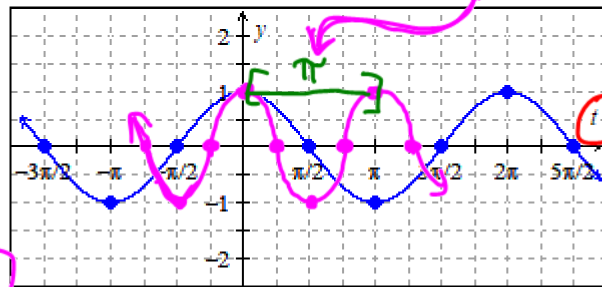
- The graphs of $y = \cos(t) + k$ and $y = \sin(t) + k$ have midline of $y = K$.

$y=\sin(1t)$ has period 2π units

EXAMPLE: The graph of $g(t) = \cos(t)$ is given below. Sketch a graph of $y = \cos(2t)$.

Compared with $g(t)=\cos(t)$, $y=\cos(2t)$ is compressed horizontally by a factor of $\frac{1}{2}$.

$\boxed{\text{Period} = 2\pi\cdot\frac{1}{2} = \pi \text{ units}}$



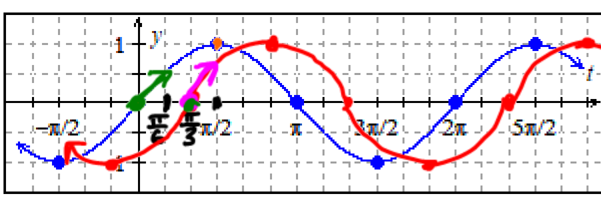
Sketch a graph of $y = \cos(2t)$.

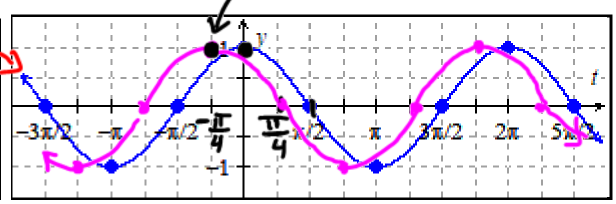
Use Desmos to graph $g(t) = \cos(t)$ & $y = \cos(\omega\cdot t)$ and $f(t) = \sin(t)$ & $y = \sin(\omega\cdot t)$ for various values of $\omega > 0$; then complete the following sentence:

- The graphs of $y = \cos(\omega\cdot t)$ and $y = \sin(\omega\cdot t)$ … will be horizontally stretched/compressed by a factor of $\frac{1}{\omega}$. Since the period is "horizontal info," the period will be stretched/compressed by factor $\frac{1}{\omega}$. Therefore the period of these functions will be $\boxed{P = 2\pi\cdot\frac{1}{\omega}}$

EXAMPLE: The graphs of $f(t) = \sin(t)$ and $g(t) = \cos(t)$ are given below; sketch graphs of $y = \sin\left(t - \frac{\pi}{3}\right)$ and $y = \cos\left(t + \frac{\pi}{4}\right)$.

$t = -\frac{\pi}{4}$





Sketch a graph of $y = \sin\left(t - \frac{\pi}{3}\right)$.

Sketch a graph of $y = \cos\left(t + \frac{\pi}{4}\right)$.

shift right $\frac{\pi}{3}$ units

shift left $\frac{\pi}{4}$ units

EXAMPLE: Use Desmos to compare $p(t) = \cos\left(2t - \frac{\pi}{3}\right)$ and $q(t) = \cos\left(2\left(t - \frac{\pi}{3}\right)\right)$.

Determine the appropriate horizontal shift for each function.

Both $y=p(t)$ & $y=q(t)$ involve the same horiz. stretch/compress factor & the same horiz. shifting constant.

NOTE THAT: $p(t) = \cos\left(2t - \frac{\pi}{3}\right)$
$= \cos\left(2\left(t - \frac{\pi}{6}\right)\right)$

Observation: *To determine the horizontal shift, first factor the input of the trig function.*

cosine wave "starts" at $t = \frac{\pi}{6}$

$y = \sin(t)$ has period 2π units but there's no "2π" in the formula

EXAMPLE: Sketch a graph of $m(t) = 2\sin\left(\frac{1}{2}t + \frac{\pi}{4}\right) + 3$. State the period, midline, and amplitude of m.

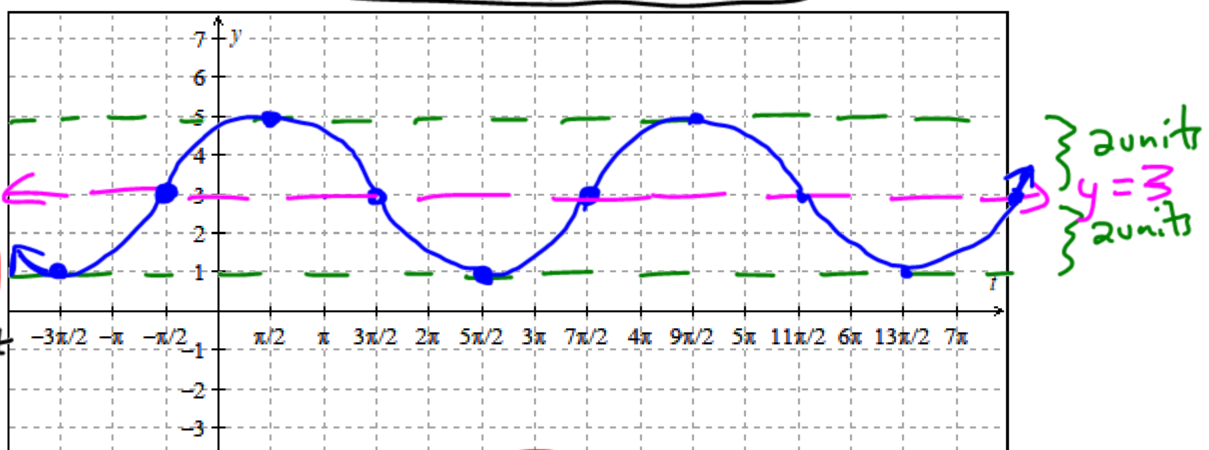
$m(t) = 2\sin\left(\frac{1}{2}\left(t + \frac{\pi}{2}\right)\right) + 3$

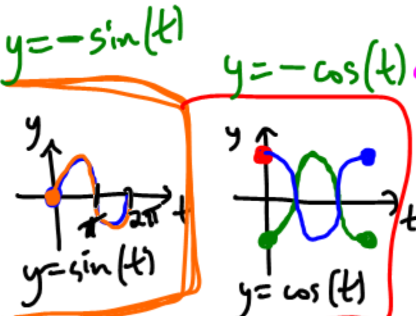
Amplitude is 2 units

Midline is $y = 3$

Period: $P = 2\pi \cdot \frac{1}{1/2}$
$= 2\pi \cdot \frac{2}{1}$
$= 4\pi$ units

we need a sine wave "starting" at $t = -\frac{\pi}{2}$





Sketch a graph of $m(t) = 2\sin\left(\frac{1}{2}t + \frac{\pi}{4}\right) + 3$.

$y = \sin(t)$ has period 2π but 2π isn't in the "formula"

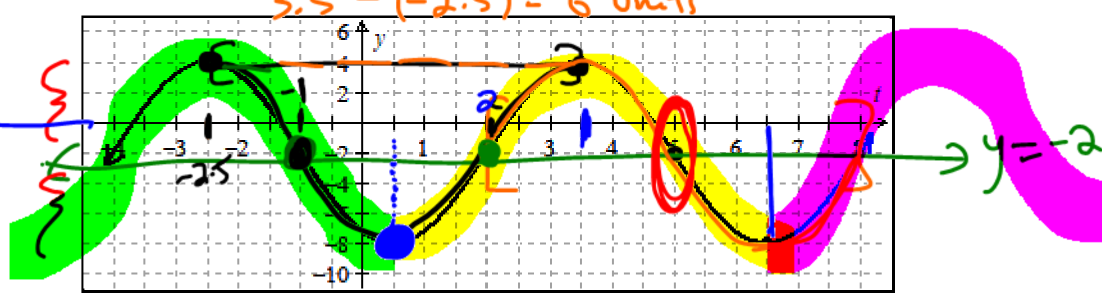
EXAMPLE: Find (at least) two algebraic rules (i.e., "formulas"), one involving sine and one involving cosine, for the sinusoidal function n whose graph is given below.

$3.5 - (-2.5) = 6$ units

Midline: $y = -2$
$\Rightarrow \boxed{K = -2}$

Amplitude: 6 units
$\Rightarrow |A| = 6$

Period: 6 units



The graph of $y = n(t)$.

$\frac{6}{6} = 2\pi \cdot \frac{1}{w}$

$w = \frac{2\pi}{6} = \frac{\pi}{3}$

$n(t) = A\sin\left(w\left(t - h\right)\right) + K$
$= 6\sin\left(\frac{\pi}{3}\left(t - 2\right)\right) - 2$
$= 6\sin\left(\frac{\pi}{3}\left(t + 16\right)\right) - 2$
$= -6\sin\left(\frac{\pi}{3}\left(t - 5\right)\right) - 2$

$n(t) = A\cos\left(w\left(t - h\right)\right) + K$
$= 6\cos\left(\frac{\pi}{3}\left(t + \frac{5}{2}\right)\right) - 2$
$= 6\cos\left(\frac{\pi}{3}\left(t - \frac{7}{2}\right)\right) - 2$
$= -6\cos\left(\frac{\pi}{3}\left(t - \frac{13}{2}\right)\right) - 2$

EXAMPLE: Sketch a graph of $f(t) = 2\sin\left(\pi t - \frac{\pi}{4}\right) - 3$ on the coordinate plane below.

Factor the input for the sine function

$$f(t) = 2\sin\left(\pi\left(t - \frac{1}{4}\right)\right) - 3$$

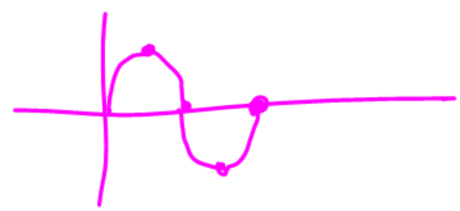
Amplitude is $|2| = 2$

Midline is $y = -3$

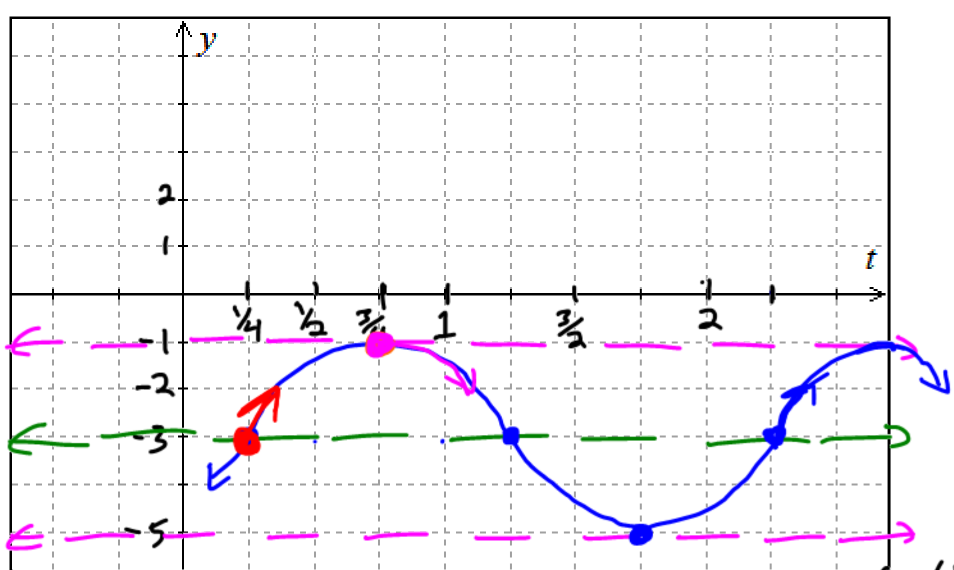
Period: $P = 2\pi \cdot \frac{1}{\pi}$

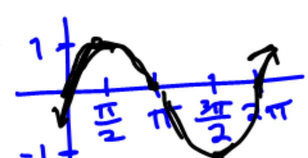
$= 2$ units

Need a sine wave "starting" at $t = \frac{1}{4}$



y = -3





Sketch a graph of $f(t) = 2\sin\left(\pi t - \frac{\pi}{4}\right) - 3 = 2\sin\left(\pi\left(t - \frac{1}{4}\right)\right) - 3$

Find rule for $y = f(t)$ that involves cosine.

$$f(t) = 2\cos\left(\pi\left(t - \frac{3}{4}\right)\right) - 3$$

$y = \sin(1t)$ has period 2π units.

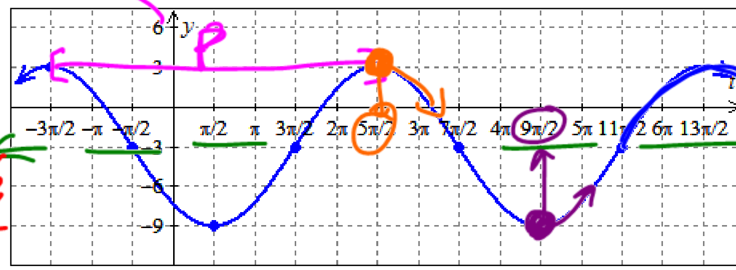
Additional Examples:

a. Find (at least) two algebraic rules (i.e., "formulas"), one involving sine and one involving cosine, for the sinusoidal function f whose graph is given below.

$A = 6$

6 units

6 units



The graph of $y = f(t)$.

$P = \frac{5\pi}{2} - \left(-\frac{3\pi}{2}\right)$

$= \frac{8\pi}{2}$

$= 4\pi$

$w \cdot 4\pi = 2\pi \cdot \frac{1}{w} \cdot w$

$w = \frac{2\pi}{4\pi} = \frac{1}{2}$

$f(t) = A \cdot \sin\left(w(t-h)\right) + k$

$= 6\sin\left(\frac{1}{2}\left(t - \frac{3\pi}{2}\right)\right) - 3$

$= -6\sin\left(\frac{1}{2}\left(t - \frac{7\pi}{2}\right)\right) - 3$

$= 6\cos\left(\frac{1}{2}\left(t - \frac{5\pi}{2}\right)\right) - 3$

$= -6\cos\left(\frac{1}{2}\left(t - \frac{9\pi}{2}\right)\right) - 3$

b. Sketch a graph of $g(t) = 3\cos\left(\frac{\pi}{2}t - \frac{\pi}{4}\right) - 1$ on the coordinate plane below. List the *period*, *amplitude*, *midline*, and *horizontal shift*.

$g(t) = 3\cos\left(\frac{\pi}{2}\left(t - \frac{1}{2}\right)\right) - 1$

Need cosine wave "starting" at $t = \frac{1}{2}$

Amp: 3 units

Midline: $y = -1$

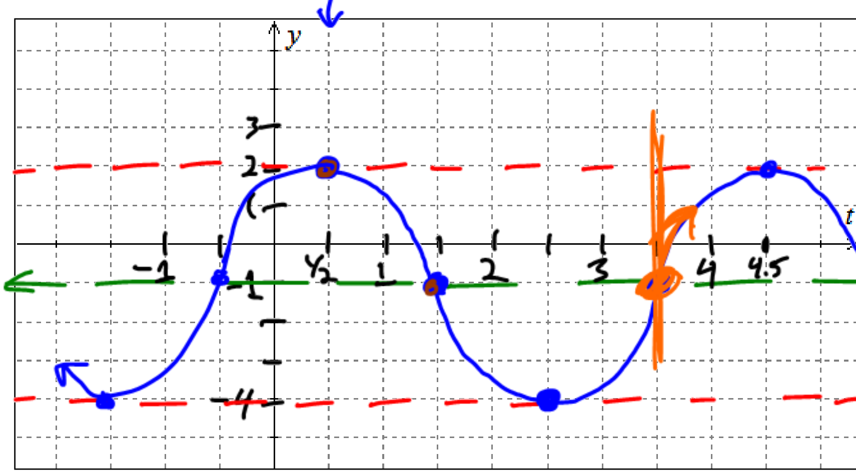
Period:

$P = 2\pi \cdot \frac{1}{\pi/2}$

$= 2\cancel{\pi} \cdot \frac{2}{\cancel{\pi}}$

$= 4$

quarter period: $\frac{4}{4} = 1$



Sketch a graph of $g(t) = 3\cos\left(\frac{\pi}{2}t - \frac{\pi}{4}\right) - 1$.

$g(t) = 3\sin\left(\frac{\pi}{2}\left(t - \frac{7}{2}\right)\right) - 1$