Section I: The Trigonometric Functions

Chapter 3, Part 1: Intro to the Trigonometric Functions

In Example 4 in Section I: Chapter 2, we observed that a circle rotating about its center (i.e., a Ferris wheel) lends itself naturally to the study of periodic functions. In fact, the two most important trigonometric functions are defined in terms of a circle – specifically a unit circle:

**DEFINITION:** A unit circle is a circle with a radius of 1 unit.

Now we can define the sine and cosine functions:

**DEFINITION:** The sine function, denoted \( \sin(\theta) \), associates each angle \( \theta \) with the vertical coordinate (i.e., the \( y \)-coordinate) of the point \( P \) specified by the angle \( \theta \) on the circumference of a unit circle.

The cosine function, denoted \( \cos(\theta) \), associates each angle \( \theta \) with the horizontal coordinate (i.e., the \( x \)-coordinate) of the point \( P \) specified by the angle \( \theta \) on the circumference of a unit circle.

So the point \( P \) in Figure 2 has coordinates \( (x, y) = (\cos(\theta), \sin(\theta)) \).

There are four other trigonometric functions. These four functions are defined in terms of the sine and cosine functions so first let’s get familiar with sine and cosine. Later in this chapter we’ll define the four other trigonometric functions.
EXAMPLE 1: The angle \( \theta \) specifies the point 
\[ P = \left( -\frac{3}{5}, \frac{4}{5} \right) \] on the circumference of a unit circle; see Figure 3. Find \( \sin(\theta) \) and \( \cos(\theta) \).

SOLUTION:

\[ \sin(\theta) = \frac{4}{5} \quad \text{and} \quad \cos(\theta) = -\frac{3}{5} \]

In order to enhance our understanding of the sine and cosine functions, we should determine some particular values for the functions and sketch their graphs. But before we confront these details, let's determine the signs (positive or negative) of the sine and cosine functions in the four different quadrants of the coordinate plane. (To review the quadrants of the coordinate plane, see Section I: Chapter 1, Figure 5.)

- When the terminal side of angle \( \theta \) is in Quadrant I, both the \( x \)- and \( y \)-coordinates of point \( P \) are positive, so \( \theta \) is in Quad. I means that \( \cos(\theta) > 0 \) and \( \sin(\theta) > 0 \).

- When the terminal side of angle \( \theta \) is in Quadrant II, the \( y \)-coordinate of point \( P \) is positive but the \( x \)-coordinate is negative, so \( \theta \) is in Quad. II means that \( \cos(\theta) < 0 \) and \( \sin(\theta) > 0 \).

- When the terminal side of angle \( \theta \) is in Quadrant III, both the \( x \)- and \( y \)-coordinates of point \( P \) are negative, so \( \theta \) is in Quad. III means that \( \cos(\theta) < 0 \) and \( \sin(\theta) < 0 \).

- When the terminal side of angle \( \theta \) is in Quadrant IV, the \( x \)-coordinate of point \( P \) is positive but the \( y \)-coordinate is negative, so \( \theta \) is in Quad. IV means that \( \cos(\theta) > 0 \) and \( \sin(\theta) < 0 \).

Let's summarize in Figure 4 what we've determined about the signs of the sine and cosine functions in the different quadrants:

**Figure 4:** The signs of sine and cosine in the four quadrants.
Now let’s find the sine and cosine of a few particular angles. Recall that the sine and cosine functions represent the coordinates of points on a unit circle, and the easiest points for us to find on the unit circle are points where the circumference of the circle intersects the coordinate axes; let’s start by finding the corresponding sine and cosine values. Keep in mind that cosine represents the $x$-coordinate and sine represents the $y$-coordinate.

[Note that this information is discussed in the videos linked from the next page.]

- The angle $\theta = 90^\circ$, i.e., $\theta = \frac{\pi}{2}$ radians, specifies the point $(0, 1)$ on the circumference of a unit circle; see Figure 5a.

  Thus, $\cos\left(\frac{\pi}{2}\right) = 0$ and $\sin\left(\frac{\pi}{2}\right) = 1$.

- The angle $\theta = 180^\circ$, i.e., $\theta = \pi$ radians, specifies the point $(-1, 0)$ on the circumference of a unit circle; see Figure 5b.

  Thus, $\cos(\pi) = -1$ and $\sin(\pi) = 0$.

- The angle $\theta = 270^\circ$, i.e, $\theta = \frac{3\pi}{2}$ radians, specifies the point $(0, -1)$ on the circumference of a unit circle; see Figure 5c.

  Thus, $\cos\left(\frac{3\pi}{2}\right) = 0$ and $\sin\left(\frac{3\pi}{2}\right) = -1$.

- The angle $\theta = 360^\circ$, i.e., $\theta = 2\pi$ radians, specifies the point $(1, 0)$ on the circumference of a unit circle; see Figure 5d.

  Thus, $\cos(2\pi) = 1$ and $\sin(2\pi) = 0$. 
Notice that angles of measure $2\pi$ radians (i.e., $\theta = 360^\circ$) and $0$ radians specify the same point: $(1, 0)$. Thus, the sine and cosine values for $2\pi$ radians and $0$ radians are the same, i.e.,

$$\cos(2\pi) = \cos(0) = 1 \quad \text{and} \quad \sin(2\pi) = \sin(0) = 0.$$ 

Since ANY angle $\theta$ and $\theta + 2\pi$ specify the same point on the unit circle, the sine and cosine values of $\theta$ and $\theta + 2\pi$ are the same; therefore, the period of the sine and cosine functions is $2\pi$ radians.

**For all $\theta$, $\sin(\theta) = \sin(\theta + 2\pi)$ and $\cos(\theta) = \cos(\theta + 2\pi)$ so the period of both $s(\theta) = \sin(\theta)$ and $c(\theta) = \cos(\theta)$ is $2\pi$ radians (i.e., $360^\circ$).**

Now we’ll sketch graphs of the sine and cosine functions.

Let’s start by organizing the function values we determined above in a table:

<table>
<thead>
<tr>
<th>$\theta$ (degrees)</th>
<th>$0^\circ$</th>
<th>$90^\circ$</th>
<th>$180^\circ$</th>
<th>$270^\circ$</th>
<th>$360^\circ$</th>
<th>$450^\circ$</th>
<th>$540^\circ$</th>
<th>$630^\circ$</th>
<th>$720^\circ$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta$ (radians)</td>
<td>$0$</td>
<td>$\frac{\pi}{2}$</td>
<td>$\pi$</td>
<td>$\frac{3\pi}{2}$</td>
<td>$2\pi$</td>
<td>$\frac{5\pi}{2}$</td>
<td>$3\pi$</td>
<td>$\frac{7\pi}{2}$</td>
<td>$4\pi$</td>
</tr>
<tr>
<td>$y = \cos(\theta)$</td>
<td>$1$</td>
<td>$0$</td>
<td>$-1$</td>
<td>$0$</td>
<td>$1$</td>
<td>$0$</td>
<td>$-1$</td>
<td>$0$</td>
<td>$1$</td>
</tr>
<tr>
<td>$y = \sin(\theta)$</td>
<td>$0$</td>
<td>$1$</td>
<td>$0$</td>
<td>$-1$</td>
<td>$0$</td>
<td>$1$</td>
<td>$0$</td>
<td>$-1$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

In Figure 6a and 6b we’ve plotted the information in the table on two coordinate planes.

**Figure 6a:** Some points on $y = \cos(\theta)$.

**Figure 6b:** Some points on $y = \sin(\theta)$. 
In the next chapters we’ll find many more values of sine and cosine, but we can go ahead and connect the dots in these graphs to obtain reasonable sketches of the graphs of the sine and cosine functions in Figures 7a and 7b. (We can use what we observed in Example 4 from Section I: Chapter 2 when we studied the Ferris wheel.)

**Figure 7a**: The graph of \( y = \cos(\theta) \).

**Figure 7b**: The graph of \( y = \sin(\theta) \).

**KEY POINT**: To get these graphs on your calculator, be sure to change the angle mode of to radians. If you want to get graphs of sine and cosine in degree mode, be sure the window has a horizontal interval like \([-300, 1440]\) since the period of the function is \(360^\circ\).

Notice that the graphs of \( y = \cos(\theta) \) and \( y = \sin(\theta) \) are very similar. In fact, if we shift \( y = \sin(\theta) \) to the left \( \frac{\pi}{2} \) units, we’ll obtain the graph of \( y = \cos(\theta) \). Using what we learned about graph transformation in MTH 111, this means that \( \cos(\theta) = \sin\left(\theta + \frac{\pi}{2}\right) \). Similarly, if we shift \( y = \cos(\theta) \) to the right \( \frac{\pi}{2} \) units, we’ll obtain the graph of \( y = \sin(\theta) \). Using what we know about graph transformation, this means that \( \sin(\theta) = \cos\left(\theta - \frac{\pi}{2}\right) \).

The two bold equations above are called identities since the left and right sides of the equations are always identical, no matter what value of \( \theta \) is used.

**DEFINITION**: An identity is an equation that is true for all values in the domains of the involved expressions.
Earlier in this chapter we observed a couple of identities but didn’t call them identities. The equations

\[
\sin(\theta) = \sin(\theta + 2\pi) \quad \text{and} \quad \cos(\theta) = \cos(\theta + 2\pi)
\]

are identities since they are true for all values of \( \theta \). We can use the definitions and graphs of sine and cosine to determine a few other important identities. Do your best to convince yourself that each of following identities is true. I strongly encourage you to graph (on your graphing calculator) both sides of the identities and notice that the graphs are identical. Also, use what you learned in MTH 111 about graph transformations and symmetry to make sense of WHY these identities are true.

### SOME IDENTITIES

- \( \sin(\theta) = \sin(\theta + 2\pi) \)
- \( \cos(\theta) = \cos(\theta + 2\pi) \)
- \( \cos(\theta) = \sin \left( \theta + \frac{\pi}{2} \right) \)
- \( \sin(\theta) = \cos \left( \theta - \frac{\pi}{2} \right) \)
- \( \cos(-\theta) = \cos(\theta) \)
- \( \sin(-\theta) = -\sin(\theta) \)
- \( \sin(\theta) = \sin(\pi - \theta) \)

The last identity on this list, \( \sin(\theta) = \sin(\pi - \theta) \), is hardest one to make sense of, but it is worth taking time to understand it since it is a useful identity. (We will use it in Section I: Chapter 7 when we solve equations involving the sine function.) The easiest way to understand it is to focus on the \( \theta \)-values between 0 and \( \frac{\pi}{2} \). If \( \theta \) is in this interval, then it should be clear (if you study the graph of \( y = \sin(\theta) \)) that \( \sin(\pi - \theta) \) is the same as \( \sin(\theta) \) due to the symmetry of the sine function between 0 and \( \pi \). Once you see why \( \sin(\theta) = \sin(\pi - \theta) \) for \( \theta \)-values between 0 and \( \frac{\pi}{2} \), it will be easier to convince yourself that the identity holds for all values of \( \theta \).

We’ll study a few more important identities at the end of this chapter and we’ll study proving trigonometric identities in Section II: Chapter 3.

Although the sine and cosine functions are defined via the unit circle, we can use sine and cosine to find the coordinates of a point on the circumference of any circle. First, let’s notice a few things about the unit circle.
Figure 8: The unit circle with a point \( P \) specified by the angle \( \theta \).

As shown in Figure 8, we can construct a right-triangle using the terminal side of angle \( \theta \) and the horizontal and vertical components of the point \( P \). By construction, this right triangle has a hypotenuse of length 1 unit (since this is the radius), a horizontal component of length \( \cos(\theta) \), and a vertical component of length \( \sin(\theta) \); see Figure 9. Now let’s consider a circle with a different radius; see Figure 10. Keep in mind that the angle \( \theta \) is the same in Figure 10 as it was in Figures 8 and 9.

Since the two triangles in Figures 9 and 10 are both right triangles (i.e., they both have a 90° angle) and both have an angle \( \theta \), basic properties of geometry can be used to prove that the two triangles are similar; see Figure 11.
A well-known fact about similar triangles is that the ratio of side-lengths is constant. For example, the ratio of the height to the hypotenuse of the respective triangles is constant. Similarly, the ratio of the horizontal length to the hypotenuse of the respective triangles is constant. We can use these facts and the triangles in Figure 11 to obtain the following equations:

\[
\frac{\cos(\theta)}{1} = \frac{x}{r} \quad \text{and} \quad \frac{\sin(\theta)}{1} = \frac{y}{r}.
\]

Solving these equations for \(x\) and \(y\), respectively, we obtain

\[
x = r \cos(\theta) \quad \text{and} \quad y = r \sin(\theta).
\]

Looking back at Figure 10, we see that what we’ve found are the coordinates of the point \(T\) specified by the angle \(\theta\) on the circumference of a circle of radius \(r\). See the box below.

Suppose that the point \(T = (x, y)\) is specified by the angle \(\theta\) on the circumference of a circle of radius \(r\). Then

\[
x = r \cos(\theta) \quad \text{and} \quad y = r \sin(\theta).
\]

**Figure 12:** Circle of radius \(r\).
EXAMPLE 2: A circle with a radius of 6 units is given in Figure 13. The point $Q$ is specified by the angle $\alpha$. Use the sine and cosine function to express the exact coordinates of point $Q$.

\[ Q = (6\cos(\alpha), 6\sin(\alpha)) \]

SOLUTION:

The point $Q$ is specified by $\alpha$ on the circumference of a circle of radius 6 units. Thus,

\[ Q = (6\cos(\alpha), 6\sin(\alpha)) \]

It turns out that the algebraic equation of a unit circle centered at the origin is $x^2 + y^2 = 1$, i.e., any point $(x, y)$ on the circumference of a unit circle centered at the origin satisfies the equation $x^2 + y^2 = 1$. Recall that the cosine and sine functions represent the horizontal and vertical coordinates of a point on the circumference of a unit circle; see Figure 14.

\[ P = (\cos(\theta), \sin(\theta)) \]

This means that ordered pairs of the form $(x, y) = (\cos(\theta), \sin(\theta))$ satisfy the equation $x^2 + y^2 = 1$, i.e.,

\[ \sin^2(\theta) + \cos^2(\theta) = 1. \]

This identity is called the Pythagorean Identity. Notice the “special notation” that we’ve employed to express the exponents for the trigonometric functions in the identity. Instead of using parentheses around the entire expressions, we can put the exponent between the letters that name the function and the input for the trig function. Thus, we can write an expression like “$(\sin(\theta))^2$” as “$\sin^2(\theta)$".
EXAMPLE 3: If $\sin(\alpha) = \frac{1}{3}$ and $\frac{\pi}{2} < \alpha < \pi$ (i.e., $\alpha$ is in Quadrant II), find $\cos(\alpha)$.

SOLUTION:

Since the Pythagorean Theorem gives us an equation involving sine and cosine, we can use it to find one of the values when we know the other value. In this case, we know the value of $\sin(\alpha)$, so we can use the Pythagorean Theorem to find $\cos(\alpha)$:

$$\sin^2(\alpha) + \cos^2(\alpha) = 1$$
$$\Rightarrow (\frac{1}{3})^2 + \cos^2(\alpha) = 1$$
$$\Rightarrow \cos^2(\alpha) = 1 - (\frac{1}{3})^2$$
$$\Rightarrow \cos(\alpha) = -\sqrt{\frac{8}{9}}$$
(we choose the negative square root since cosine is negative in Quadrant II.)
$$\Rightarrow \cos(\alpha) = -\frac{2\sqrt{2}}{3}$$