Section I: Periodic Functions and Trigonometry

Chapter 1: Angles and Arc-Length

In this chapter we will study a few definitions and concepts related to angles inside circles, (like angle θ in Figure 1) that we'll use throughout the course.

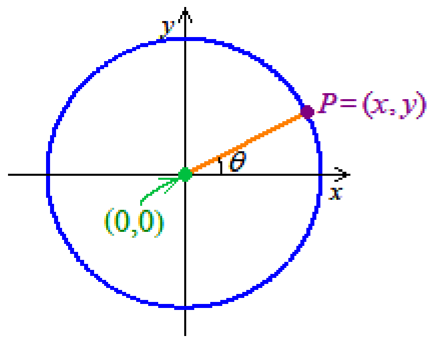


Figure 1

- The angle θ is measured **counterclockwise from the positive x-axis**.

- The line segment between the origin, $(0, 0)$, and the point P is the ***terminal side of angle θ***. (An angle in standard position "starts" at the positive x-axis and rotates in the counterclockwise direction until it "ends" at its terminal side).

- The point P on the circumference of the circle is said to be ***specified by the angle θ***.

- Two angles with the same terminal side are said to be ***co-terminal angles***. Co-terminal angles specify the same point on the circumference of a circle.

- Angle θ corresponds with a portion of the circumference of the circle called the **arc spanned by θ**; see Figure 2.

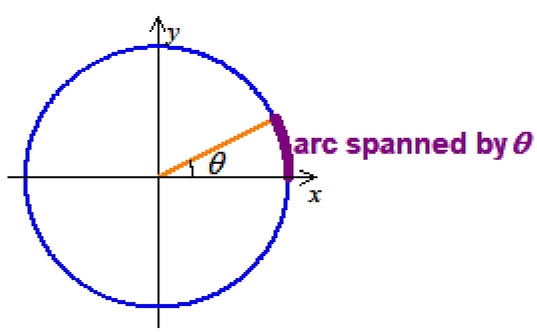


Figure 2

Thus far in your mathematics careers you have probably measured angles in **degrees**. Three hundred and sixty degrees (360°) represents a complete rotation around a circle, so 1° corresponds to $1/360^{\text{th}}$ of a complete rotation. Soon we'll discuss a different unit for measuring angles (namely, **radians**) but first let's use degrees to familiarize ourselves with *negative angles* and *co-terminal angles*.

As noted previously, angles are measured counterclockwise from the positive x-axis; consequently, *negative angles* are measured *clockwise* from the positive x-axis; see Figure 3.

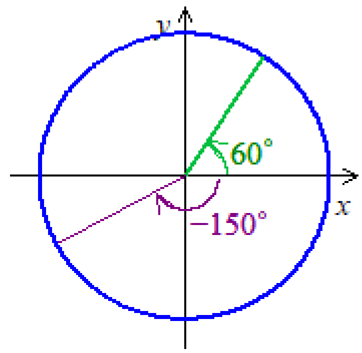


Figure 3

Recall that *co-terminal angles* share the same terminal side. Since 360° represents a complete rotation about the circle, if we add any integer multiple of 360° to an angle, we'll obtain a co-terminal angle. In other words, the angles

$$\theta_1 \text{ and } \theta_2 = \theta_1 + 360^\circ \cdot k \text{ where } k \in \mathbb{Z}$$

are co-terminal. For example, the angles 45° and $45^\circ + 360^\circ = 405^\circ$ are co-terminal; see Figure 4.

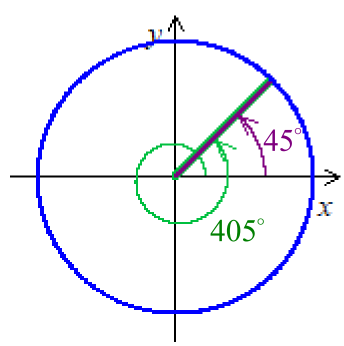


Figure 4: The angles 45° and 405° are co-terminal.

Traditionally, the coordinate plane is divided into ***four quadrants***; see Figure 5. We will often use the names of these quadrants to describe the location of the terminal side of different angles.

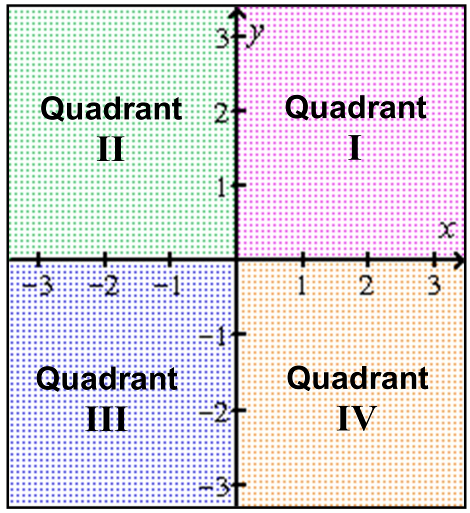


Figure 5

For example, consider the angles given in Figure 3: the angle 60° is in Quadrant I while -150° is in Quadrant III.

Next we'll discuss an alternative to degrees for measuring angles: **radians**.

DEFINITION: The **radian** measure of an angle is the ratio of the length of the arc on the circumference of the circle spanned by the angle, s, and the radius, r, of the circle; see Figure 6. Since a radian is a ratio of two lengths, the length-units cancel; thus, radians are considered a ***unit-less measure***.

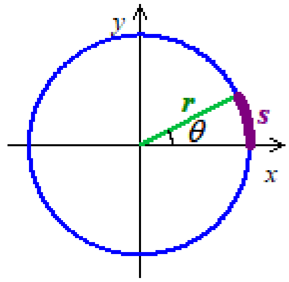


Figure 6: The angle θ measures $\frac{s}{r}$ radian.

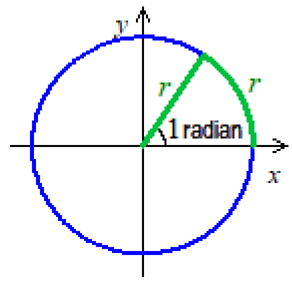
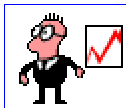


Figure 7: An angle that measures 1 radian.

An alternative yet equivalent definition is that an angle that measures 1 **radian** spans an arc whose length is equal to the length of the radius, r; see Figure 7.



CLICK HERE to see a video about radians.

Since a complete rotation around a circle, 360°, spans an arc equivalent to the entire circumference of the circle, we can find the radian equivalent of 360° by comparing a circle's circumference to its radius. A well-known formula from geometry tells us that the circumference, c, of a circle is given by $c = 2\pi r$ where r is the radius of the circle. Therefore

$$\begin{aligned} 360^\circ &= \frac{s}{r} \text{ rad} \\ &= \frac{2\pi r}{r} \text{ rad} \qquad \text{[since } s = c = 2\pi r \text{, the entire circumference]} \\ &= 2\pi \text{ rad}, \end{aligned}$$

so 360° is equivalent to 2π radians. This implies that the following two ratios equal 1; we can use these ratios to convert from degrees to radians, and vise versa:

$$\frac{2\pi \text{ rad}}{360^\circ} = \frac{360^\circ}{2\pi \text{ rad}} = 1.$$

EXAMPLE: **a.** How many degrees are 8 radians?

b. How many radians are 8 degrees?

SOLUTION:

a. In order to convert 8 radians into degrees, we can multiply 8 radians by $\frac{360^\circ}{2\pi \text{ rad}}$. (Since this equals 1, multiplying by it won't change the value of our angle-measure.)

$$8 \text{ rad} \cdot \frac{360^\circ}{2\pi \text{ rad}} = \frac{8 \cdot 360^\circ}{2\pi}$$
$$= \frac{1440^\circ}{\pi}$$
$$\approx 458.37^\circ$$

Therefore, 8 radians is about 458.37°.

b. In order to convert 8 degrees into radians, we can multiply 8° by $\frac{2\pi \text{ rad}}{360^\circ}$. (Since this equals 1, multiplying by it won't change the value of our angle-measure.)

$$8^\circ \cdot \frac{2\pi \text{ rad}}{360^\circ} = \frac{16\pi}{360} \text{ rad}$$
$$= \frac{2\pi}{45} \text{ rad}$$
$$\approx 0.14 \text{ rad}$$

So 8° is about 0.14 radians.

EXAMPLE: **a.** Convert 1 radian into degrees.

b. Convert 90° into radians.

SOLUTION:

a. In order to convert 1 radian into degrees, we can multiply 1 radian by $\frac{360^\circ}{2\pi \text{ rad}}$.

$$1 \cancel{\text{rad}} \cdot \frac{360^\circ}{2\pi \cancel{\text{rad}}} = \frac{360^\circ}{2\pi}$$
$$= \frac{180^\circ}{\pi}$$
$$\approx 57.3^\circ$$

Thus, 1 radian is about 57.3°.

b. In order to convert 90° into radians, we can multiply 90° by $\frac{2\pi \text{ rad}}{360^\circ}$.

$$90\cancel{^\circ} \cdot \frac{2\pi \text{ rad}}{360\cancel{^\circ}} = \frac{180\pi}{360} \text{ rad}$$
$$= \frac{\pi}{2} \text{ rad}$$

Thus, 90° is equivalent to $\frac{\pi}{2}$ radians.

EXAMPLE: Complete the table below:

θ (degrees)	0°	30°	45°	60°	90°	180°	270°	360°
θ (radians)								

CLICK HERE to see a video of this example.

SOLUTION:

θ (degrees)	0°	30°	45°	60°	90°	180°	270°	360°
θ (radians)	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π

Recall the definition of *radian*: the radian measure of an angle is the ratio of the length of the arc on the circumference of the circle spanned by the angle and the radius of the circle. Applying this fact to the circle in Figure 9 if θ is measured in radians, then

$$\theta = \frac{\text{arc-length}}{\text{radius}} = \frac{s}{r}$$

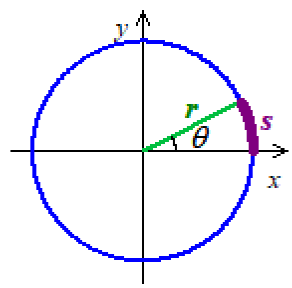


Figure 8: Circle of radius r with an angle θ spanning an arc-length s.

By solving the equation $\theta = \frac{s}{r}$ for s, we obtain the following definition:

DEFINITION: The **arc-length**, s, spanned in a circle of radius r by an angle θ radians is given by

$$s = r|\theta|.$$

Note that we need the absolute value of θ so that we obtain a positive arc-length if θ is negative. (Lengths are always positive!) Also, note that this formula only applies if θ is measured in radians.)

EXAMPLE: **a.** What is the arc-length spanned by an angle of 2 radians on a circle of radius 5 inches?

b. What is the arc-length spanned by an angle of 30° on a circle of radius 20 meters?

SOLUTION:

a. To find the arc-length, we can use the formula $s = r|\theta|$.

$$\begin{aligned} s &= r|\theta| \\ &= 5 \cdot 2 \\ &= 10 \end{aligned}$$

Thus, the arc-length spanned by an angle of 2 radians on a circle of radius 5 inches is 10 inches.

b. Before we can use the formula $s = r|\theta|$, we need to convert the angle into radians. In order to convert 30° into radians, we can multiply 30° by $\frac{2\pi \text{ rad}}{360^{\circ}}$ (which equals 1). (Of course we could use the table we created earlier in this chapter, but we will go ahead and show the computation here.)

$$\begin{aligned} 30\cancel{^{\circ}} \cdot \frac{2\pi \text{ rad}}{360\cancel{^{\circ}}} &= \frac{60\pi}{360} \text{ rad} \\ &= \frac{\pi}{6} \text{ rad} \end{aligned}$$

Thus, 30° is equivalent to $\frac{\pi}{6}$ radians. Now we can find the desired arc-length:

$$\begin{aligned} s &= r|\theta| \\ &= 20 \cdot \frac{\pi}{6} \\ &= \frac{10\pi}{3} \end{aligned}$$

Thus, the arc-length spanned by an angle of 30° on a circle of radius 20 meters is $\frac{10\pi}{3}$ meters.