**Section I: Periodic Functions and Trigonometry**

**Chapter 0: Sets and Numbers**

**DEFINITION:** A set is a collection of objects specified in a manner that enables one to determine if a given object is or is not in the set.

In other words, a set is a well-defined collection of objects.

**EXAMPLE:** Which of the following represent a set?

a. The students registered for MTH 112 at PCC this quarter.

b. The good students registered for MTH 112 at PCC this quarter.

**SOLUTION:**

a. This represents a set since it is “well defined”: We all know what it means to be registered for a class.

b. This does NOT represent a set since it is not well defined: There are many different understandings of what it means to be a good student (get an A or pass the class or attend class or avoid falling asleep in class).

**EXAMPLE:** Which of the following represent a set?

a. All of the really big numbers.

b. All the whole numbers between 3 and 10.

**SOLUTION:**

a. It should be obvious why this does NOT represent a set. (What does it mean to be a “big number”?)

b. This represents a set. We can represent sets like b in **roster notation** by using “curly brackets”.

“All of the whole numbers between 3 and 10" = \{4, 5, 6, 7, 8, 9\}

**Roster Notation** involves listing the elements in a set within **curly brackets**: “{ }”. 
**DEFINITION:** An object in a set is called an **element** of the set. (symbol: “∈”)

**EXAMPLE:** 5 is an element of the set \( \{4, 5, 6, 7, 8, 9\} \). We can express this symbolically:

\[ 5 \in \{4, 5, 6, 7, 8, 9\} \]

**DEFINITION:** Two sets are considered **equal** if they have the same elements.

We used this definition earlier when we wrote:

“All of the whole numbers between 3 and 10” = \( \{4, 5, 6, 7, 8, 9\} \).

**DEFINITION:** A set \( S \) is a **subset** of a set \( T \), denoted \( S \subseteq T \), if all elements of \( S \) are also elements of \( T \).

If \( S \) and \( T \) are sets and \( S = T \), then \( S \subseteq T \). Sometimes it is useful to consider a subset \( S \) of a set \( T \) that is not equal to \( T \). In such a case, we write \( S \subset T \) and say that \( S \) is a **proper subset** of \( T \).

**EXAMPLE:** \( \{4, 7, 8\} \) is a subset of the set \( \{4, 5, 6, 7, 8, 9\} \).

We can express this fact symbolically by \( \{4, 7, 8\} \subseteq \{4, 5, 6, 7, 8, 9\} \).

Since these two sets are not equal, \( \{4, 7, 8\} \) is a **proper** subset of \( \{4, 5, 6, 7, 8, 9\} \), so we can write

\( \{4, 7, 8\} \subset \{4, 5, 6, 7, 8, 9\} \).
**DEFINITION:** The empty set, denoted \( \emptyset \), is the set with no elements.

\[ \emptyset = \{ \} \quad \text{There are NO elements in} \ \emptyset. \]

The empty set is a subset of all sets. Note that \( 0 \neq \emptyset \).

**DEFINITION:** The union of two sets \( A \) and \( B \), denoted \( A \cup B \), is the set containing all of the elements in either \( A \) or \( B \) (or both \( A \) and \( B \)).

**EXAMPLE:** Consider the sets \( \{4, 7, 8\} \), \( \{0, 2, 4, 6, 8\} \), and \( \{1, 3, 5, 7\} \). Then...

a. \( \{4, 7, 8\} \cup \{1, 3, 5, 7\} = \{1, 3, 4, 5, 7, 8\} \)

b. \( \{4, 7, 8\} \cup \{0, 2, 4, 6, 8\} = \{0, 2, 4, 6, 7, 8\} \)

c. \( \{0, 2, 4, 6, 8\} \cup \{1, 3, 5, 7\} = \{0, 1, 2, 3, 4, 5, 6, 7, 8\} \)

**DEFINITION:** The intersection of two sets \( A \) and \( B \), denoted \( A \cap B \), is the set containing all of the elements in both \( A \) and \( B \).

**EXAMPLE:** Consider the sets \( \{4, 7, 8\} \), \( \{0, 2, 4, 6, 8\} \), and \( \{1, 3, 5, 7\} \). Then...

a. \( \{4, 7, 8\} \cap \{0, 2, 4, 6, 8\} = \{4, 8\} \)

b. \( \{4, 7, 8\} \cap \{1, 3, 5, 7\} = \{7\} \)

c. \( \{0, 2, 4, 6, 8\} \cap \{1, 3, 5, 7\} = \emptyset \quad \text{These sets have no elements in common, so their intersection is the empty set.} \)
EXAMPLE: All of the whole numbers (positive and negative) form a set. This set is called the integers, and is represented by the symbol \( \mathbb{Z} \). We can express the set of integers in roster notation:

\[
\mathbb{Z} = \{ \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \}
\]

Note that \( \mathbb{Z} \) is used to represent the integers because the German word for “number” is “zahlen.”

Now that we have the integers, we can represent sets like “All of the whole numbers between 3 and 10” using set-builder notation:

**SET-BUILDER NOTATION:**

"All the whole numbers between 3 and 10" = \( \{ x \mid x \in \mathbb{Z} \text{ and } 3 < x < 10 \} \)

This vertical line means "such that"

Armed with set-builder notation, we can define important sets of numbers:

**DEFINITIONS:** The set of natural numbers: \( \mathbb{N} = \{ 1, 2, 3, 4, 5, \ldots \} \)

The set of integers: \( \mathbb{Z} = \{ \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \} \)

The set of rational numbers (i.e., the set of fractions): \( \mathbb{Q} = \{ x \mid x = \frac{p}{q} \text{ and } p, q \in \mathbb{Z} \text{ and } q \neq 0 \} \)

The set of real numbers: \( \mathbb{R} \) (All the numbers on the number line.)

The set of complex numbers:

\[
\mathbb{C} = \{ x \mid x = a + bi \text{ and } a, b \in \mathbb{R} \text{ and } i = \sqrt{-1} \}
\]

Note that \( \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C} \), i.e., the set of natural numbers is a subset of the set of integers which is a subset of the set of rational numbers which is a subset of the real numbers which is a subset of the set of complex numbers.

Throughout this course, we will assume that the number-set in question is the real numbers, \( \mathbb{R} \), unless we are specifically asked to consider an alternative set.
Since we use the real numbers so often, we have special notation for subsets of the real numbers: **interval notation**. Interval notation involves square or round brackets. Use the examples below to understand how interval notation works.

**EXAMPLE:**

\[ \{ x \mid x \in \mathbb{R} \text{ and } -2 \leq x \leq 3 \} = [-2, 3] \]

- **Set-builder Notation:** \( \{ x \mid x \in \mathbb{R} \text{ and } -2 \leq x \leq 3 \} \)
- **Interval Notation:** \([-2, 3]\)

We use square brackets here since the endpoints are included.

**EXAMPLE:**

\[ \{ x \mid x \in \mathbb{R} \text{ and } -2 < x < 3 \} = (-2, 3) \]

- **Set-builder Notation:** \( \{ x \mid x \in \mathbb{R} \text{ and } -2 < x < 3 \} \)
- **Interval Notation:** \((-2, 3)\)

We use round brackets here since the endpoints are NOT included.

**EXAMPLE:**

\[ \{ x \mid x \in \mathbb{R} \text{ and } -2 < x \leq 3 \} = (-2, 3] \]

- **Set-builder Notation:** \( \{ x \mid x \in \mathbb{R} \text{ and } -2 < x \leq 3 \} \)
- **Interval Notation:** \((-2, 3]\)

We use a round bracket on the left since \(-2\) is NOT included.

**EXAMPLE:**

\[ \{ x \mid x \in \mathbb{R} \text{ and } -2 \leq x < 3 \} = [-2, 3) \]

- **Set-builder Notation:** \( \{ x \mid x \in \mathbb{R} \text{ and } -2 \leq x < 3 \} \)
- **Interval Notation:** \([-2, 3)\)

We use a round bracket on the right since \(3\) is NOT included.

**EXAMPLE:** When the interval has no upper (or lower) bound, the symbol \(\infty\) (or \(-\infty\)) is used.

\[ \{ x \mid x \in \mathbb{R} \text{ and } x \leq 4 \} = (-\infty, 4] \]

- **Set-builder Notation:** \( \{ x \mid x \in \mathbb{R} \text{ and } x \leq 4 \} \)
- **Interval Notation:** \((-\infty, 4]\)

We ALWAYS use a round bracket with \(-\infty\) since it is NOT a number in the set.

\[ \{ x \mid x \in \mathbb{R} \text{ and } x \geq 4 \} = [4, \infty) \]

- **Set-builder Notation:** \( \{ x \mid x \in \mathbb{R} \text{ and } x \geq 4 \} \)
- **Interval Notation:** \([4, \infty)\)

We ALWAYS use a round bracket with \(\infty\) since it is NOT a number in the set.
EXAMPLE: Simplify the following expressions.

a. \((-4, \infty) \cup [-8, 3]\)

b. \((-4, \infty) \cup (-\infty, 2]\)

c. \((-4, \infty) \cap (-\infty, 2]\)

d. \((-4, \infty) \cap [-10, -5]\)

SOLUTION:

a. \((-4, \infty) \cup [-8, 3] = [-8, \infty)\)

b. \((-4, \infty) \cup (-\infty, 2] = (-\infty, \infty) = \mathbb{R}\)

c. \((-4, \infty) \cap (-\infty, 2] = (-4, 2]\)

d. \((-4, \infty) \cap [-10, -5] = \emptyset\)