$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

Complex Numbers and Polar Coordinates

Recall from Section I: Chapter 0 the definition of the set of complex numbers:

$$\mathbb{C} = \left\{ x \mid x = a + bi \text{ and } a, b \in \mathbb{R} \text{ and } i = \sqrt{-1} \right\}.$$

You've probably seen complex numbers in a course like MTH 95 or Algebra 2 when solving quadratic equations like $x^2 - 4x + 13 = 0$: $\Rightarrow$ $1x^2 - 4x + 13 = 0$

Recall the quadratic formula:
If $ax^2 + bx + c = 0$, then
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot (1)(13)}}{2 \cdot 1} = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i$$

For a complex number of the form $a + bi$, its *real part* is a and its *imaginary part* is b.

Because a complex number has *two* parts, we can use the *two dimensional* rectangular coordinate plane to plot complex numbers. We use the horizontal axis to represent the real part of the number and the vertical axis to represent the complex part of the number. Thus, the complex number $a + bi$ can be represented by the point (a, b) on the rectangular coordinate plane; see Figure 1.

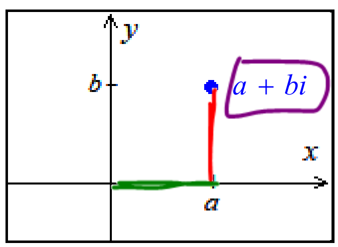
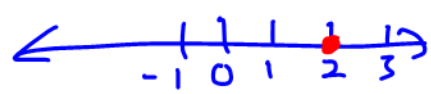


Figure 1

Plot $x = 2$ on number line



EXAMPLE: Plot the following complex numbers on the coordinate plane in Figure 2.

a. $s = 2 + 5i \Rightarrow (2, 5)$

b. $t = \frac{3}{2} - 3i \Rightarrow \left(\frac{3}{2}, -3\right)$

c. $u = 3i \Rightarrow (0, 3)$

d. $v = -4 \Rightarrow (-4, 0)$

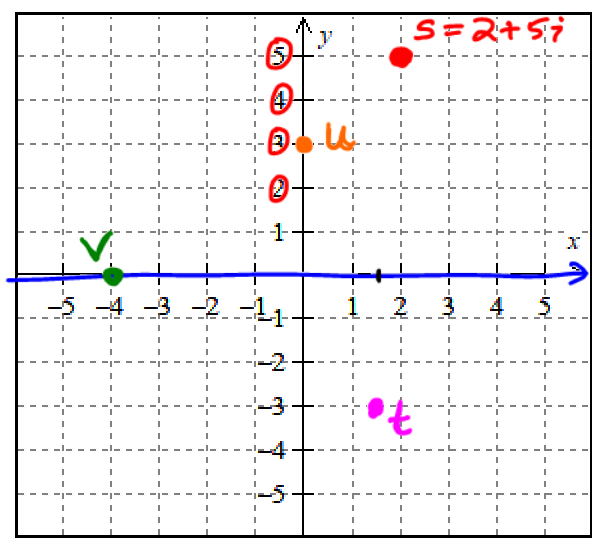


Figure 2

As we've studied, the rectangular ordered pair (a, b) can be represented in polar coordinates (r, θ) where r represents the distance the point is from the origin and θ represents the angle between the positive x-axis and the segment connecting the origin and the point; see Figure 3. Let's recall how we can find r and θ when we know a and b:

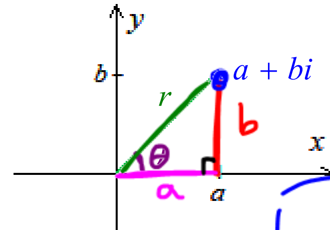
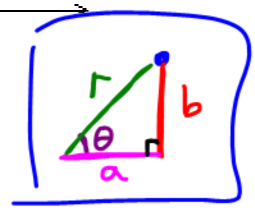


Figure 3



$$r = \sqrt{a^2 + b^2}$$

$$\boxed{\tan(\theta) = \frac{b}{a}}$$

Let's also recall how we can find a and b when we know r and θ:

$$\cos(\theta) = \frac{a}{r} \qquad \sin(\theta) = \frac{b}{r}$$

$$a = r\cos(\theta) \qquad b = r\sin(\theta)$$

Using this information, we can write the complex number $a + bi$ in terms of r and θ:

$$a + bi = r\cos(\theta) + r\sin(\theta)\cdot i$$

$$e^x = \exp(x)$$

Now we can establish a surprising connection between exponential function $y = e^x$ and complex numbers: Euler's Formula. (If you continue studying mathematics and take a calculus sequence, you have an opportunity to see why this equation is true but, for now, you need to just accept it and learn to work with it.)

Euler vs. Euclid

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EULER'S FORMULA

$$e^{i\theta} = \cos(\theta) + \sin(\theta)\cdot i$$

By multiplying both sides of Euler's formula by r, we obtain the following formula that allows us to write any complex number in **polar form**.

$$r\cdot(e^{i\theta}) = (\cos(\theta) + \sin(\theta)\cdot i)\cdot r$$

$$\Rightarrow \boxed{re^{i\theta} = r\cos(\theta) + r\sin(\theta)\cdot i}$$

The **polar form** of the complex number $z = a + bi$ is $z = re^{i\theta}$:

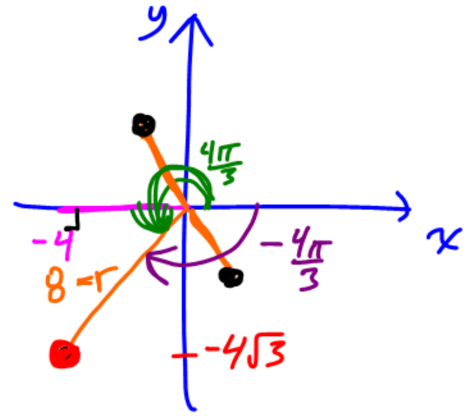
$$\begin{aligned} z &= a + bi \\ &= r\cos(\theta) + r\sin(\theta)\cdot i \\ &= r\cdot e^{i\theta} \end{aligned}$$

$$\therefore \quad a + bi = re^{i\cdot\theta}$$

$x^2 = 4$

$\Rightarrow x = 2$ or $x = -2$

ex. Convert $z = -4 - 4\sqrt{3}\,i$ into polar form:



$$r = \sqrt{(-4)^2 + (-4\sqrt{3})^2}$$
$$= \sqrt{16\cdot 1 + 16\cdot 3}$$
$$= \sqrt{16\cdot(1+3)}$$
$$= \sqrt{16\cdot 4}$$
$$= 4\cdot 2$$
$$= 8$$

$$\tan(\theta) = \frac{-\cancel{4}\sqrt{3}}{-\cancel{4}}$$
$$\tan(\theta) = \sqrt{3}$$
$$\theta = \tan^{-1}(\sqrt{3}) + \pi$$
$$= \frac{\pi}{3} + \pi$$
$$= \frac{4\pi}{3}$$

$$\therefore\ -4 - 4\sqrt{3}\,i = \boxed{8e^{i\cdot\left(\frac{4\pi}{3}\right)}}$$

If want $\sqrt{\boxed{-4 - 4\sqrt{3}\,i}} = \left(-4 - 4\sqrt{3}\right)^{1/2}$

$$= \left(8\cdot e^{i\frac{4\pi}{3}}\right)^{1/2}$$
$$= \sqrt{8}\cdot\left(e^{i\frac{4\pi}{3}}\right)^{1/2}$$
$$= \boxed{2\sqrt{2}\, e^{i\frac{2\pi}{3}}}$$
$$= 2\sqrt{2}\,\underbrace{\cancel{\cos\left(\tfrac{2\pi}{3}\right)}}_{-1/2} + \cancel{2}\sqrt{2}\,\underbrace{\cancel{\sin\left(\tfrac{2\pi}{3}\right)}}_{\sqrt{3}/\cancel{2}}\, i$$
$$= \boxed{-\sqrt{2} + \sqrt{6}\, i}$$

$$\boxed{re^{i\theta} = r\cos(\theta) + r\sin(\theta)\cdot i}$$

This is polar form

EXAMPLE: Express the complex number $z = 6e^{i\cdot\frac{5\pi}{6}}$ in rectangular form $z = a + bi$.

$$z = 6e^{i\cdot\frac{5\pi}{6}}$$

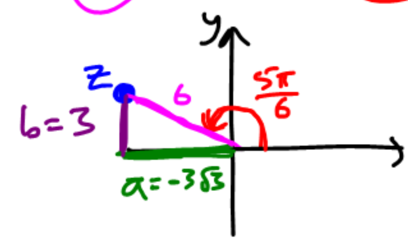
$$= 6\cos\left(\tfrac{5\pi}{6}\right) + 6\sin\left(\tfrac{5\pi}{6}\right)\cdot i$$

$$= 6\left(-\tfrac{\sqrt{3}}{2}\right) + 6\left(\tfrac{1}{2}\right)i$$

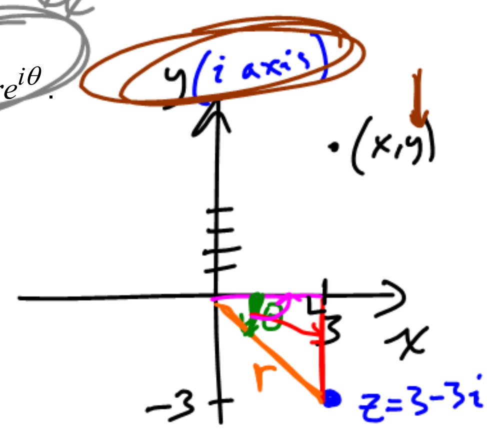
$$= \underbrace{-3\sqrt{3}}_{\text{"a"}} + \underbrace{3}_{\text{"b"}}i$$

$z = 6e^{i\cdot\left(\frac{5\pi}{6}\right)}$ has

$r = 6$ & $\theta = \frac{5\pi}{6}$



EXAMPLE: Express the complex number $z = 3 + (-3)i$ in polar form $z = re^{i\theta}$.



In $z = 3-3i$, $a = 3$ & $b = -3$

So $z = 3-3i$ can be identified with point $\boxed{(3, -3)}$.

$$r = \sqrt{(3)^2 + (-3)^2}$$

$$= \sqrt{9+9}$$

$$= \sqrt{9\cdot 2}$$

$$= 3\sqrt{2}\ \checkmark$$

$$\tan(\theta) = \frac{-3}{3}$$

$$\theta = \tan^{-1}(-1)$$

$$= -\frac{\pi}{4}$$

$$\therefore\ 3-3i = 3\sqrt{2}e^{i\left(-\frac{\pi}{4}\right)}$$

Check:

$$3\sqrt{2}e^{i\left(-\frac{\pi}{4}\right)} = 3\sqrt{2}\cos\left(-\tfrac{\pi}{4}\right) + 3\sqrt{2}\sin\left(-\tfrac{\pi}{4}\right)i$$

$$= 3\sqrt{2}\cdot\left(+\tfrac{\sqrt{2}}{2}\right) + 3\sqrt{2}\cdot\left(-\tfrac{\sqrt{2}}{2}\right)i$$

$$= \frac{3\cdot\cancel{2}}{\cancel{2}} - \frac{3\cdot\cancel{2}}{\cancel{2}}i$$

$$= 3 - 3i$$