Section I: The Trigonometric Functions

Chapter 7: Solving Trig Equations

Let's start by solving a couple of equations that involve the sine function.

EXAMPLE 1a: Solve the equation $\sin(t) = \frac{1}{2}$.

SOLUTION:

The inverse functions we constructed in Chapter 6 can be used to solve equations like $\sin(t) = \frac{1}{2}$ but the fraction $\frac{1}{2}$ is a "friendly" sine value so we don't need to use the inverse sine function: our experience with the sine function tells us that that $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$, so we know that $t = \frac{\pi}{6}$ is a solution to $\sin(t) = \frac{1}{2}$. We also know that the sine function is periodic with period 2π, so its values repeat every 2π units, so angles like

$$t = \frac{\pi}{6} + 2\pi = \frac{13\pi}{6} \quad \textit{and} \quad t = \frac{\pi}{6} + 4\pi = \frac{25\pi}{6} \quad \textit{and} \quad t = \frac{\pi}{6} - 2\pi = -\frac{11\pi}{6}$$

are also solutions. We can represent multiples of the period with the expression $2k\pi$ where k is any integer, i.e., $k \in \mathbb{Z}$, so we can represent all of the solutions that are "related" to $\frac{\pi}{6}$ with the expression $\frac{\pi}{6} + 2k\pi$, $k \in \mathbb{Z}$. This expression represents *infinitely many solutions*, but it still doesn't represent all of the solutions; see Figure 1.

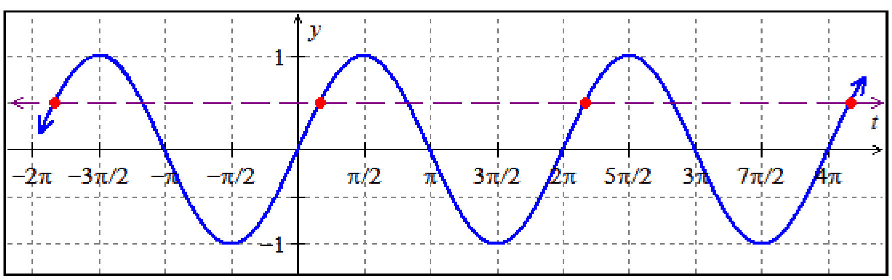

Figure 1: The graph of $y = \sin(t)$ and the line $y = \frac{1}{2}$. The red dots represent points with horizontal coordinates of the form $t = \frac{\pi}{6} + 2k\pi$, $k \in \mathbb{Z}$. The other instances where the blue graph intersects the line $y = \frac{1}{2}$ are also solutions to the equation $\sin(t) = \frac{1}{2}$ but they are NOT represented by $t = \frac{\pi}{6} + 2k\pi$.

Notice that one of the solutions that we are missing is just as close to π as our original solution, $t = \frac{\pi}{6}$, is to 0. Recall the identity $\sin(t) = \sin(\pi - t)$ that we first noticed in Part 1 of Chapter 3: this identity tells us that the angles t and $\pi - t$ always have the same sine value. This means that whenever we've found a solution, t, to an equation involving sine, we can find another solution by computing $\pi - t$. Now let's apply this observation to find the rest of the solutions to $\sin(t) = \frac{1}{2}$: since we know that $t = \frac{\pi}{6}$ is a solution to $\sin(t) = \frac{1}{2}$, we know that $t = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$ is another solution. And now we can again utilize the fact that the period of the sine function is 2π so we can express the rest of the solutions with $t = \frac{5\pi}{6} + 2k\pi$, $k \in \mathbb{Z}$; in Figure 2, these solutions are colored green. So the complete solution to the equation $\sin(t) = \frac{1}{2}$ is:

$$t = \frac{\pi}{6} + 2k\pi \quad \text{or} \quad t = \frac{5\pi}{6} + 2k\pi \quad \text{for all } k \in \mathbb{Z}.$$

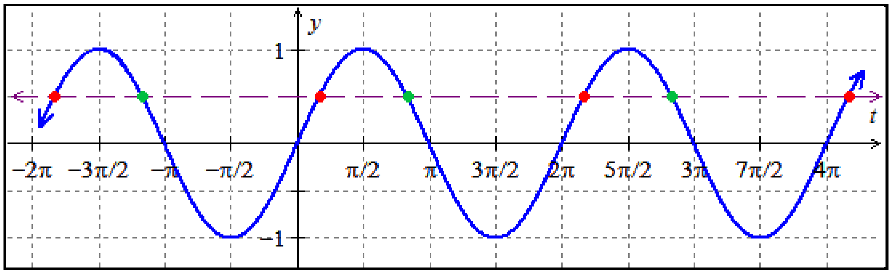

Figure 2: The red dots represent points with horizontal coordinates of the form $t = \frac{\pi}{6} + 2k\pi$, $k \in \mathbb{Z}$, while the green dots represent points with horizontal coordinates of the form $t = \frac{5\pi}{6} + 2k\pi$, $k \in \mathbb{Z}$. The red and green dots together represent all of the solutions to $\sin(t) = \frac{1}{2}$.

EXAMPLE 1b: Solve the equation $\sin(t) = -0.555$.

SOLUTION:

Unlike Example 1a where the equation involved a "friendly" sine value, -0.555 isn't a "friendly" sine value: we don't know what input for the sine function is related to the output -0.555, so we need to utilize the inverse sine function that we constructed in Chapter 6 in order to solve the equation:

$$\sin(t) = -0.555$$

$$\Rightarrow \sin^{-1}(\sin(t)) = \sin^{-1}(-0.555)$$ (apply the inverse sine function to both sides of the equation)

$$\Rightarrow t = \sin^{-1}(-0.555) \approx -0.588$$

(Note that you can utilize a calculator to obtain an approximation for $\sin^{-1}(-0.555)$ by accessing a button labeled "$\sin^{-1}$".)

Although we've found *one* solution to the equation, **we aren't done yet!** The inverse sine inverse only gives us one value, but we know that the periodic nature of the sine function suggests that there are *infinitely many solutions* to an equation like this; see Figure 3.

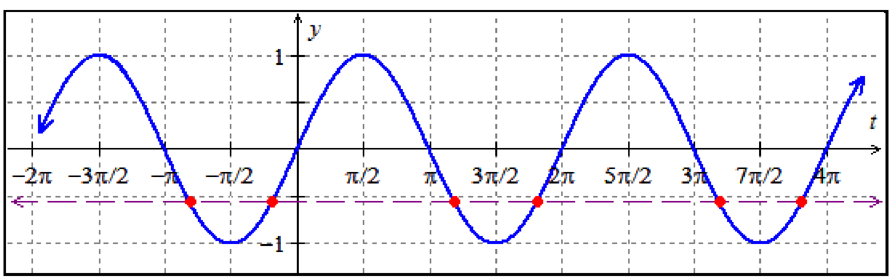

Figure 3: The graph of $y = \sin(t)$ intersecting the line $y = -0.555$ many, many times. Each point of intersection represents a solution to $\sin(t) = -0.555$.

We can find all of the solutions by using the solution that we found using the inverse sine function along with the fact that the sine function has period 2π: since the sine function has period 2π units, we know that the outputs repeat every 2π units. So if $t \approx -0.588$ is a solution, the values represented by $t \approx -0.588 + 2k\pi$, $k \in \mathbb{Z}$ must also be solutions. This gives us LOTS of solutions, but we are still missing *half* of them. (Recall we had the same problem in Example 1a.) In order to get the rest of the solutions, can use the identity $\sin(t) = \sin(\pi - t)$, and subtract our original solution $(t \approx -0.588)$ from π: $t \approx \pi - (-0.588) + 2k\pi$, $k \in \mathbb{Z}$. Therefore, the complete solution to the equation is:

$$t \approx -0.588 + 2k\pi \text{ or } t \approx \pi + 0.588 + 2k\pi \text{ for all } k \in \mathbb{Z}.$$

EXAMPLE 2a: Solve the equation $\cos(t) = -\frac{\sqrt{3}}{2}$.

SOLUTION:

Like Example 1a, $-\frac{\sqrt{3}}{2}$ is a "friendly" cosine value so we can use our knowledge about the cosine function, rather than the inverse cosine function, to solve the equation. Our experience with the cosine function tells us that that $\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$, so we know that $t = \frac{5\pi}{6}$ is a solution to $\cos(t) = -\frac{\sqrt{3}}{2}$. We also know that the cosine function is periodic with period 2π, so its values repeat every 2π units, so angles like

$$t = \frac{5\pi}{6} + 2\pi = \frac{17\pi}{6} \quad \textit{and} \quad t = \frac{5\pi}{6} + 4\pi = \frac{29\pi}{6} \quad \textit{and} \quad t = \frac{5\pi}{6} - 2\pi = -\frac{7\pi}{6}$$

are also solutions. We can represent all of the solutions that are "related" to $\frac{5\pi}{6}$ with the expression $\frac{5\pi}{6} + 2k\pi$, $k \in \mathbb{Z}$. This expression represents *infinitely many solutions*, but it still doesn't represent all of the solutions; see Figure 4.

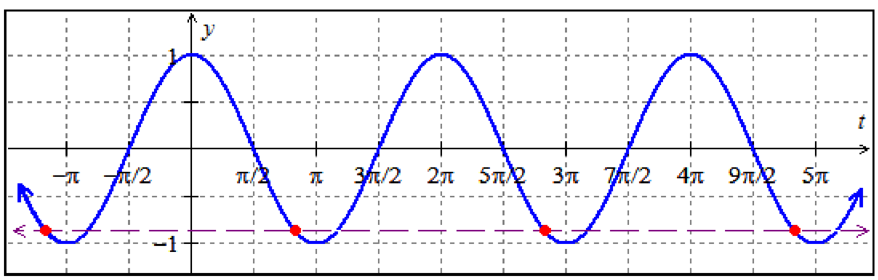

Figure 4: The graph of $y = \cos(t)$ and the line $y = -\frac{\sqrt{3}}{2}$. The red dots represent points with horizontal coordinates of the form $t = \frac{5\pi}{6} + 2k\pi$, $k \in \mathbb{Z}$. The other instances where the blue graph intersects the line $y = -\frac{\sqrt{3}}{2}$ are also solutions to the equation $\cos(t) = -\frac{\sqrt{3}}{2}$ but they are NOT represented by $t = \frac{5\pi}{6} + 2k\pi$.

Recall the identity $\cos(t) = \cos(-t)$ that we noticed in Part 1 of Chapter 3: this identity tells us that the angles t and $-t$ always have the same cosine value. This means that whenever we've found a solution, t, to an equation involving cosine, we can find another solution by computing $-t$. Now let's apply this observation to find the rest of the solutions to $\cos(t) = -\frac{\sqrt{3}}{2}$: since we know that $t = \frac{5\pi}{6}$ is a solution to $\cos(t) = -\frac{\sqrt{3}}{2}$, we know that $t = -\frac{5\pi}{6}$ is another solution. Now we can again utilize the fact that the period of cosine is

2π so we can express the rest of the solutions with $t = -\frac{5\pi}{6} + 2k\pi$, $k \in \mathbb{Z}$; in Figure 5, these solutions are colored green. So the complete solution to the equation $\cos(t) = -\frac{\sqrt{3}}{2}$ is:

$$t = \frac{5\pi}{6} + 2k\pi \quad \text{or} \quad t = -\frac{5\pi}{6} + 2k\pi \quad \text{for all } k \in \mathbb{Z}.$$

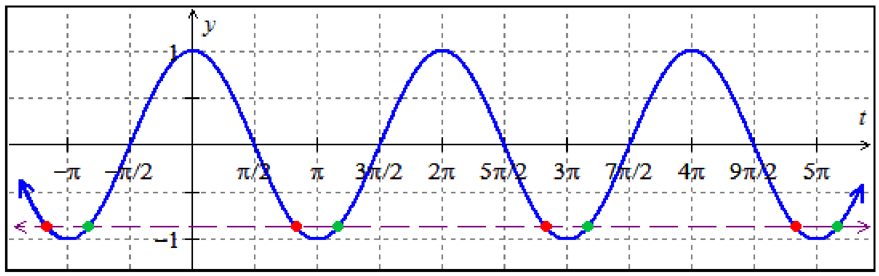

Figure 5: The red dots represent points with horizontal coordinates of the form $t = \frac{5\pi}{6} + 2k\pi$, $k \in \mathbb{Z}$, while the green dots represent points with horizontal coordinates of the form $t = -\frac{5\pi}{6} + 2k\pi$, $k \in \mathbb{Z}$. The red and green dots together represent all of the solutions to $\cos(t) = -\frac{\sqrt{3}}{2}$.

EXAMPLE 2b: Solve the equation $\cos(t) = 0.4$.

SOLUTION:

Like in Example 1b, 0.4 isn't a "friendly" cosine value so we need to utilize the inverse cosine function that we constructed in Chapter 6 in order to solve the equation:

$$\cos(t) = 0.4$$

$$\Rightarrow \cos^{-1}\left(\cos(t)\right) = \cos^{-1}(0.4) \qquad \text{(apply the inverse cosine function to both sides of the equation)}$$

$$\Rightarrow \qquad t = \cos^{-1}(0.4) \approx 1.16$$

(Note that you can utilize a calculator to obtain an approximation for $\cos^{-1}(0.4)$ by accessing a button labeled "$\cos^{-1}$".)

Although we have found a solution to the given equation, **we aren't done yet!** The inverse cosine function only gives us *one* value but we know that the periodic nature of the cosine function suggests that there are *infinitely many solutions* to an equation like this; see Figure 6.

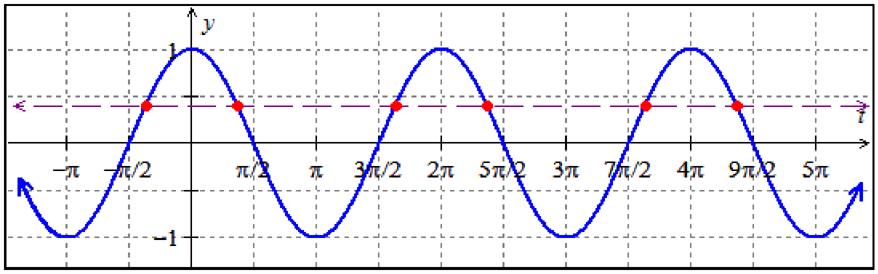

Figure 6: The graph of $y = \cos(t)$ intersecting the line $y = 0.4$ many, many times. Each point of intersection represents a solution to $\cos(t) = 0.4$.

We can find all of the solutions by using the solution that we found using the inverse cosine function along with the fact that the cosine function has period 2π : since the cosine function has period 2π units, we know that the outputs repeat every 2π units. So if $t \approx 1.16$ is a solution, the values represented by $t \approx 1.16 + 2k\pi$, $k \in \mathbb{Z}$ must also be solutions. This gives us LOTS of solutions, but we are still missing *half* of them. (Recall we had the same problem in Example 2a.) In order to get the rest of the solutions, we can use the identity $\cos(t) = \cos(-t)$ and take the opposite of original solution to find a second "family" of solutions: $t \approx -1.16 + 2k\pi$, $k \in \mathbb{Z}$. Therefore, the complete solution to the equation $\cos(t) = 0.4$ is:

$$t \approx 1.16 + 2k\pi \text{ or } t \approx -1.16 + 2k\pi \text{ for all } k \in \mathbb{Z}.$$

EXAMPLE 3a: Solve the equation $2\cos(t) = -1$.

SOLUTION:

CLICK HERE to see a video of this example.

$$2\cos(t) = -1$$

$$\Rightarrow \cos(t) = -\tfrac{1}{2}$$

$$\Rightarrow \cos^{-1}\left(\cos(t)\right) = \cos^{-1}\left(-\tfrac{1}{2}\right)$$ (this step is optional since $-\frac{1}{2}$ is a "friendly" cosine value)

$$\Rightarrow t = \tfrac{2\pi}{3} + 2k\pi \text{ or } t = -\tfrac{2\pi}{3} + 2k\pi \text{ for } k \in \mathbb{Z}$$ (since $\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$ and $\cos(t) = \cos(-t)$)

[Since $-\frac{1}{2}$ is a "friendly" cosine value, we didn't need to use the inverse cosine function as we did in the third step – the inverse trig functions are available when solving equations but we don't need to use them if the values are friendly.]

EXAMPLE 3b: Solve the equation $8\cos(x) + 9 = 10$.

SOLUTION:

$$8\cos(x) + 9 = 10$$

$$\Rightarrow 8\cos(x) = 1$$

$$\Rightarrow \cos(x) = \tfrac{1}{8}$$

$$\Rightarrow \cos^{-1}\left(\cos(x)\right) = \cos^{-1}\left(\tfrac{1}{8}\right)$$

$$\Rightarrow t = \cos^{-1}\left(\tfrac{1}{8}\right) + 2k\pi \text{ or } t = -\cos^{-1}\left(\tfrac{1}{8}\right) + 2k\pi \text{ for } k \in \mathbb{Z}$$

$$\Rightarrow t \approx 1.445 + 2k\pi \text{ or } t \approx -1.445 + 2k\pi \text{ for } k \in \mathbb{Z}$$ (since $\cos^{-1}\left(\frac{1}{8}\right) \approx 1.445$)

[Since $\frac{1}{8}$ isn't a "friendly" cosine value, we need to use the inverse cosine function as we've done in the fourth step. Also note that in the last step we've approximated the solutions: this requires a calculator, so it's not something that we would need to do on no-calculator exams.]

EXAMPLE 4a: Solve the equation $4\sin(\theta) + \sqrt{3} = -\sqrt{3}$.

SOLUTION:

CLICK HERE to see a video of this example.

$$4\sin(\theta) + \sqrt{3} = -\sqrt{3}$$

$$\Rightarrow \quad 4\sin(\theta) = -2\sqrt{3}$$

$$\Rightarrow \quad \sin(\theta) = -\frac{2\sqrt{3}}{4} = -\frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad \sin^{-1}(\sin(\theta)) = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$$ (this step is optional since $-\frac{\sqrt{3}}{2}$ is a "friendly" sine value)

$$\Rightarrow \theta = -\frac{\pi}{3} + 2k\pi \text{ or } \theta = \pi - \left(-\frac{\pi}{3}\right) + 2k\pi \text{ for } k \in \mathbb{Z}$$ (since $\sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$ and $\sin(t) = \sin(\pi - t)$)

$$\Rightarrow \theta = -\frac{\pi}{3} + 2k\pi \text{ or } \theta = \frac{4\pi}{3} + 2k\pi \text{ for } k \in \mathbb{Z}$$

[Since $-\frac{\sqrt{3}}{2}$ is a "friendly" value, we didn't need to employ the inverse sine function as we did in the fourth step – the inverse trig functions are always available when solving trig equations but we don't need to use them when the values are friendly.]

EXAMPLE 4b: Solve the equation $13\sin(t) = 6$.

SOLUTION:

$$13\sin(t) = 6$$

$$\Rightarrow \quad \sin(t) = \frac{6}{13}$$

$$\Rightarrow \quad \sin^{-1}(\sin(t)) = \sin^{-1}\left(\frac{6}{13}\right)$$

$$\Rightarrow t = \sin^{-1}\left(\frac{6}{13}\right) + 2k\pi \text{ or } t = \pi - \sin^{-1}\left(\frac{6}{13}\right) + 2k\pi \text{ for all } k \in \mathbb{Z}$$

$$\Rightarrow t \approx 0.48 + 2k\pi \text{ or } t \approx \pi - 0.48 + 2k\pi \text{ for all } k \in \mathbb{Z}$$ (since $\sin^{-1}\left(\frac{6}{13}\right) \approx 0.48$)

$$\Rightarrow t \approx 0.48 + 2k\pi \text{ or } t \approx 2.66 + 2k\pi \text{ for all } k \in \mathbb{Z}$$

[Since $\frac{6}{13}$ isn't a "friendly" sine value, we need to employ the inverse sine function as we've done in the third step. Also note that in the second-to-last step we've approximated the solutions: this requires a calculator, so it's not something that we need to do on a no-calculator activity.]

EXAMPLE 5a: Solve the equation $\sqrt{3}\tan(x) = 1$.

SOLUTION:

$$\sqrt{3}\tan(x) = 1$$

$$\Rightarrow \quad \tan(x) = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad \tan^{-1}(\tan(x)) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$ (this step is optional since $\frac{1}{\sqrt{3}}$ is a "friendly" tangent value)

$$\Rightarrow \quad x = \frac{\pi}{6} + k\pi \quad \text{for all } k \in \mathbb{Z}$$ (since $\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$)

[Notice that we add $k\pi$ (rather than $2k\pi$) to our solutions since, unlike sine and cosine, the period of tangent is π units. Also, there's only one "family" of solutions since tangent only reaches each output value once in each period.]

EXAMPLE 5b: Solve the equation $5\tan(\theta) - 10 = -6$.

SOLUTION:

$$5\tan(\theta) - 10 = -6$$

$$\Rightarrow \quad 5\tan(\theta) = 4$$

$$\Rightarrow \quad \tan(\theta) = \frac{4}{5}$$

$$\Rightarrow \quad \tan^{-1}(\tan(\theta)) = \tan^{-1}\left(\frac{4}{5}\right)$$

$$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{4}{5}\right) + k\pi \quad \text{for all } k \in \mathbb{Z}$$

[Since $\frac{4}{5}$ isn't a "friendly" tangent value, we need to use the inverse tangent function as we've done in the fourth step. As mentioned above in Example 5a, we add $k\pi$ (rather than $2k\pi$) to our solutions since the period of tangent is π units and there's only one "family" of solutions since tangent only reaches each output value once in each period.]

EXAMPLE 6: **a.** Find all of the solutions to the equation $6\sin(2x) = 3\sqrt{2}$.

b. Find the solutions to $6\sin(2x) = 3\sqrt{2}$ that are in the interval $[0, 2\pi]$.

SOLUTION:

a. Notice that the trigonometric function involved in the given equation is $\sin(2x)$, and recall that $\sin(2x)$ has period π units, i.e., the values for $\sin(2x)$ repeat every π units. This means that once we find a solution to the given equation we'll be able to add to it any integer multiple of π and obtain another solution. Thus, we should expect the phrase "$k\pi$ for all $k \in \mathbb{Z}$" to be involved in our solutions. You'll see in the work below that we add $2k\pi$ to our solutions after applying the inverse sine function since the period of the sine function is 2π units. In the last step, we finish solving for x and obtain the desired period-shift of $k\pi$ units.

$$6\sin(2x) = 3\sqrt{2}$$

$$\Rightarrow \quad \sin(2x) = \frac{\sqrt{2}}{2}$$

$$\Rightarrow \quad \sin^{-1}\left(\sin(2x)\right) = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$$

$$\Rightarrow \quad 2x = \frac{\pi}{4} + 2k\pi \quad \text{or} \quad 2x = \pi - \frac{\pi}{4} + 2k\pi \quad \text{for all } k \in \mathbb{Z}$$

$$\Rightarrow \quad x = \frac{1}{2}\left(\frac{\pi}{4} + 2k\pi\right) \quad \text{or} \quad x = \frac{1}{2}\left(\frac{3\pi}{4} + 2k\pi\right) \quad \text{for all } k \in \mathbb{Z}$$

$$\Rightarrow \quad x = \frac{\pi}{8} + k\pi \quad \text{or} \quad x = \frac{3\pi}{8} + k\pi \quad \text{for all } k \in \mathbb{Z}$$

b. Now we need to substitute specific values of k into the solutions we found in part (a) and determine which solutions are in the interval $[0, 2\pi]$.

$\boldsymbol{k = -1}$: $x = \frac{\pi}{8} + (\mathbf{-1}) \cdot \pi \quad$ or $\quad x = \frac{3\pi}{8} + (\mathbf{-1}) \cdot \pi$

$\qquad = -\frac{7\pi}{8} \qquad\qquad\qquad = -\frac{5\pi}{8}$

Both of these values are negative so they aren't in the interval $[0, 2\pi]$. Smaller values of k will produce even smaller values of x so we don't need to try smaller values of k.

$\boldsymbol{k = 0}$: $x = \frac{\pi}{8} + \mathbf{0} \cdot \pi \quad$ or $\quad x = \frac{3\pi}{8} + \mathbf{0} \cdot \pi$

$\qquad = \frac{\pi}{8} \qquad\qquad\qquad = \frac{3\pi}{8}$

Both of these values are in the interval $[0, 2\pi]$.

$\boldsymbol{k=1}$: $x = \frac{\pi}{8} + \mathbf{1} \cdot \pi$ or $x = \frac{3\pi}{8} + \mathbf{1} \cdot \pi$

$= \frac{9\pi}{8}$ $= \frac{11\pi}{8}$

Both of these values are in the interval $[0, 2\pi]$.

$\boldsymbol{k=2}$: $x = \frac{\pi}{8} + \mathbf{2} \cdot \pi$ or $x = \frac{3\pi}{8} + \mathbf{2} \cdot \pi$

$= \frac{17\pi}{8}$ $= \frac{19\pi}{8}$

Since both of these values are greater than 2π, they aren't in the interval $[0, 2\pi]$. Certainly larger values of k will produce even larger values of x so we don't need to try larger values of k.

Therefore, the solution set to the equation $6\sin(2x) = 3\sqrt{2}$ on the interval $[0, 2\pi]$ is $\left\{\frac{\pi}{8}, \frac{3\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}\right\}$.

EXAMPLE 7: **a.** Find all of the solutions to the equation $2\cos(3t) = -1$.

b. Find the solutions in the interval $[0, 2\pi]$ to the equation $2\cos(3t) = -1$.

SOLUTION:

a. Notice that the trigonometric function involved in the given equation is $\cos(3t)$, and recall that $\cos(3t)$ has period $\frac{2\pi}{3}$ units, i.e., the values for $\cos(3t)$ repeat every $\frac{2\pi}{3}$ units. This means that once we find a solution to the given equation we'll be able to add to it any integer multiple of $\frac{2\pi}{3}$ and obtain another solution. Therefore, we should expect the phrase "$\frac{2k\pi}{3}$ for all $k \in \mathbb{Z}$" to be involved in our solutions. You'll see in the work below that we add $2k\pi$ to our solutions after applying the inverse cosine function since the period of the cosine function is 2π units. In the last step, we finish solving for t and obtain the desired period-shift of $\frac{2k\pi}{3}$ units.

$$2\cos(3t) = -1$$

$$\Rightarrow \quad \cos(3t) = -\tfrac{1}{2}$$

$$\Rightarrow \quad \cos^{-1}(\cos(3t)) = \cos^{-1}\left(-\tfrac{1}{2}\right)$$

$$\Rightarrow \quad 3t = \tfrac{2\pi}{3} + 2k\pi \quad \text{or} \quad 3t = -\tfrac{2\pi}{3} + 2k\pi \quad \text{for all } k \in \mathbb{Z}$$

$$\Rightarrow \quad t = \tfrac{1}{3}\left(\tfrac{2\pi}{3} + 2k\pi\right) \quad \text{or} \quad t = \tfrac{1}{3}\left(-\tfrac{2\pi}{3} + 2k\pi\right) \quad \text{for all } k \in \mathbb{Z}$$

$$\Rightarrow \quad t = \tfrac{2\pi}{9} + \tfrac{2k\pi}{3} \quad \text{or} \quad t = -\tfrac{2\pi}{9} + \tfrac{2k\pi}{3} \quad \text{for all } k \in \mathbb{Z}$$

b. Now we need to substitute specific values of k into the solutions we found in part (a) and determine which solutions are in the interval $[0, 2\pi]$.

$\boldsymbol{k = -1}$: $t = \tfrac{2\pi}{9} + \tfrac{2(\mathbf{-1})\pi}{3} = -\tfrac{4\pi}{9}$ or $t = -\tfrac{2\pi}{9} + \tfrac{2(\mathbf{-1})\pi}{3} = -\tfrac{8\pi}{9}$

Both of these values are negative so they aren't in the interval $[0, 2\pi]$. Smaller values of k will produce even smaller values of t so we don't need to try smaller values of k.

$\boldsymbol{k = 0}$: $t = \tfrac{2\pi}{9} + \tfrac{2(\mathbf{0})\pi}{3} = \tfrac{2\pi}{9}$ or $t = -\tfrac{2\pi}{9} + \tfrac{2(\mathbf{0})\pi}{3} = -\tfrac{2\pi}{9}$

Only $\tfrac{2\pi}{9}$ is in the interval $[0, 2\pi]$ since $-\tfrac{2\pi}{9}$ is negative.

$\boldsymbol{k = 1}$: $t = \tfrac{2\pi}{9} + \tfrac{2(\mathbf{1})\pi}{3} = \tfrac{8\pi}{9}$ or $t = -\tfrac{2\pi}{9} + \tfrac{2(\mathbf{1})\pi}{3} = \tfrac{4\pi}{9}$

Both of these values are in the interval $[0, 2\pi]$.

$\boldsymbol{k = 2}$: $t = \tfrac{2\pi}{9} + \tfrac{2(\mathbf{2})\pi}{3} = \tfrac{14\pi}{9}$ or $t = -\tfrac{2\pi}{9} + \tfrac{2(\mathbf{2})\pi}{3} = \tfrac{10\pi}{9}$

Both of these values are in the interval $[0, 2\pi]$.

$\boldsymbol{k = 3}$: $t = \frac{2\pi}{9} + \frac{2(\mathbf{3})\pi}{3} = \frac{20\pi}{9}$ or $t = -\frac{2\pi}{9} + \frac{2(\mathbf{3})\pi}{3} = \frac{16\pi}{9}$

Only $\frac{16\pi}{9}$ is in the interval $[0, 2\pi]$ since $\frac{20\pi}{9}$ is greater than 2π.

$\boldsymbol{k = 4}$: $t = \frac{2\pi}{9} + \frac{2(\mathbf{4})\pi}{3} = \frac{26\pi}{9}$ or $t = -\frac{2\pi}{9} + \frac{2(\mathbf{4})\pi}{3} = \frac{22\pi}{9}$

Since both of these values are greater than 2π, they aren't in the interval $[0, 2\pi]$. Certainly larger values of k will produce even larger values of t so we don't need to try larger values of k.

Therefore, the solution set to the equation $6\sin(2x) = 3\sqrt{2}$ on the interval $[0, 2\pi]$ is $\left\{\frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}, \frac{10\pi}{9}, \frac{14\pi}{9}, \frac{16\pi}{9}\right\}$.

EXAMPLE 8: Find the solutions to the equation $3\cos(2x) - 2 = 0$ on the interval $[-\pi, \pi]$.

SOLUTION:

$$3\cos(2x) - 2 = 0$$

$$\Rightarrow \quad \cos(2x) = \frac{2}{3}$$

$$\Rightarrow \quad \cos^{-1}(\cos(2x)) = \cos^{-1}\left(\frac{2}{3}\right)$$

$$\Rightarrow 2x = \cos^{-1}\left(\frac{2}{3}\right) + 2k\pi \quad \text{or} \quad 2x = -\cos^{-1}\left(\frac{2}{3}\right) + 2k\pi \quad \text{for all } k \in \mathbb{Z}$$

$$\Rightarrow \quad x = \frac{1}{2}\left(\cos^{-1}\left(\frac{2}{3}\right) + 2k\pi\right) \quad \text{or} \quad x = \frac{1}{2}\left(-\cos^{-1}\left(\frac{2}{3}\right) + 2k\pi\right) \quad \text{for all } k \in \mathbb{Z}$$

$$\Rightarrow \quad x = \frac{1}{2}\cos^{-1}\left(\frac{2}{3}\right) + k\pi \quad \text{or} \quad x = -\frac{1}{2}\cos^{-1}\left(\frac{2}{3}\right) + k\pi \quad \text{for all } k \in \mathbb{Z}$$

Notice that we were asked to find the solutions in the interval $[-\pi, \pi]$, so we need to find which of the infinitely many solutions we've found are on the interval. It might help if we approximate the values we found above:

$$x = \frac{1}{2}\cos^{-1}\left(\frac{2}{3}\right) + k\pi \approx 0.42 + k\pi$$

or

$$x = -\frac{1}{2}\cos^{-1}\left(\frac{2}{3}\right) + k\pi \approx -0.42 + k\pi \quad \text{for all } k \in \mathbb{Z}$$

We know that $\pi \approx 3.14$ so we need to find values that satisfy the equation as above and are between -3.14 and 3.14.

$\boldsymbol{k = -1}$: $x \approx 0.42 + (\mathbf{-1})\cdot\pi \approx -2.72$ or $x \approx -0.42 + (\mathbf{-1})\cdot\pi \approx -3.56$

Only -2.72 is in the interval $[-\pi, \pi]$.

$\boldsymbol{k = 0}$: $x \approx 0.42 + \mathbf{0}\cdot\pi \approx 0.42$ or $x \approx -0.42 + \mathbf{0}\cdot\pi \approx -0.42$

Both of these values are in the interval $[-\pi, \pi]$.

$\boldsymbol{k = 1}$: $x \approx 0.42 + \mathbf{1}\cdot\pi \approx 3.56$ or $x \approx -0.42 + \mathbf{1}\cdot\pi \approx 2.72$

Only 2.72 is in the interval $[-\pi, \pi]$.

$\boldsymbol{k = 2}$: $x \approx 0.42 + \mathbf{2}\cdot\pi \approx 6.7$ or $x \approx -0.42 + \mathbf{2}\cdot\pi \approx 5.86$

Neither of these values is in the interval $[-\pi, \pi]$.

We could try more values of k but we can tell from the work we've done thus far that no other values of k will give us solutions that are on the interval $[-\pi, \pi]$. Thus, the solution set to the equation $3\cos(2x) - 2 = 0$ on the interval $[-\pi, \pi]$ is $\{-2.72, -0.42, 0.42, 2.72\}$.