SUPPLEMENT to §5.1

Coterminal Angles

DEFINITION: Two angles are **coterminal** if they have the same terminal side when in standard position.

Since 360° represents a complete revolution, if we add integer-multiples of 360° to an angle measured in degrees we'll obtain a coterminal angle. Similarly, since 2π represent a complete revolution in radians, if we add integer-multiples of 2π to an angle measured in radians, we'll obtain a conterminal angle. We can summarize this information as follows:

- if θ is measured in degrees, θ and $\theta + 360^\circ \cdot k$, where $k \in \mathbb{Z}$, are coterminal.
- if θ is measured in radians, θ and $\theta + 2\pi \cdot k$, where $k \in \mathbb{Z}$, are coterminal.

EXAMPLE 1: The angles 45°, 405°, and -315° are coterminal; see Figure 1.

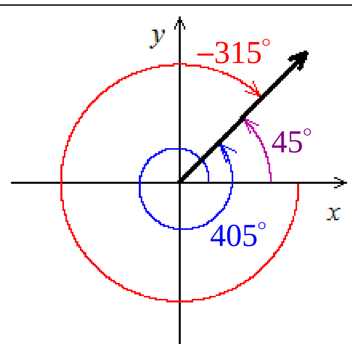


Figure 1: Coterminal Angles

Reference Angles

DEFINITION: The **reference angle** for an angle in standard position is the positive acute angle formed by the x-axis and the terminal side of the angle.

Depending on the location of the angle's terminal side, we'll have to use a different calculation to determine the angle's reference angle.

EXAMPLE 2: The $\frac{\pi}{3}$ and 30° are their own reference angles since they are acute angles; see Figures 2a and 2b.

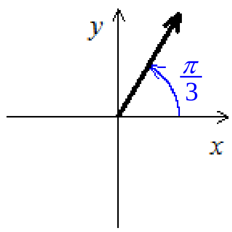


Figure 2a

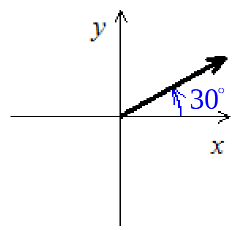


Figure 2b

EXAMPLE 3: The reference angle for $\frac{2\pi}{3}$ is $\pi - \frac{2\pi}{3} = \frac{\pi}{3}$ (see Figure 3a) while the reference angle for 150° is $180^\circ - 150^\circ = 30^\circ$ (see Figure 3b).

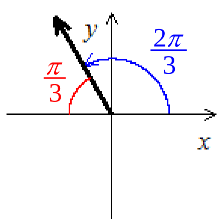


Figure 3a

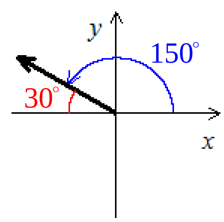


Figure 3b

EXAMPLE 4: The reference angle for $\frac{4\pi}{3}$ is $\frac{4\pi}{3} - \pi = \frac{\pi}{3}$ (see Figure 4a) while the reference angle for 210° is $210^\circ - 180^\circ = 30^\circ$ (see Figure 4b).

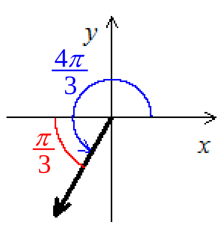


Figure 4a

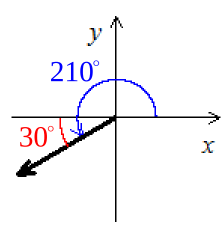


Figure 4b

EXAMPLE 5: The reference angle for $\frac{5\pi}{3}$ is $2\pi - \frac{5\pi}{3} = \frac{\pi}{3}$ (see Figure 5a) while the reference angle for 330° is $360^\circ - 330^\circ = 30^\circ$ (see Figure 5b).

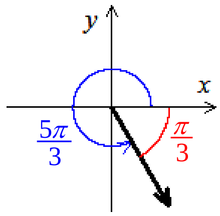


Figure 5a

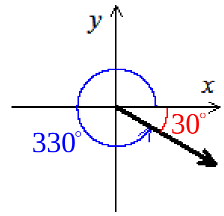


Figure 5b

EXAMPLE 6: The reference angle for 7.5 radians is $7.5 - 2\pi \approx 1.2$ radians (see Figure 6a) and the reference angle for -137° is $180^\circ + (-137^\circ) = 43^\circ$ (see Figure 6b).

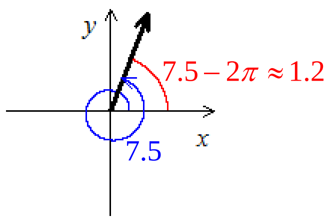


Figure 6a

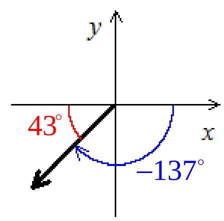


Figure 6b

EXERCISES:

1. Find both a positive and negative angle that is coterminal angle with the following angles.

a. 63° **b.** $\frac{\pi}{9}$ **c.** $\frac{13\pi}{8}$

2. Find the reference angle for the following angles.

a. 120° **b.** $\frac{5\pi}{4}$ **c.** $\frac{13\pi}{8}$

d. 400° **e.** 2 **f.** $\frac{10\pi}{11}$

g. $-\frac{9\pi}{5}$ **h.** 2000° **i.** -100°

SUPPLEMENT TO §5.6

Graphing Sinsoidal Functions: Phase Shift vs. Horizontal Shift

Let's consider the function $g(x) = \sin\left(2x - \frac{2\pi}{3}\right)$. Using what we study in MTH 111 about graph transformations, it should be apparent that the graph of $g(x) = \sin\left(2x - \frac{2\pi}{3}\right)$ can be obtained by transforming the graph of $f(x) = \sin(x)$. (To confirm this, notice that $g(x)$ can be expressed in terms of $f(x) = \sin(x)$ as $g(x) = f\left(2x - \frac{2\pi}{3}\right)$.) Since the constants "$2$" and "$\frac{2\pi}{3}$" are multiplied by and subtracted from the input variable, x, what we study in MTH 111 tells us that these constants represent a horizontal stretch/compression and a horizontal shift, respectively.

It is often recommended in MTH 111 that we factor-out the horizontal stretching/compressing factor before transforming the graph, i.e., it's often recommended that we first re-write $g(x) = \sin\left(2x - \frac{2\pi}{3}\right)$ as $g(x) = \sin\left(2\left(x - \frac{\pi}{3}\right)\right)$. After writing g in this format, we can draw its graph by performing the following sequence of transformations of the "base function" $f(x) = \sin(x)$:

1st compress horizontally by a factor of $\frac{1}{2}$

2nd shift to the right $\frac{\pi}{3}$ units

The advantage of this method is that the y-intercept of $f(x) = \sin(x)$, $(0, 0)$, ends-up exactly where the horizontal shift suggests: when we compress the x-coordinate of $(0, 0)$ by a factor of $\frac{1}{2}$, it doesn't move since $\frac{1}{2} \cdot 0 = 0$; then, when we shift the graph right $\frac{\pi}{3}$ units, the point $(0, 0)$ ends up at $\left(\frac{\pi}{3}, 0\right)$; so the y-intercept ends up moving to right $\frac{\pi}{3}$ units, exactly how far we shifted.

Compare this with the alternative method: we can leave $g(x) = \sin\left(2x - \frac{2\pi}{3}\right)$ as-is and skip factoring-out the horizontal stretching/compressing factor, but then we need the following sequence to transform $f(x) = \sin(x)$ into the graph of g:

1st shift to the right $\frac{2\pi}{3}$ units

2nd compress horizontally by a factor of $\frac{1}{2}$

The disadvantage of this method is that the y-intercept of $f(x) = \sin(x)$ **doesn't** end-up where the horizontal shift suggests: When we shift $(0, 0)$ to the right $\frac{2\pi}{3}$ units, it moves to

$\left(\frac{2\pi}{3}, 0\right)$; then, when we compress the x-coordinate of this point by a factor of $\frac{1}{2}$, it changes to $\frac{1}{2}\cdot\frac{2\pi}{3} = \frac{\pi}{3}$ and the point moves to $\left(\frac{\pi}{3}, 0\right)$ so the y-intercept **doesn't** end up shifted to right $\frac{2\pi}{3}$ units.

In Figure 7, we've graphed $y = g(x)$. Notice that this graph behaves like the graph of $f(x) = \sin(x)$ at $x = \frac{\pi}{3}$, i.e., $y = g(x)$ appears to have been shifted to the right $\frac{\pi}{3}$ units. For this reason, $\frac{\pi}{3}$ is called the ***horizontal shift*** of $g(x) = \sin\left(2x - \frac{2\pi}{3}\right) = \sin\left(2\left(x - \frac{\pi}{3}\right)\right)$.

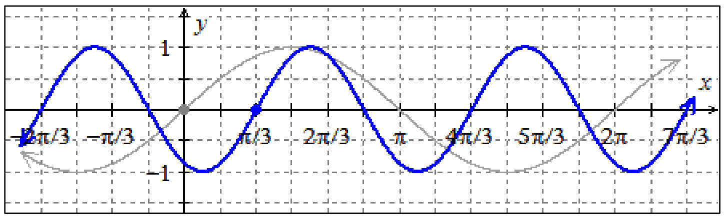


Figure 7: $y = g(x)$ with $f(x) = \sin(x)$.

The constant $\frac{2\pi}{3}$ is given a different name, ***phase shift***, since it can be used to determine how far "out-of-phase" a sinusoidal function is in comparison with $y = \sin(x)$ or $y = \cos(x)$. To determine how far out-of-phase a sinusoidal function is, we can determine the ratio of the phase shift and 2π. (We use 2π is because it's the period of $y = \sin(x)$ and $y = \cos(x)$.) Since $\frac{2\pi}{3}$ is the phase shift for $g(x) = \sin\left(2x - \frac{2\pi}{3}\right)$, the graph of $y = g(x)$ is out-of-phase $\frac{2\pi/3}{2\pi} = \frac{1}{3}$ of a period. (Since this number is positive, it represents a horizontal shift to the right $\frac{1}{3}$ of a period.)

Phase Shift vs. Horizontal Shift

Given a sinusoidal function of the form $y = A\sin(\omega x - C) + k$ or $y = A\cos(\omega x - C) + k$, the ***phase shift*** is C and $\frac{|C|}{2\pi}$ represents the fraction of a period that the graph has been shifted (shift to the right if C is positive or to the left if C is negative).

If we re-write the function as $y = A\sin\left(\omega\left(x - \frac{C}{\omega}\right)\right) + k$ or $y = A\cos\left(\omega\left(x - \frac{C}{\omega}\right)\right) + k$, we can see that the ***horizontal shift*** is $\frac{C}{\omega}$ units (shift to the right if $\frac{C}{\omega}$ is positive or to the left if $\frac{C}{\omega}$ is negative).

EXAMPLE 7: Identify the phase shift and horizontal shift of $g(x) = \cos\left(3x - \frac{\pi}{4}\right)$.

SOLUTION:

- The phase shift of $g(x) = \cos\left(3x - \frac{\pi}{4}\right)$ is $\frac{\pi}{4}$. This tells us that the graph of $y = g(x)$ is out-of-phase $\frac{\left|\pi/4\right|}{2\pi} = \frac{1}{8}$ of a period, i.e., compared with $y = \cos(x)$, the graph of $g(x) = \cos\left(3x - \frac{\pi}{4}\right)$ has been shifted one-eighth of a period to the right.

- To find the horizontal shift, we need to factor-out 3 from $3x - \frac{\pi}{4}$:

$$\begin{aligned} g(x) &= \cos\left(3x - \frac{\pi}{4}\right) \\ &= \cos\left(3\left(x - \frac{\pi}{3\cdot 4}\right)\right) \\ &= \cos\left(3\left(x - \frac{\pi}{12}\right)\right) \end{aligned}$$

So the horizontal shift is $\frac{\pi}{12}$. This tells us that, compared with $y = \cos(x)$, the graph of $g(x) = \cos\left(3x - \frac{\pi}{4}\right)$ has been shifted $\frac{\pi}{12}$ units to the right.

Notice that the period of $g(x) = \cos\left(3x - \frac{\pi}{4}\right)$ is $2\pi \cdot \frac{1}{3} = \frac{2\pi}{3}$, and that one-eighth of $\frac{2\pi}{3}$ is $\frac{2\pi}{3} \cdot \frac{1}{8} = \frac{\pi}{12}$, so a shift of one-eighth of a period is the same as a shift of $\frac{\pi}{12}$ units!

EXAMPLE 8: Draw a graph $q(t) = 2\sin(4t + \pi) + 1$. First, find its amplitude, period, midline, phase shift, and horizontal shift.

SOLUTION:

- Amplitude: $|A| = |2| = 2$
- Period: $P = 2\pi \cdot \frac{1}{|\omega|} = \frac{2\pi}{4} = \frac{\pi}{2}$
- Midline: $y = 1$
- Phase shift: $-\pi$ (this tells us that the graph is out-of-phase $\frac{|-\pi|}{2\pi} = \frac{1}{2}$ of a period)

- Horizontal shift: $\frac{\pi}{4}$ units to the left since:

$$\begin{aligned} q(t) &= 2\sin\left(4t+\pi\right)+1 \\ &= 2\sin\left(4\left(t+\tfrac{\pi}{4}\right)\right)+1 \\ &= 2\sin\left(4\left(t-\left(-\tfrac{\pi}{4}\right)\right)\right)+1 \end{aligned}$$

Now we can draw a graph of $q(t)=2\sin\left(4t+\pi\right)+1$ by drawing a sinusoidal function with the necessary features; see Figure 8.

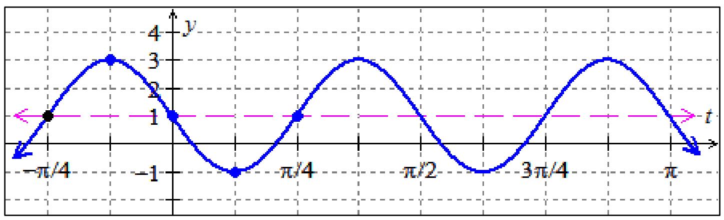


Figure 8: $y=q(t)$

EXERCISES:

1. Draw a graph of each of the following functions. List the amplitude, midline, period, phase shift, and horizontal shift.

a. $f(x)=3\sin\left(3x-\frac{\pi}{2}\right)$

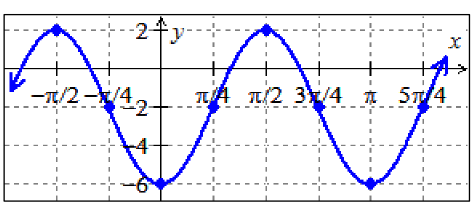
b. $g(t)=\cos\left(4t+\pi\right)+3$

c. $m(\theta)=2\cos(2\pi\theta-\pi)+4$

d. $n(x)=-4\sin\left(\pi x+\frac{\pi}{4}\right)-2$

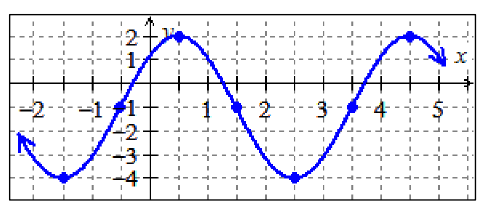
2. Find two algebraic rules (one involving sine and one involving cosine) for each of the functions graphed below.

a.



A graph of $y=p(t)$

b.



A graph of $y=q(x)$

SUPPLEMENT TO §8.3

Complex Numbers and Polar Coordinates

Recall that a *complex number* has the form $a + bi$ where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$. Complex numbers have two parts: a real part and an imaginary part. For the number $a + bi$, the real part is a and the imaginary part is b. Because of they have *two* parts, we can use the *two dimensional* rectangular coordinate plane to represent complex numbers. We use the horizontal axis to represent the real part and the vertical axis to represent the complex part. Thus, the complex number $a + bi$ can be represented by the point (a, b) on the rectangular coordinate plane; see Figure 9.

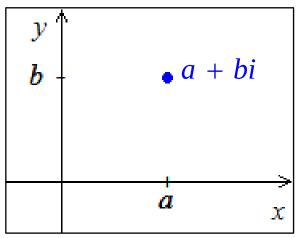


Figure 9

As we've studied in this course, the rectangular ordered pair (a, b) can be represented in polar coordinates (r, θ) where r represents the distance the point is from the origin and θ represents the angle between the positive x-axis and the segment connecting the origin and the point; see Figure 10.

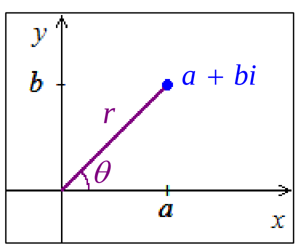


Figure 10

We know that if the rectangular pair (a, b) represents the same point as the polar pair (r, θ), then $a = r\cos(\theta)$ and $b = r\sin(\theta)$. Thus,

$$\begin{aligned} a + bi &= r\cos(\theta) + r\sin(\theta) \cdot i \\ &= r(\cos(\theta) + i \cdot \sin(\theta)) \end{aligned}$$

i.e., we can express a complex number using the "polar information" r and θ.

The expression "$r\left(\cos(\theta) + i \cdot \sin(\theta)\right)$" is what our textbook describes as the "polar form of a complex number." But a more appropriate expression to label as "the polar form of a complex number" involves *Euler's Formula*. Euler's Formula is an identity that establishes a surprising connection between the exponential function e^x and complex numbers.

EULER'S FORMULA:

$$e^{i\theta} = \cos(\theta) + i \cdot \sin(\theta)$$

Notice that if we multiply both sides of Euler's formula by *r* we obtain a formula that allows us to write any complex number in **polar form**:

$$\begin{aligned} & e^{i\theta} = \cos(\theta) + i \cdot \sin(\theta) \\ \Rightarrow\ & r \cdot \left(e^{i\theta}\right) = r \cdot (\cos(\theta) + i \cdot \sin(\theta)) \\ \Rightarrow\ & re^{i\theta} = r\cos(\theta) + r\sin(\theta) \cdot i \end{aligned}$$

The **polar form** of the complex number $z = r\cos(\theta) + r\sin(\theta) \cdot i$ is:

$$z = re^{i\theta}.$$

Let's review what we've established: First, we observed that we can write a complex number of the form "$a + bi$" in the form "$r(\cos(\theta) + i \cdot \sin(\theta))$". Then we noticed that we can write an expression of the form "$r(\cos(\theta) + i \cdot \sin(\theta))$" in the form "$re^{i\theta}$". Finally, we realized that we can write a complex number "$a + bi$" in the form "$re^{i\theta}$" so we defined "$re^{i\theta}$" as being *the polar form* of the complex number $a + bi$.

EXAMPLE 9: Express in "rectangular form" (i.e., in the form $z = a + bi$) the complex number $z = 6e^{\frac{5\pi}{6} \cdot i}$ given in polar form.

SOLUTION:

$$\begin{aligned} z &= 6e^{\frac{5\pi}{6} \cdot i} \\ &= 6\cos\left(\tfrac{5\pi}{6}\right) + 6\sin\left(\tfrac{5\pi}{6}\right) \cdot i \\ &= 6\left(-\tfrac{\sqrt{3}}{2}\right) + 6\left(\tfrac{1}{2}\right) \cdot i \\ &= -3\sqrt{3} + 3i \end{aligned}$$

Thus, the complex number $z = 6e^{\frac{5\pi}{6} \cdot i}$ can be expressed in "rectangular form" as $z = -3\sqrt{3} + 3i$.

EXAMPLE 10: Express in polar form (i.e., in the form $z = re^{i\theta}$) the complex number $z = 3 - 3i$ given in "rectangular form."

SOLUTION:

We can associate the complex number $z = 3 - 3i$ with the rectangular ordered pair $(3, -3)$, and then translate this ordered pair into polar coordinates (r, θ), and finally use this polar ordered pair to obtain the polar form $z = re^{i\theta}$. First, let's find r:

$$\begin{aligned} r &= \sqrt{(3)^2 + (-3)^2} \\ &= \sqrt{9 + 9} \\ &= 3\sqrt{2}. \end{aligned}$$

Now, let's find θ:

$$\begin{aligned} &\tan(\theta) = \tfrac{-3}{3} \\ \Rightarrow \quad &\theta = \tan^{-1}(-1) \\ \Rightarrow \quad &\theta = -\tfrac{\pi}{4} \end{aligned}$$

Thus, the complex number $z = 3 - 3i$ can be expressed in polar form $z = 3\sqrt{2}\, e^{-\frac{\pi}{4} \cdot i}$.

Using Polar Form to find Complex Roots

EXAMPLE 11: Find the two square roots of $-1 + i\sqrt{3}$ using the polar form of $-1 + i\sqrt{3}$.

SOLUTION:

Recall that there are two distinct square roots of any positive real number (e.g., the two square roots of 4 are 2 and –2). The same is true for any complex number. We can find two different square roots of a complex number by using two different polar forms of the number.

To find polar forms of $-1 + i\sqrt{3}$, we first associate the number with the rectangular ordered pair $\left(-1, \sqrt{3}\right)$ and then translate it into polar coordinates (r, θ):

$$\begin{aligned} r &= \sqrt{(-1)^2 + (\sqrt{3})^2} \\ &= \sqrt{1+3} \\ &= 2 \end{aligned}$$

$$\tan(\theta) = -\sqrt{3}, \text{ with } \theta \text{ in Quadrant II.}$$

Both $\theta = \frac{2\pi}{3}$ and $\theta = -\frac{4\pi}{3}$ satisfy this condition, so we'll use these two angles to obtain two polar forms of $-1 + i\sqrt{3}$:

$$-1 + i\sqrt{3} = 2e^{\frac{2\pi}{3}\cdot i} \quad \text{and} \quad -1 + i\sqrt{3} = 2e^{-\frac{4\pi}{3}\cdot i}$$

Therefore,

$$\begin{aligned} \left(-1 + i\sqrt{3}\right)^{1/2} &= \left(2e^{\frac{2\pi}{3}\cdot i}\right)^{1/2} \\ &= 2^{\frac{1}{2}} e^{\frac{2\pi}{3}\cdot\frac{1}{2} i} \\ &= \sqrt{2}\, e^{\frac{\pi}{3}\cdot i} \\ &= \sqrt{2}\left(\cos\left(\frac{\pi}{3}\right) + i\cdot\sin\left(\frac{\pi}{3}\right)\right) \\ &= \sqrt{2}\cdot\left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) \\ &= \frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{2} i \end{aligned}$$

and

$$\begin{aligned}\left(-1+i\sqrt{3}\right)^{1/2} &= \left(2e^{-\frac{4\pi}{3}\cdot i}\right)^{1/2}\\ &= 2^{\frac{1}{2}}\,e^{-\frac{4\pi}{3}\cdot\frac{1}{2}\,i}\\ &= \sqrt{2}\,e^{-\frac{2\pi}{3}\cdot i}\\ &= \sqrt{2}\left(\cos\left(-\frac{2\pi}{3}\right)+i\cdot\sin\left(-\frac{2\pi}{3}\right)\right)\\ &= \sqrt{2}\cdot\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\\ &= -\frac{\sqrt{2}}{2}-\frac{\sqrt{6}}{2}i.\end{aligned}$$

So both $\frac{\sqrt{2}}{2}+\frac{\sqrt{6}}{2}i$ and $-\frac{\sqrt{2}}{2}-\frac{\sqrt{6}}{2}i$ are square roots of $-1+i\sqrt{3}$. But just as 2, not –2, is called the ***principal square root*** of 4, only one of the two roots that we found is the *principal* square root of $-1+i\sqrt{3}$. The principal root of a complex number is the one found by using an angle in the interval $(-\pi,\ \pi]$ to represent the complex number in polar form, so the first root we found (i.e., the one we found using $\theta=\frac{2\pi}{3}$) is the principal root of $-1+i\sqrt{3}$. The principal root is the one represented by the radical symbol, so we can write

$$\sqrt{-1+i\sqrt{3}}=\frac{\sqrt{2}}{2}+\frac{\sqrt{6}}{2}i.$$

EXAMPLE 12: Find $\sqrt[3]{-4\sqrt{2}+4\sqrt{2}\,i}$ using the polar form of $-4\sqrt{2}+4\sqrt{2}\,i$.

SOLUTION:

To find polar forms of $-4\sqrt{2}+4\sqrt{2}\,i$, we first associate the number with the rectangular ordered pair $\left(-4\sqrt{2},\ 4\sqrt{2}\right)$ and then translate it into polar coordinates $(r,\ \theta)$. First, let's find r:

$$\begin{aligned} r &= \sqrt{\left(-4\sqrt{2}\right)^2 + \left(4\sqrt{2}\right)^2} \\ &= \sqrt{4^2 \cdot 2 + 4^2 \cdot 2} \\ &= 4\sqrt{2+2} \\ &= 8 \end{aligned}$$

Now, let's find θ:

$$\tan(\theta) = \tfrac{4\sqrt{2}}{-4\sqrt{2}}$$

$$\Rightarrow \quad \theta = \tan^{-1}(-1) + \pi$$ (we add π since $\left(-4\sqrt{2},\, 4\sqrt{2}\right)$ is in Quad. 2, outside the range of arctangent)

$$\Rightarrow \quad \theta = -\tfrac{\pi}{4} + \pi$$

$$\Rightarrow \quad \theta = \tfrac{3\pi}{4}$$

So the polar form of $-4\sqrt{2} + 4\sqrt{2}\,i$ is $z = 8e^{\frac{3\pi}{4}\cdot i}$. Therefore:

$$\begin{aligned} \sqrt[3]{-4\sqrt{2} + 4\sqrt{2}\,i} &= \left(8e^{\frac{3\pi}{4}\cdot i}\right)^{1/3} \\ &= \sqrt[3]{8} \cdot e^{\frac{3\pi}{4}\cdot\frac{1}{3}i} \\ &= 2e^{\frac{\pi}{4}\cdot i} \\ &= 2\left(\cos\left(\frac{\pi}{4}\right) + i \cdot \sin\left(\frac{\pi}{4}\right)\right) \\ &= 2 \cdot \left(\frac{\sqrt{2}}{2} + i \cdot \frac{\sqrt{2}}{2}\right) \\ &= \sqrt{2} + i\sqrt{2} \end{aligned}$$

EXERCISES:

1. Find the polar form $z = re^{i\theta}$ of the following complex numbers given in rectangular form.

a. $z = 6 + 6\sqrt{3}\,i$ **b.** $z = -2\sqrt{3} + 2i$ **b.** $z = 5\sqrt{2} - 5\sqrt{2}\,i$

2. Find the rectangular form $z = a + bi$ of the following complex numbers given in polar form.

a. $z = 8e^{\frac{\pi}{6}\cdot i}$ **b.** $z = 4e^{i\cdot\pi}$ **b.** $z = 5e^{\frac{4\pi}{3}\cdot i}$

3. Find the following principal roots by first converting to the polar form of complex number.

a. $\sqrt{18 - 18\sqrt{3}\,i}$ **b.** $\sqrt[3]{-16 + 16i}$

c. $\sqrt{-i}$ **d.** $\sqrt[5]{-16\sqrt{3} - 16i}$

4. **a.** Find all three of the cube roots of $27i$.

b. Find both of the square roots of $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$.

5. Find all three solutions to the equation $z^3 + 1 = 0$.

SOLUTIONS: Supplement to §5.1

1. **a.** 423° and -297° are coterminal with 63°.

b. $\frac{19\pi}{9}$ and $-\frac{17\pi}{9}$ are coterminal with $\frac{\pi}{9}$.

c. $\frac{29\pi}{8}$ and $-\frac{3\pi}{8}$ are coterminal with $\frac{13\pi}{8}$.

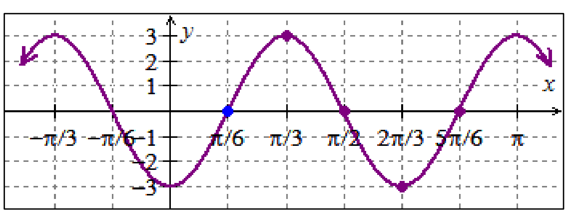
2. **a.** 60° **b.** $\frac{\pi}{4}$ **c.** $\frac{3\pi}{8}$

d. 40° **e.** $\pi - 2 \approx 1.14$ **f.** $\frac{\pi}{11}$

g. $\frac{\pi}{5}$ **h.** 20° **i.** 80°

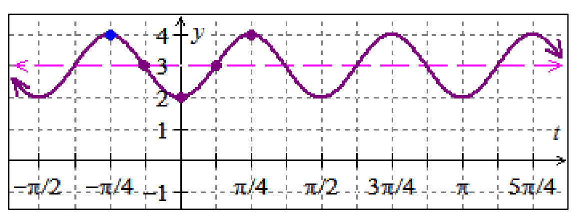
SOLUTIONS: Supplement to §5.6

1.a. the *amplitude* is 3 units; the *period* is $\frac{2\pi}{3}$ units; the *midline* is $y = 0$; the *phase shift* is $\frac{\pi}{2}$; the *horizontal shift* is $\frac{\pi}{6}$ units to the right.



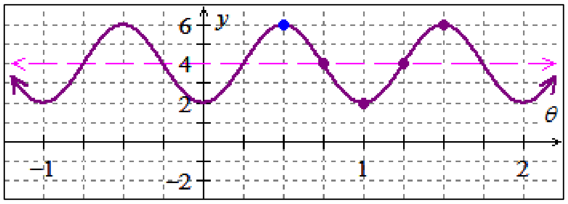
A graph of $f(x) = 3\sin\left(3x - \frac{\pi}{2}\right)$.

1.b. the *amplitude* is 1 unit; the *period* is $\frac{\pi}{2}$ units; the *midline* is $y = 3$; the *phase shift* is $-\pi$; the *horizontal shift* is $\frac{\pi}{4}$ units to the left.



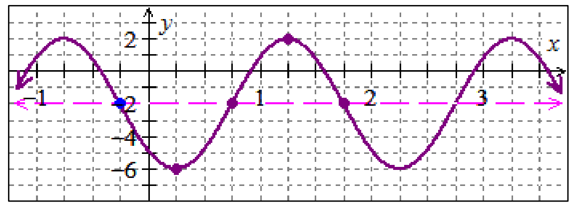
A graph of $g(t) = \cos\left(4t + \pi\right) + 3$.

1.c. the *amplitude* is 2 units; the *period* is 1 unit; the *midline* is $y = 4$; the *phase shift* is π; the *horizontal shift* is $\frac{1}{2}$ of a unit to the right.



A graph of $m(\theta) = 2\cos(2\pi\theta - \pi) + 4$.

1.d. the *amplitude* is 4 units; the *period* is 2 units; the *midline* is $y = -2$; the *phase shift* is $-\frac{\pi}{4}$; the *horizontal shift* is $\frac{1}{4}$ of a unit to the left.



A graph of $n(x) = -4\sin\left(\pi x + \frac{\pi}{4}\right) - 2$.

2. a. $p(t) = 4\,\mathbf{sin}\left(2\left(x - \frac{\pi}{4}\right)\right) - 2$, $\quad p(t) = 4\,\mathbf{cos}\left(2\left(x - \frac{\pi}{2}\right)\right) - 2$

b. $q(x) = 3\,\mathbf{sin}\left(\frac{\pi}{2}\left(x + \frac{1}{2}\right)\right) - 1$, $\quad q(x) = 3\,\mathbf{cos}\left(\frac{\pi}{2}\left(x - \frac{1}{2}\right)\right) - 1$

SOLUTIONS: Supplement to §8.3

1. a. $z = 12e^{\frac{\pi}{3}\cdot i}$ **b.** $z = 4e^{\frac{5\pi}{6}\cdot i}$ **c.** $z = 10e^{-\frac{\pi}{4}\cdot i}$

2. a. $z = 4\sqrt{3} + 4i$ **b.** $z = -4$ **c.** $z = -\frac{5}{2} - \frac{5\sqrt{3}}{2}i$

3. a. $\sqrt{18 - 18\sqrt{3}\,i} = 3\sqrt{3} - 3i$ **b.** $\sqrt[3]{-16 + 16i} = 2 + 2i$

c. $\sqrt{-i} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$ **d.** $\sqrt[5]{-16\sqrt{3} - 16i} = \sqrt{3} - i$

4. a. The three cube roots of $27i$ are $\frac{3\sqrt{3}}{2} + \frac{3}{2}i$, $-3i$, and $-\frac{3\sqrt{3}}{2} + \frac{3}{2}i$.

b. The two square roots of $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$ are $\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$.

5. The solutions to $z^3 + 1 = 0$ are $z = -1$, $z = \frac{1}{2} + \frac{\sqrt{3}}{2}i$, and $z = \frac{1}{2} - \frac{\sqrt{3}}{2}i$.