Section II: Trigonometric Identities

Chapter 3: Proving Trigonometric Identities

This quarter we've studied many important trigonometric identities. Because these identities are so useful, it is worthwhile to learn (or memorize) most of them. But there are many other identities that aren't particularly important (so they aren't worth memorizing) but they exist and they offer us an opportunity to learn another skill: proving mathematical statements. In this chapter we will *prove* that some equations are in fact identities.

Recall that an **identity** is an equation that is true for all values in the domains of the involved expressions. Thus, to prove an identity we need to show that the two sides of the equation are *always* equal. To accomplish this, we need to start with the expression on one side of the equation and use the rules of algebra as well as the identities that we've already studied to manipulate the expression until it's identical to the expression on the other side of the equation. Let's look at a few examples to help you make sense of this procedure.

EXAMPLE 1: Prove the identity $\sin(x) = \dfrac{\tan(x)}{\sec(x)}$.

As we mentioned above, we prove identities by manipulating the expression on one side of the equation (using the rules of algebra and the identities we already know) until it looks like the expression on the other side of the equation. We can start with either side of the equation, but it's usually most sensible to start with the "more complicated" side since it will be easier to manipulate it. In this example, $\dfrac{\tan(x)}{\sec(x)}$ is "more complicated" than $\sin(x)$, so let's start with $\dfrac{\tan(x)}{\sec(x)}$ and try to manipulate it until it looks like $\sin(x)$.

$$\frac{\tan(x)}{\sec(x)} = \frac{\;\frac{\sin(x)}{\cos(x)}\;}{\frac{1}{\cos(x)}} \qquad \leftarrow \text{since } \tan(x) = \frac{\sin(x)}{\cos(x)} \text{ and } \sec(x) = \frac{1}{\cos(x)}$$

$$= \frac{\sin(x)}{\cos(x)} \cdot \frac{\cos(x)}{1} \qquad \leftarrow \text{using the rules of algebra}$$

$$= \sin(x) \qquad \leftarrow \text{more algebra}$$

This is a *proof* of the identity $\sin(x) = \dfrac{\tan(x)}{\sec(x)}$ since we've shown that the right side is equivalent to the left side.

EXAMPLE 2: Prove the identity $\cot(x) + \tan(x) = \csc(x)\sec(x)$.

Here, both sides are equally "complicated" so it's not obvious which side we should start with. In such a case, just start with *either* side and see what happens. If you get stuck, start over using the other side. Let's start with the left side:

$$\cot(x) + \tan(x) = \frac{\cos(x)}{\sin(x)} + \frac{\sin(x)}{\cos(x)} \quad \leftarrow \text{since } \cot(x) = \frac{\cos(x)}{\sin(x)} \text{ and } \tan(x) = \frac{\sin(x)}{\cos(x)}$$

$$= \frac{\cos(x)}{\sin(x)} \cdot \frac{\cos(x)}{\cos(x)} + \frac{\sin(x)}{\cos(x)} \cdot \frac{\sin(x)}{\sin(x)} \quad \leftarrow \text{using the rules of algebra to obtain a common denominator}$$

$$= \frac{\cos^2(x) + \sin^2(x)}{\sin(x)\cos(x)}$$

$$= \frac{1}{\sin(x)\cos(x)} \quad \leftarrow \text{since } \cos^2(x) + \sin^2(x) = 1$$

$$= \frac{1}{\sin(x)} \cdot \frac{1}{\cos(x)} \quad \leftarrow \text{using more algebra}$$

$$= \csc(x)\sec(x) \quad \leftarrow \text{since } \csc(x) = \frac{1}{\sin(x)} \text{ and } \sec(x) = \frac{1}{\cos(x)}$$

This is a proof that $\cot(x) + \tan(x) = \csc(x)\sec(x)$ since we've shown that the left side is equivalent to the right side.

For the next examples, we'll need to use variations of the Pythagorean Identity discussed at the end of Section I: Chapter 3.

EXAMPLE 3: Prove the identity $\dfrac{(1 + \cos(t))(1 - \cos(t))}{\sin(t)} = \sin(t)$.

The left side is more "complicated", so we'll start with it:

$$\frac{(1 + \cos(t))(1 - \cos(t))}{\sin(t)} = \frac{1 - \cos(t) + \cos(t) - \cos^2(t)}{\sin(t)} \quad \leftarrow \text{using algebra}$$

$$= \frac{1 - \cos^2(t)}{\sin(t)} \quad \leftarrow \text{using algebra}$$

$$= \frac{\sin^2(t)}{\sin(t)} \quad \leftarrow \text{since } 1 - \cos^2(t) = \sin^2(t)$$

$$= \sin(t) \quad \leftarrow \text{using more algebra}$$

This is a proof that $\dfrac{\big(1+\cos(t)\big)\big(1-\cos(t)\big)}{\sin(t)} = \sin(t)$ since we've shown that the left side is equivalent to the right side.

EXAMPLE 4: Prove the identity $\dfrac{\cos(\theta)}{1-\sin(\theta)} = \dfrac{1+\sin(\theta)}{\cos(\theta)}$.

The proof of this identity requires a relatively commonly employed "trick" that is important to become familiar with. Sometimes it is useful to use the **conjugate** of some part (often the denominator) of the expression. The **conjugate** of the expression $a+b$ is the expression $a-b$, and vice versa. As you can see in the proof below, by multiplying the denominator by its conjugate, allows us to manipulate the left side so that it is equivalent to the right side.

$$
\begin{aligned}
\frac{\cos(\theta)}{1-\sin(\theta)} &= \frac{\cos(\theta)}{1-\sin(\theta)} \cdot \frac{1+\sin(\theta)}{1+\sin(\theta)} && \leftarrow \text{using the conjugate of } 1-\sin(\theta) \\
&= \frac{\cos(\theta)\big(1+\sin(\theta)\big)}{\big(1-\sin(\theta)\big)\big(1+\sin(\theta)\big)} \\
&= \frac{\cos(\theta)\big(1+\sin(\theta)\big)}{1-\sin(\theta)+\sin(\theta)-\sin^2(\theta)} \\
&= \frac{\cos(\theta)\big(1+\sin(\theta)\big)}{1-\sin^2(\theta)} \\
&= \frac{\cos(\theta)\big(1+\sin(\theta)\big)}{\cos^2(\theta)} && \leftarrow \text{since } 1-\sin^2(\theta)=\cos^2(\theta) \\
&= \frac{\cancel{\cos(\theta)}\big(1+\sin(\theta)\big)}{\cos^{\cancel{2}}(\theta)} \\
&= \frac{1+\sin(\theta)}{\cos(\theta)}
\end{aligned}
$$

This is a proof that $\dfrac{\cos(\theta)}{1-\sin(\theta)} = \dfrac{1+\sin(\theta)}{\cos(\theta)}$ since we've shown that the left side is equivalent to the right side.

For the Example 5 we'll need to use a variation of the identity $\tan^2(\theta) + 1 = \sec^2(\theta)$ discussed at the end of Section I, Chapter 3.

EXAMPLE 5: Prove the identity $\dfrac{1}{\sec(u) - \tan(u)} - \dfrac{1}{\sec(u) + \tan(u)} = 2\tan(u)$.

To prove this identity, we will again use the **conjugate** of the denominators of the expressions on the left side of the equation:

$$\begin{aligned}
\frac{1}{\sec(u)-\tan(u)} - \frac{1}{\sec(u)+\tan(u)} &= \frac{1}{\sec(u)-\tan(u)} \cdot \frac{\sec(u)+\tan(u)}{\sec(u)+\tan(u)} - \frac{1}{\sec(u)+\tan(u)} \cdot \frac{\sec(u)-\tan(u)}{\sec(u)-\tan(u)} \\
&= \frac{\sec(u) + \tan(u) - \big(\sec(u) - \tan(u)\big)}{\big(\sec(u) - \tan(u)\big)\big(\sec(u) + \tan(u)\big)} \\
&= \frac{\sec(u) + \tan(u) - \sec(u) + \tan(u)}{\sec^2(u) - \sec(u)\tan(u) + \sec(u)\tan(u) - \tan^2(u)} \\
&= \frac{2\tan(u)}{\sec^2(u) - \tan^2(u)} \\
&= \frac{2\tan(u)}{1} \qquad \leftarrow \text{since } \sec^2(u) - \tan^2(u) = 1 \\
&= 2\tan(u)
\end{aligned}$$