Angles and Arc-Length

In this set of notes, we'll study some conventions and terminology that we will use when discussing angles within circles, like angle θ in Figure 1.

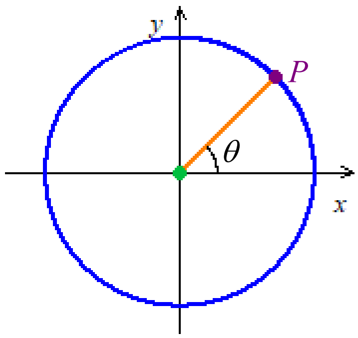


Figure 1

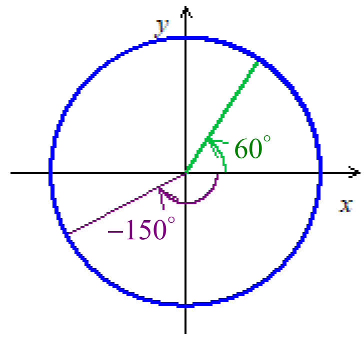


Figure 2

- The angle θ is measured **counterclockwise from the positive x-axis**. Consequently, negative angles are measured *clockwise* from the positive x-axis; see Figure 2.

- The segment between the origin, $(0, 0)$, and the point P is the ***terminal side of angle θ***.

- The point P on the circumference of the circle is said to be ***specified by the angle θ***.

- Angle θ corresponds with a portion of the circumference of the circle called the **arc spanned by θ**. (We'll discuss finding the ***length*** of this arc.)

- Two angles with the same terminal side are said to be ***co-terminal angles***.

Thus far in your mathematics careers you've probably measured angles in **degrees**: 360° represents a complete trip around a circle (i.e., a full rotation), so 1° corresponds to $1/360^{\text{th}}$ of a full rotation; see Fig. 3.

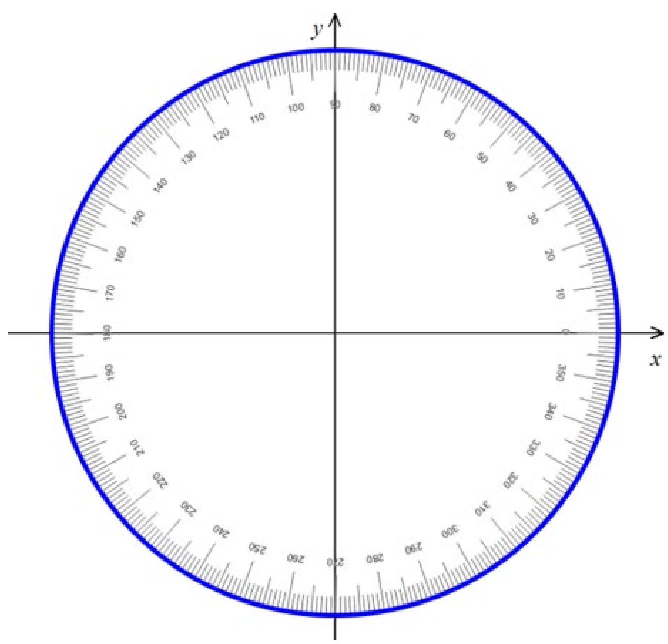


Figure 3

We mentioned above that *co-terminal angles* share the same terminal side. Since 360° represents a full rotation about the circle, if we add any integer multiple of 360° to an angle θ_1, we'll obtain an angle co-terminal to θ_1. In other words, the angles

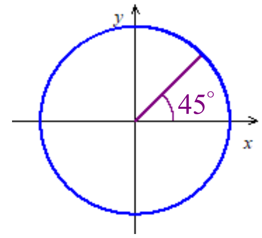
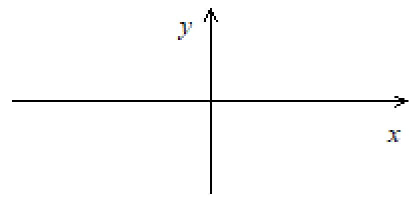


Figure 4

are co-terminal. So 45° and ______________________________ are co-terminal.

Traditionally, the coordinate plane is divided into ***four quadrants***. We will often use the names of these quadrants to describe the location of the terminal side of different angles.



DEFINITION: The **radian** measure of an angle is the ratio of the length of the arc on the circumference of the circle spanned by the angle, s, and the radius, r, of the circle; see Figure 5.

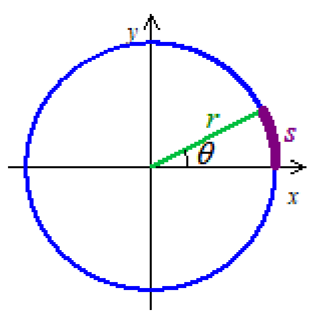


Figure 5: θ measures $\frac{s}{r}$ radian.

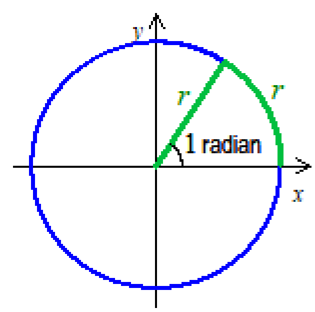


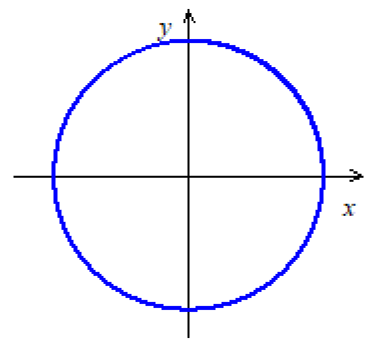
Figure 6: 1 radian.

An alternative yet equivalent definition is that an angle that measures 1 **radian** spans an arc whose length is equal to the length of the radius, r; see Figure 6.

Since a radian is a ratio of two lengths, the length-units cancel; thus, radians are considered a ***unit-less measure***.

(For more on radians, see this video that's linked from the online lecture notes: https://youtu.be/93cTjSTS4Gs.)

DEFINITION: The number π represents the ratio of the *circumference of a circle* to the *radius of the circle*.



EXAMPLE: Convert 8 radians into degrees and 8° into radians.

EXAMPLE: Convert the given angles into radians in order to complete the table below.

θ (degrees)	0°	30°	45°	60°	90°	180°	270°	360°
θ (radians)								

Recall that the *radian measure* of an angle is the ratio of the length of the arc on the circumference of the circle spanned by the angle and the radius of the circle. Applying this fact to the circle in Figure 7 if θ is measured in radians, then...

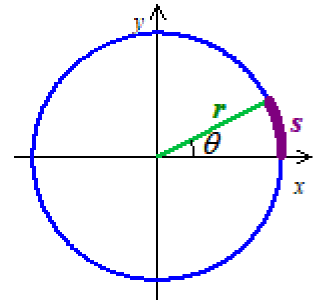


Figure 7: Circle of radius r with an angle θ spanning an arc of length s.

DEFINITION: The **arc length** s spanned in a circle of radius r by an angle θ measured in radians is given by:

Note that this formula only applies if θ *is measured in radians*!

EXAMPLE: What is the arc length spanned by a 40° angle on a circle of radius 30 meters?