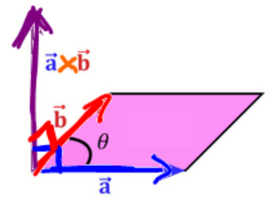
Cross Product



The Dot Product

We've studied how to add and subtract vectors and how to multiply vectors by scalars. Now we'll study how to multiply one vector by another. This type of multiplication is called the **dot product**. Since we are focusing on two-dimensional vectors in this class, we will define the dot product in terms of two-dimensional vectors:

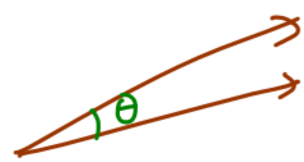
DEFINITION: If $\vec{u} = \langle u_1, u_2 \rangle$ and $\vec{v} = \langle v_1, v_2 \rangle$, then **the dot product of $\vec{u}$ and $\vec{v}$**, denoted $\vec{u} \cdot \vec{v}$, is defined as follows:

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2$$

Thus, to compute the dot product of two vectors, we simply multiply the horizontal components of the two vectors and the vertical components of the two vectors and then add the results. It is important to note that the dot product produces a **scalar**.

EXAMPLE 1: If $\vec{a} = \langle 3, -9 \rangle$ and $\vec{b} = \langle 6, -1 \rangle$, find $\vec{a} \cdot \vec{b}$.

$$\vec{a} \cdot \vec{b} = 3 \cdot 6 + (-9)(-1)$$
$$= 18 + 9$$
$$= 27 \checkmark$$



Properties of the Dot Product

If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors then the following properties hold true:

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$ (commutative property)
2. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$ (distributive property)
3. $\boxed{\vec{v} \cdot \vec{v} = \|\vec{v}\|^2}$
4. $\boxed{\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cos(\theta)}$ where θ is the angle between $\vec{u}$ and $\vec{v}$.

$$\cos(\theta) = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \cdot \|\vec{v}\|}$$

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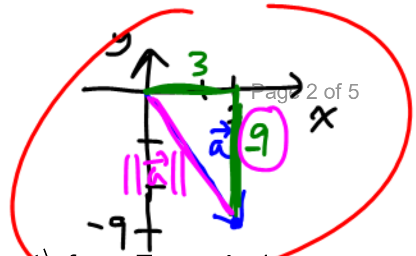


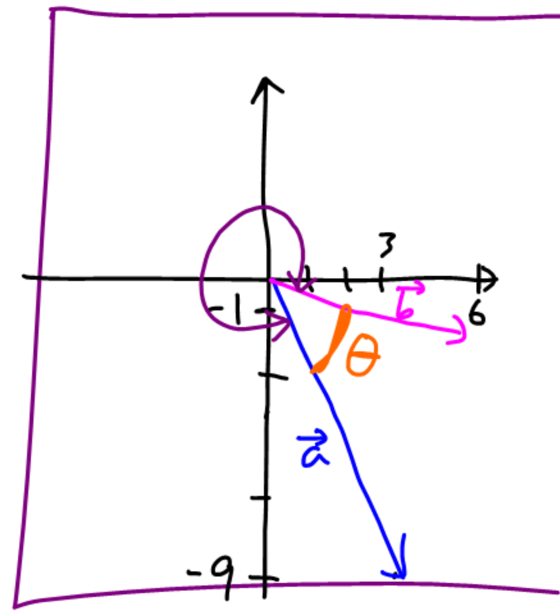
The dot product can be used to find the angle between two vectors.

EXAMPLE 2: Find the angle between vectors $\vec{a} = \langle 3, -9\rangle$ and $\vec{b} = \langle 6, -1\rangle$ from Example 1.

$\vec{a}\cdot\vec{b} = 27$

$$\|\vec{a}\| = \sqrt{3^2 + (-9)^2} = \sqrt{9+81} = \sqrt{90} = 3\sqrt{10}$$

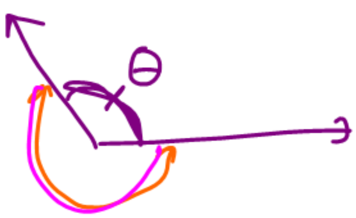
$$\|\vec{b}\| = \sqrt{(6)^2 + (-1)^2} = \sqrt{37}$$



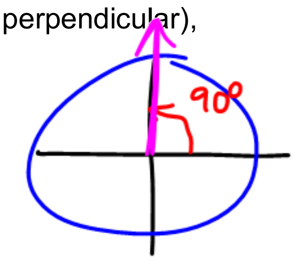
$$\cos(\theta) = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\cdot\|\vec{b}\|}$$

$$\cos(\theta) = \frac{27}{3\sqrt{10}\cdot\sqrt{37}}$$

$$\theta = \boxed{\cos^{-1}\left(\frac{27}{3\sqrt{10}\cdot\sqrt{37}}\right)} \approx 62.1^\circ$$

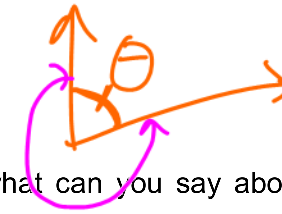


EXAMPLE 3: If the angle between $\vec{u}$ and $\vec{v}$ is $\theta = 90^\circ$ (i.e., if $\vec{u}$ and $\vec{v}$ are perpendicular), find $\vec{u}\cdot\vec{v}$.



$$\vec{u}\cdot\vec{v} = \boxed{\|\vec{u}\|\cdot\|\vec{v}\|\cos(90^\circ)} = \boxed{\|\vec{u}\|}\boxed{\|\vec{v}\|}\cdot 0 = 0$$

If vectors are perpendicular, then their dot product equals 0. Also, if dot equals 0, then vectors must be perpendicular.



EXAMPLE 4: If $\vec{u}$ and $\vec{v}$ are non-zero vectors and $\vec{u}\cdot\vec{v} > 0$, what can you say about the angle θ between vectors $\vec{u}$ and $\vec{v}$. What if $\vec{u}\cdot\vec{v} < 0$?

Recall $\boxed{\vec{u}\cdot\vec{v} = \|\vec{u}\|\cdot\|\vec{v}\|\cos(\theta)}$.

Also $\|\vec{u}\| > 0$ & $\|\vec{v}\| > 0$.

So sign of $\cos(\theta)$ determines the sign of $\vec{u}\cdot\vec{v}$.

$\cos(\theta) > 0$

$0 < \theta < 90°$

Angle between is acute

$\cos(\theta) < 0$

$90° < \theta < 180°$

Angle between vectors is obtuse.

Now we'll prepare to derive $\boxed{\vec{u}\cdot\vec{v} = \|\vec{u}\|\cdot\|\vec{v}\|\cos(\theta)}$. We'll need to use the **difference of two vectors** so first let's explore that:

$\vec{v} = \langle v_1, v_2\rangle$ & $\vec{u} = \langle u_1, u_2\rangle$

Then $\vec{v} - \vec{u} = \langle v_1 - u_1, v_2 - u_2\rangle$

EXAMPLE 5: Let $\vec{v} = \langle 4, 3\rangle$ and $\vec{s} = \langle 2, -6\rangle$; find $\vec{v} - \vec{s}$.

$$\vec{v} - \vec{s} = \langle 4, 3\rangle - \langle 2, -6\rangle$$
$$= \langle 4-2, 3-(-6)\rangle$$
$$= \langle 2, 9\rangle$$

We can also subtract vectors by using arrows on the coordinate plane. Notice that $\boxed{\vec{v} - \vec{s} = \vec{v} + (-\vec{s})}$, so in Figure 1 we can subtract $\vec{s}$ from $\vec{v}$ by adding $-\vec{s}$ to $\vec{v}$:

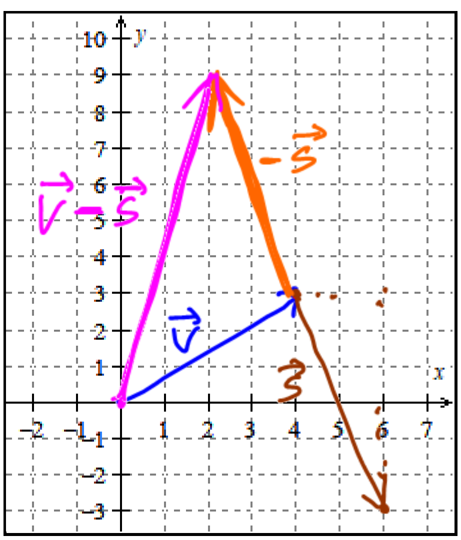


Figure 1

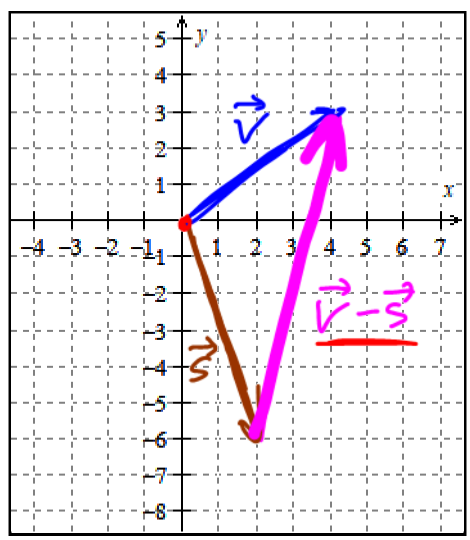


Figure 2

Interestingly, $\vec{v} - \vec{s}$ is the vector that we need to add to $\vec{s}$ in order to create $\vec{v}$; we can see this symbolically and then use it obtain another way to draw $\vec{v} - \vec{s}$ in Figure 2.

$$\vec{s} + (\vec{v} - \vec{s}) = \vec{v}$$

We'll also need to use the identity $\boxed{\vec{v}\cdot\vec{v} = \|\vec{v}\|^2}$ so let's prove it by computing $\vec{v}\cdot\vec{v}$ for a generic vector $\boxed{\vec{v} = \langle v_1, v_2\rangle}$ and showing that the result is equal to $\|\vec{v}\|^2$ (this proof will have the same structure as our proofs of trig identities).

vector dot product

real number multiplication

$$\vec{v}\cdot\vec{v} = \langle v_1, v_2\rangle \cdot \langle v_1, v_2\rangle$$

$$= v_1 \cdot v_1 + v_2 \cdot v_2$$

$$= v_1^2 + v_2^2$$

$$= \left(\sqrt{v_1^2 + v_2^2}\right)^2 \quad ☆$$

$$= \|\vec{v}\|^2$$

Explore other side of identity:

$$\|\vec{v}\|^2 = \left(\sqrt{v_1^2 + v_2^2}\right)^2$$

$$= v_1^2 + v_2^2$$

Now we're ready to derive $\boxed{\vec{u}\cdot\vec{v} = \|\vec{u}\|\cdot\|\vec{v}\|\cos(\theta)}$ where θ is the angle between $\vec{u}$ and $\vec{v}$:

First let's draw two vectors $\vec{u}$ and $\vec{v}$ so that θ is the angle between the vectors:

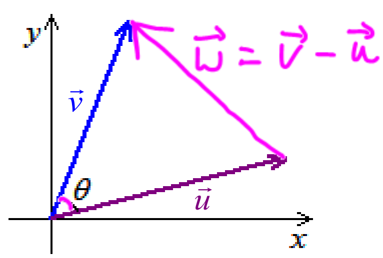


Figure 3

Now let's construct the vector $\boxed{\vec{w} = \vec{v} - \vec{u}}$. Recall that we can obtain the vector $\vec{v} - \vec{u}$ by drawing an arrow that starts at the tip of $\vec{u}$ and ends at the tip of $\vec{v}$.

If we think of the arrows as being line segments instead of arrows, we have a triangle with side lengths $\|\vec{u}\|$, $\|\vec{v}\|$, and $\|\vec{w}\|$ and angle θ opposite the side $\|\vec{w}\|$.

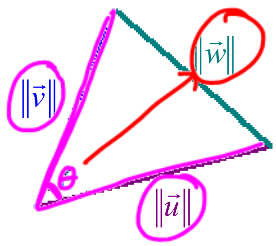


Figure 4

Now we can use the $\boxed{\text{Law of Cosines}}$ to obtain an equation that relates the magnitudes of the vectors and the angle θ.

$$\|\vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{u}\|^2 - 2\cdot\underbrace{\|\vec{v}\|\cdot\|\vec{u}\|\cos(\theta)}$$

We can use this equation to obtain the statement given in fourth property in the table above. First, let's find $\|\vec{w}\|^2$. Recall that $\boxed{\vec{w} = \vec{v} - \vec{u}}$. So...

$$\|\vec{w}\|^2 = \vec{w}\cdot\vec{w}$$
$$= (\vec{v}-\vec{u})\cdot(\vec{v}-\vec{u})$$
$$= \vec{v}\cdot\vec{v} \underbrace{- \vec{v}\cdot\vec{u} - \vec{u}\cdot\vec{v}} + \vec{u}\cdot\vec{u}$$
$$= \underbrace{\|\vec{v}\|^2 - 2\vec{v}\cdot\vec{u} + \|\vec{u}\|^2}$$

We can now substitute this expression for $\|\vec{w}\|^2$ in the equation we found above, and simplify to obtain $\vec{u}\cdot\vec{v} = \|\vec{u}\|\cdot\|\vec{v}\|\cos(\theta)$:

$$\cancel{\|\vec{v}\|^2} - 2\vec{v}\cdot\vec{u} + \cancel{\|\vec{u}\|^2} = \cancel{\|\vec{v}\|^2} + \cancel{\|\vec{u}\|^2} - 2\cdot\|\vec{v}\|\cdot\|\vec{u}\|\cos(\theta)$$
$$\frac{-\cancel{2}\vec{v}\cdot\vec{u}}{-\cancel{2}} = \frac{-\cancel{2}\cdot\|\vec{v}\|\cdot\|\vec{u}\|\cos(\theta)}{-\cancel{2}}$$
$$\boxed{\vec{v}\cdot\vec{u} = \|\vec{v}\|\cdot\|\vec{u}\|\cos(\theta)}$$