Section II: Trigonometric Identities

Chapter 4: Sum and Difference Identities

In this chapter we'll study identities that allow us to change the form of expressions involving the sine or cosine of the sum or difference of two angles. Although we have the tools necessary to prove these identities, we won't bother proving them here. (You can see proofs in §6.5 in our textbook.) We'll just state the identities and then see some examples of how these identities can be used.

THE SUM- AND DIFFERENCE-OF-ANGLES IDENTITIES

sine :

$$\sin(A+B) = \sin(A)\cos(B) + \sin(B)\cos(A)$$
$$\sin(A-B) = \sin(A)\cos(B) - \sin(B)\cos(A)$$

cosine :

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$
$$\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B)$$

EXAMPLE 1: Use the sum-of-angles or the difference-of-angles identities to calculate the following.

a. $\cos(75^\circ)$ **b.** $\sin(-15^\circ)$ **c.** $\sin\left(\frac{11\pi}{12}\right)$

SOLUTIONS:

a. Since $75^\circ = 45^\circ + 30^\circ$, we can use the cosine of a sum-of-angles identity to calculate $\cos(75^\circ)$:

$$\begin{aligned}
\cos(75^\circ) &= \cos(45^\circ + 30^\circ) \\
&= \cos(45^\circ)\cos(30^\circ) - \sin(45^\circ)\sin(30^\circ) \\
&= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2} \\
&= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} \\
&= \frac{\sqrt{6}-\sqrt{2}}{4}
\end{aligned}$$

b. Since $-15^\circ = 30^\circ - 45^\circ$, we can use the sine of a difference-of-angles identity to calculate $\sin(-15^\circ)$:

$$\begin{aligned}
\sin(-15^\circ) &= \sin(30^\circ - 45^\circ) \\
&= \sin(30^\circ)\cos(45^\circ) - \sin(45^\circ)\cos(30^\circ) \\
&= \frac{1}{2}\cdot\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} \\
&= \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4} \\
&= \frac{\sqrt{2}-\sqrt{6}}{4}
\end{aligned}$$

c. Since

$$\begin{aligned}
\frac{11\pi}{12} &= \frac{3\pi}{12} + \frac{8\pi}{12} \\
&= \frac{\pi}{4} + \frac{2\pi}{3},
\end{aligned}$$

we can use the sine of a sum-of-angles identity to calculate $\sin\left(\frac{11\pi}{12}\right)$:

$$\begin{aligned}
\sin\left(\frac{11\pi}{12}\right) &= \sin\left(\frac{\pi}{4} + \frac{2\pi}{3}\right) \\
&= \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{2\pi}{3}\right) + \sin\left(\frac{2\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) \\
&= \frac{\sqrt{2}}{2}\cdot\left(-\frac{1}{2}\right) + \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} \\
&= -\frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} \\
&= \frac{-\sqrt{2}+\sqrt{6}}{4}.
\end{aligned}$$

EXAMPLE 2: Use the sum-of-angles or the difference-of-angles identities to calculate the following $\tan\left(\frac{7\pi}{12}\right)$.

SOLUTION:

To calculate $\tan\left(\frac{7\pi}{12}\right)$, we need to use the fact that $\tan\left(\frac{7\pi}{12}\right) = \dfrac{\sin\left(\frac{7\pi}{12}\right)}{\cos\left(\frac{7\pi}{12}\right)}$ along with the sum-of-angles identities since

$$\begin{aligned}\frac{7\pi}{12} &= \frac{3\pi}{12} + \frac{4\pi}{12}\\ &= \frac{\pi}{4} + \frac{\pi}{3}.\end{aligned}$$

Therefore,

$$\begin{aligned}\tan\left(\frac{7\pi}{12}\right) &= \frac{\sin\left(\frac{7\pi}{12}\right)}{\cos\left(\frac{7\pi}{12}\right)}\\
&= \frac{\sin\left(\frac{\pi}{4}+\frac{\pi}{3}\right)}{\cos\left(\frac{\pi}{4}+\frac{\pi}{3}\right)}\\
&= \frac{\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}\right)+\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right)}{\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}\right)-\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{3}\right)}\\
&= \frac{\frac{\sqrt{2}}{2}\cdot\frac{1}{2}+\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}\cdot\frac{1}{2}-\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}}\\
&= \frac{\frac{\sqrt{2}+\sqrt{6}}{4}}{\frac{\sqrt{2}-\sqrt{6}}{4}}\\
&= \frac{\sqrt{2}+\sqrt{6}}{\sqrt{2}-\sqrt{6}}\cdot\frac{\sqrt{2}+\sqrt{6}}{\sqrt{2}+\sqrt{6}} \qquad \text{(use the conjugate of the denominator to rationalize the denominator)}\\
&= \frac{2+2\sqrt{12}+6}{2-6}\\
&= \frac{8+4\sqrt{3}}{-4}\\
&= -2-\sqrt{3}\end{aligned}$$

In §6.5 in our textbook, sum and difference identities are given for tangent. We could have derived those identities here and then employed one of them in Example 2 but instead we've chosen to focus on the identities for sine and cosine, and then utilize these identities when working with tangent (i.e., there's no reason to waste our energy learning identities involving tangent if we can instead use what we already know about sine and cosine).