Section I: Periodic Functions and Trigonometry

Chapter 0: Sets and Numbers

DEFINITION: A **set** is a collection of objects specified in a manner that enables one to determine if a given object is or is not in the set.

In other words, a set is a well-defined collection of objects.

EXAMPLE: Which of the following represent a set?

a. The students registered for MTH 112 at PCC this quarter.

b. The good students registered for MTH 112 at PCC this quarter.

SOLUTION:

a. This represents a set since it is "well defined": We all know what it means to be registered for a class.

b. This does NOT represent a set since it is not well defined: There are many different understandings of what it means to be a good student (get an A or pass the class or attend class or avoid falling asleep in class).

EXAMPLE: Which of the following represent a set?

a. All of the really big numbers. **b.** All the whole numbers between 3 and 10.

SOLUTION:

a. It should be obvious why this does NOT represent a set. (What does it mean to be a "big number"?)

b. This represents a set. We can represent sets like **b** in **roster notation** by using "curly brackets".

"All of the whole numbers between 3 and 10" $= \{4, 5, 6, 7, 8, 9\}$

Roster Notation involves listing the elements in a set within *curly brackets*: "$\{\ \}$".

DEFINITION: An object in a set is called an **element** of the set. (symbol: "$\in$")

EXAMPLE: 5 is an element of the set $\{4, 5, 6, 7, 8, 9\}$. We can express this symbolically:

$$5 \in \{4, 5, 6, 7, 8, 9\}$$

DEFINITION: Two sets are considered **equal** if they have the same elements.

We used this definition earlier when we wrote:

"All of the whole numbers between 3 and 10" $= \{4, 5, 6, 7, 8, 9\}$.

DEFINITION: A set S is a **subset** of a set T, denoted $S \subseteq T$, if all elements of S are also elements of T.

If S and T are sets and $S = T$, then $S \subseteq T$. Sometimes it is useful to consider a subset S of a set T that is not equal to T. In such a case, we write $S \subset T$ and say that S is a **proper subset** of T.

EXAMPLE: $\{4, 7, 8\}$ is a subset of the set $\{4, 5, 6, 7, 8, 9\}$.

We can express this fact symbolically by $\{4, 7, 8\} \subseteq \{4, 5, 6, 7, 8, 9\}$.

Since these two sets are not equal, $\{4, 7, 8\}$ is a *proper* subset of $\{4, 5, 6, 7, 8, 9\}$, so we can write

$$\{4, 7, 8\} \subset \{4, 5, 6, 7, 8, 9\}.$$

DEFINITION: The **empty set**, denoted Ø, is the set with no elements.

$$\varnothing = \{\ \ \}$$ There are NO elements in Ø.

The empty set is a subset of all sets. Note that $0 \neq \varnothing$.

DEFINITION: The **union** of two sets A and B, denoted $A \cup B$, is the set containing all of the elements in either A or B (or both A and B).

EXAMPLE: Consider the sets $\{4, 7, 8\}$, $\{0, 2, 4, 6, 8\}$, and $\{1, 3, 5, 7\}$. Then…

a. $\{4, 7, 8\} \cup \{1, 3, 5, 7\} = \{1, 3, 4, 5, 7, 8\}$

b. $\{4, 7, 8\} \cup \{0, 2, 4, 6, 8\} = \{0, 2, 4, 6, 7, 8\}$

c. $\{0, 2, 4, 6, 8\} \cup \{1, 3, 5, 7\} = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$

DEFINITION: The **intersection** of two sets A and B, denoted $A \cap B$, is the set containing all of the elements in both A and B.

EXAMPLE: Consider the sets $\{4, 7, 8\}$, $\{0, 2, 4, 6, 8\}$, and $\{1, 3, 5, 7\}$. Then…

a. $\{4, 7, 8\} \cap \{0, 2, 4, 6, 8\} = \{4, 8\}$

b. $\{4, 7, 8\} \cap \{1, 3, 5, 7\} = \{7\}$

c. $\{0, 2, 4, 6, 8\} \cap \{1, 3, 5, 7\} = \varnothing$ These sets have no elements in common, so their intersection is the empty set.

EXAMPLE: All of the whole numbers (positive and negative) form a set. This set is called the **integers**, and is represented by the symbol $\mathbb{Z}$. We can express the set of integers in roster notation:

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

Note that $\mathbb{Z}$ is used to represent the integers because the German word for "number" is "zahlen."

Now that we have the integers, we can represent sets like "All of the whole numbers between 3 and 10" using **set-builder notation**:

SET-BUILDER NOTATION:

$$\text{"All the whole numbers between 3 and 10"} = \{x \mid x \in \mathbb{Z} \text{ and } 3 < x < 10\}$$

This vertical line means "such that"

Armed with set-builder notation, we can define important **sets of numbers**:

DEFINITIONS: The set of **natural numbers**: $\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$

The set of **integers**: $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

The set of **rational numbers** (i.e., the set of fractions):

$$\mathbb{Q} = \left\{x \,\middle|\, x = \tfrac{p}{q} \text{ and } p, q \in \mathbb{Z} \text{ and } q \neq 0\right\}$$

The set of **real numbers**: $\mathbb{R}$ (All the numbers on the number line.)

The set of **complex numbers**:

$$\mathbb{C} = \left\{x \,\middle|\, x = a + bi \text{ and } a, b \in \mathbb{R} \text{ and } i = \sqrt{-1}\right\}$$

Note that $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$, i.e., the set of natural numbers is a subset of the set of integers which is a subset of the set of rational numbers which is a subset of the real numbers which is a subset of the set of complex numbers.

Throughout this course we will assume that the number-set in question is the real numbers, $\mathbb{R}$, unless we are specifically asked to consider an alternative set.

Since we use the real numbers so often, we have special notation for subsets of the real numbers: **interval notation**. Interval notation involves square or round brackets. Use the examples below to understand how interval notation works.

EXAMPLE:

a. $\{x \mid x \in \mathbb{R} \text{ and } -2 \le x \le 3\} = [-2, 3]$

↑ Set-builder Notation ↑ Interval Notation

We use square brackets here since the endpoints are included

b. $\{x \mid x \in \mathbb{R} \text{ and } -2 < x < 3\} = (-2, 3)$

↑ Set-builder Notation ↑ Interval Notation

We use round brackets here since the endpoints are NOT included.

c. $\{x \mid x \in \mathbb{R} \text{ and } -2 < x \le 3\} = (-2, 3]$

↑ Set-builder Notation ↑ Interval Notation

We use a round bracket on the left since -2 is NOT included.

d. $\{x \mid x \in \mathbb{R} \text{ and } -2 \le x < 3\} = [-2, 3)$

↑ Set-builder Notation ↑ Interval Notation

We use a round bracket on the right since 3 is NOT included.

EXAMPLE: When the interval has no upper (or lower) bound, the symbol ∞ (or $-\infty$) is used.

a. $\{x \mid x \in \mathbb{R} \text{ and } x \le 4\} = (-\infty, 4]$

↑ Set-builder Notation ↑ Interval Notation

We ALWAYS use a round bracket with $-\infty$ since it is NOT a number in the set.

b. $\{x \mid x \in \mathbb{R} \text{ and } x \ge 4\} = [4, \infty)$

↑ Set-builder Notation ↑ Interval Notation

We ALWAYS use a round bracket with ∞ since it is NOT a number in the set.

EXAMPLE: Simplify the following expressions.

a. $(-4,\ \infty) \cup [-8,\ 3]$

b. $(-4,\ \infty) \cup (-\infty,\ 2]$

c. $(-4,\ \infty) \cap (-\infty,\ 2]$

d. $(-4,\ \infty) \cap [-10,\ -5]$

SOLUTION:

a. $(-4,\ \infty) \cup [-8,\ 3] = [-8,\ \infty)$

b. $(-4,\ \infty) \cup (-\infty,\ 2] = (-\infty,\ \infty) = \mathbb{R}$

c. $(-4,\ \infty) \cap (-\infty,\ 2] = (-4,\ 2]$

d. $(-4,\ \infty) \cap [-10,\ -5] = \emptyset$

Section I: Periodic Functions and Trigonometry

Chapter 1: Angles and Arc-Length

In this chapter we will study a few definitions and concepts related to angles inside circles, (like angle θ in Figure 1) that we'll use throughout the course.

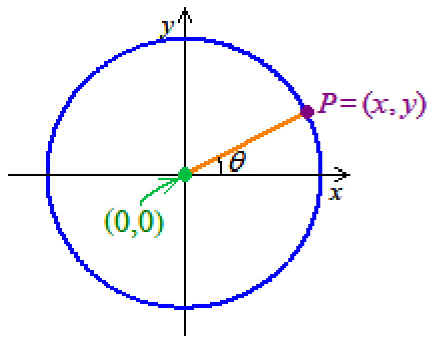


Figure 1

- The angle θ is measured **counterclockwise from the positive x-axis**.

- The line segment between the origin, $(0, 0)$, and the point P is the ***terminal side of angle θ***. (An angle in standard position "starts" at the positive x-axis and rotates in the counterclockwise direction until it "ends" at its terminal side).

- The point P on the circumference of the circle is said to be ***specified by the angle θ***.

- Two angles with the same terminal side are said to be ***co-terminal angles***. Co-terminal angles specify the same point on the circumference of a circle.

- Angle θ corresponds with a portion of the circumference of the circle called the **arc spanned by θ**; see Figure 2.

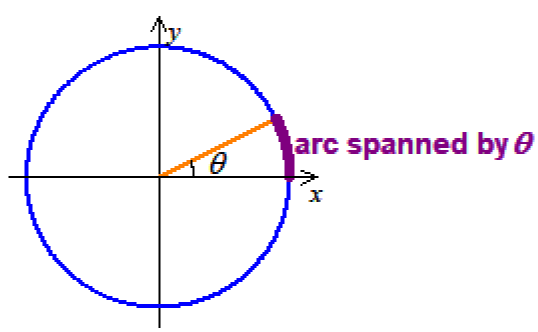


Figure 2

Thus far in your mathematics careers you have probably measured angles in **degrees**. Three hundred and sixty degrees (360°) represents a complete rotation around a circle, so 1° corresponds to $1/360^{\text{th}}$ of a complete rotation. Soon we'll discuss a different unit for measuring angles (namely, **radians**) but first let's use degrees to familiarize ourselves with *negative angles* and *co-terminal angles*.

As noted previously, angles are measured counterclockwise from the positive x-axis; consequently, *negative angles* are measured *clockwise* from the positive x-axis; see Figure 3.

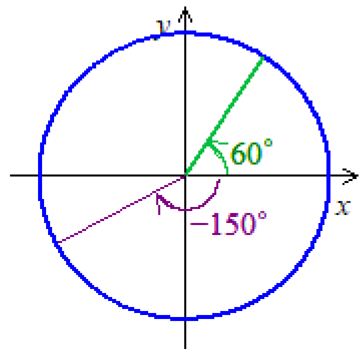


Figure 3

Recall that *co-terminal angles* share the same terminal side. Since 360° represents a complete rotation about the circle, if we add any integer multiple of 360° to an angle, we'll obtain a co-terminal angle. In other words, the angles

$$\theta_1 \text{ and } \theta_2 = \theta_1 + 360^\circ \cdot k \text{ where } k \in \mathbb{Z}$$

are co-terminal. For example, the angles 45° and $45^\circ + 360^\circ = 405^\circ$ are co-terminal; see Figure 4.

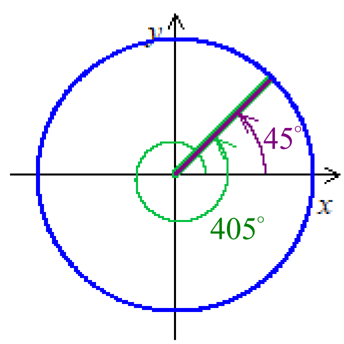


Figure 4: The angles 45° and 405° are co-terminal.

Traditionally, the coordinate plane is divided into ***four quadrants***; see Figure 5. We will often use the names of these quadrants to describe the location of the terminal side of different angles.

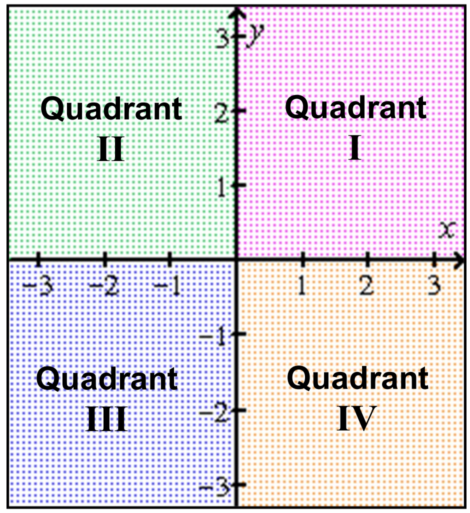


Figure 5

For example, consider the angles given in Figure 3: the angle 60° is in Quadrant I while -150° is in Quadrant III.

Next we'll discuss an alternative to degrees for measuring angles: **radians**.

DEFINITION: The **radian** measure of an angle is the ratio of the length of the arc on the circumference of the circle spanned by the angle, s, and the radius, r, of the circle; see Figure 6. Since a radian is a ratio of two lengths, the length-units cancel; thus, radians are considered a ***unit-less measure***.

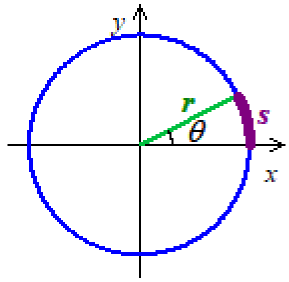


Figure 6: The angle θ measures $\frac{s}{r}$ radian.

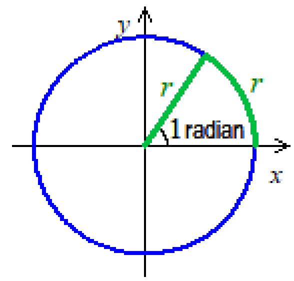


Figure 7: An angle that measures 1 radian.

An alternative yet equivalent definition is that an angle that measures 1 **radian** spans an arc whose length is equal to the length of the radius, r; see Figure 7.

CLICK HERE to see a video about radians.

Since a complete rotation around a circle, 360°, spans an arc equivalent to the entire circumference of the circle, we can find the radian equivalent of 360° by comparing a circle's circumference to its radius. A well-known formula from geometry tells us that the circumference, c, of a circle is given by $c = 2\pi r$ where r is the radius of the circle. Therefore

$$\begin{aligned} 360^\circ &= \frac{s}{r} \text{ rad} \\ &= \frac{2\pi r}{r} \text{ rad} \qquad \text{[since } s = c = 2\pi r, \text{ the entire circumference]} \\ &= 2\pi \text{ rad}, \end{aligned}$$

so 360° is equivalent to 2π radians. This implies that the following two ratios equal 1; we can use these ratios to convert from degrees to radians, and vise versa:

$$\frac{2\pi \text{ rad}}{360^\circ} = \frac{360^\circ}{2\pi \text{ rad}} = 1.$$

EXAMPLE: a. How many degrees are 8 radians?

b. How many radians are 8 degrees?

SOLUTION:

a. In order to convert 8 radians into degrees, we can multiply 8 radians by $\frac{360^\circ}{2\pi \text{ rad}}$. (Since this equals 1, multiplying by it won't change the value of our angle-measure.)

$$\begin{aligned} 8 \cancel{\text{rad}} \cdot \frac{360^\circ}{2\pi \cancel{\text{rad}}} &= \frac{8 \cdot 360^\circ}{2\pi} \\ &= \frac{1440^\circ}{\pi} \\ &\approx 458.37^\circ \end{aligned}$$

Therefore, 8 radians is about 458.37°.

b. In order to convert 8 degrees into radians, we can multiply 8° by $\frac{2\pi \text{ rad}}{360^\circ}$. (Since this equals 1, multiplying by it won't change the value of our angle-measure.)

$$\begin{aligned} 8\cancel{^\circ} \cdot \frac{2\pi \text{ rad}}{360\cancel{^\circ}} &= \frac{16\pi}{360} \text{ rad} \\ &= \frac{2\pi}{45} \text{ rad} \\ &\approx 0.14 \text{ rad} \end{aligned}$$

So 8° is about 0.14 radians.

EXAMPLE: **a.** Convert 1 radian into degrees.

b. Convert 90° into radians.

SOLUTION:

a. In order to convert 1 radian into degrees, we can multiply 1 radian by $\frac{360^\circ}{2\pi \text{ rad}}$.

$$
\begin{aligned}
1 \cancel{\text{rad}} \cdot \frac{360^\circ}{2\pi \cancel{\text{rad}}} &= \frac{360^\circ}{2\pi} \\
&= \frac{180^\circ}{\pi} \\
&\approx 57.3^\circ
\end{aligned}
$$

Thus, 1 radian is about 57.3°.

b. In order to convert 90° into radians, we can multiply 90° by $\frac{2\pi \text{ rad}}{360^\circ}$.

$$
\begin{aligned}
90\cancel{^\circ} \cdot \frac{2\pi \text{ rad}}{360\cancel{^\circ}} &= \frac{180\pi}{360} \text{ rad} \\
&= \frac{\pi}{2} \text{ rad}
\end{aligned}
$$

Thus, 90° is equivalent to $\frac{\pi}{2}$ radians.

EXAMPLE: Complete the table below:

θ (degrees)	0°	30°	45°	60°	90°	180°	270°	360°
θ (radians)								

CLICK HERE to see a video of this example.

SOLUTION:

θ (degrees)	0°	30°	45°	60°	90°	180°	270°	360°
θ (radians)	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π

Recall the definition of *radian*: the radian measure of an angle is the ratio of the length of the arc on the circumference of the circle spanned by the angle and the radius of the circle. Applying this fact to the circle in Figure 9 if θ is measured in radians, then

$$\theta = \frac{\text{arc-length}}{\text{radius}} = \frac{s}{r}$$

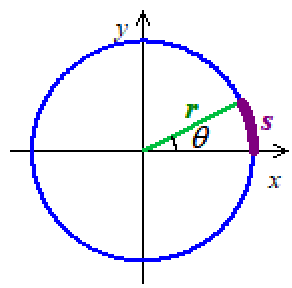


Figure 8: Circle of radius r with an angle θ spanning an arc-length s.

By solving the equation $\theta = \frac{s}{r}$ for s, we obtain the following definition:

DEFINITION: The **arc-length**, s, spanned in a circle of radius r by an angle θ radians is given by

$$s = r|\theta|.$$

Note that we need the absolute value of θ so that we obtain a positive arc-length if θ is negative. (Lengths are always positive!) Also, note that this formula only applies if θ is measured in radians.)

EXAMPLE: **a.** What is the arc-length spanned by an angle of 2 radians on a circle of radius 5 inches?

b. What is the arc-length spanned by an angle of 30° on a circle of radius 20 meters?

SOLUTION:

a. To find the arc-length, we can use the formula $s = r|\theta|$.

$$\begin{aligned} s &= r|\theta| \\ &= 5 \cdot 2 \\ &= 10 \end{aligned}$$

Thus, the arc-length spanned by an angle of 2 radians on a circle of radius 5 inches is 10 inches.

b. Before we can use the formula $s = r|\theta|$, we need to convert the angle into radians. In order to convert 30° into radians, we can multiply 30° by $\frac{2\pi \text{ rad}}{360^\circ}$ (which equals 1). (Of course we could use the table we created earlier in this chapter, but we will go ahead and show the computation here.)

$$\begin{aligned} 30\cancel{^\circ} \cdot \frac{2\pi \text{ rad}}{360\cancel{^\circ}} &= \frac{60\pi}{360} \text{ rad} \\ &= \frac{\pi}{6} \text{ rad} \end{aligned}$$

Thus, 30° is equivalent to $\frac{\pi}{6}$ radians. Now we can find the desired arc-length:

$$\begin{aligned} s &= r|\theta| \\ &= 20 \cdot \frac{\pi}{6} \\ &= \frac{10\pi}{3} \end{aligned}$$

Thus, the arc-length spanned by an angle of 30° on a circle of radius 20 meters is $\frac{10\pi}{3}$ meters.

Section I: Periodic Functions and Trigonometry

Chapter 2: Introduction to Periodic Functions

DEFINITION: A function f is ***periodic*** if its values repeat on regular intervals. So f is periodic if there exists some constant c such that

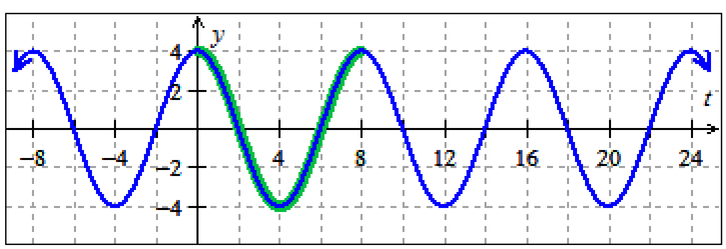
$$f(t+c)=f(t)$$

for all t in the domain of f. (This means that if the graph of $y=f(t)$ is shifted horizontally c units then it will appear unaffected.)

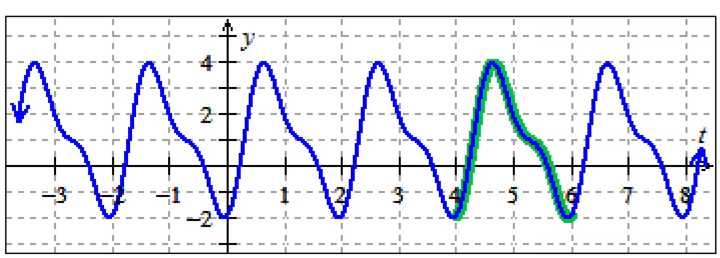
Any activity that repeats on a regular time interval can be described as *periodic*. For example, if the bell at a local church rings once every-hour-on-the-hour, then the function that relates the time of day to whether or not the bell will ring is a *periodic* function. Similarly if you take your dogs on a one-hour walk every day at 10 am, then the function that associates the time of day with whether or not you're on a walk with your dogs is a *periodic* function.

EXAMPLE 1: The following are graphs of periodic functions. (We know that they are periodic since an interval of each graph repeats over-and-over-and-over; that interval has been highlighted green in the graphs below.)

a.



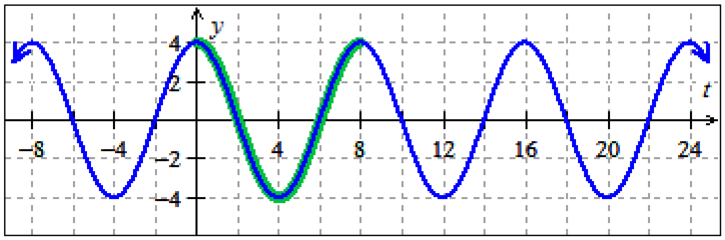
b.



DEFINITION: The ***period*** of a periodic function f is the smallest value $|c|$ such that $f(t+c) = f(t)$ for all t in the domain of f.

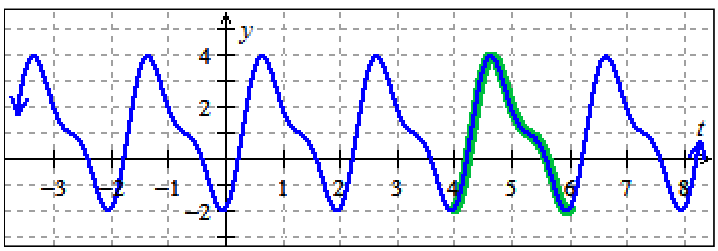
EXAMPLE 2: Find the period of the functions graphed below.

a.



The period of this function is 8 units since we can shift the graph horizontally 8 units, the graph will appear unaffected. (Notice that the "green interval" represents one period and is 8 units long.)

b.



The period of this function is 2 units since we can shift the graph horizontally 2 units, the graph will appear unaffected. (Notice that the "green interval" represents one period and is 2 units long.)

DEFINITIONS:

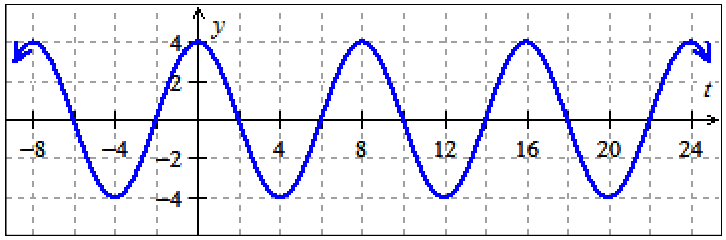
- The ***midline*** of a periodic function is the horizontal line midway between the function's minimum and maximum values.

  If $y = f(t)$ is periodic and $f_{\max}$ and $f_{\min}$ are the maximum and minimum values of f, respectively, then the equation of the midline is $y = \frac{f_{\max} + f_{\min}}{2}$.

- The ***amplitude*** of a periodic function is the distance between the function's maximum value and the midline (or the function's minimum value and the midline).

EXAMPLE 3: Find the midline and the amplitude of the functions graphed below.

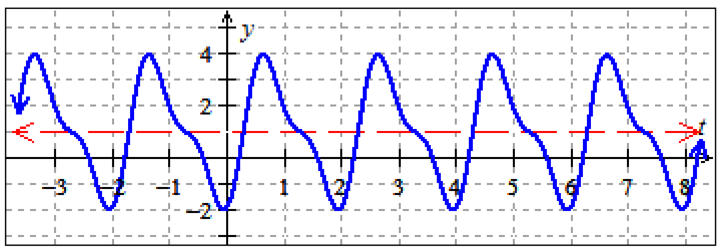
a.



The ***midline*** of this function is the t-axis (i.e., the line $y = 0$) since the maximum output for the function is 4 while the minimum output is -4, and $\frac{4+(-4)}{2} = 0$.

The ***amplitude*** of this function is 4 units.

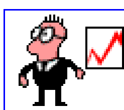
b.



The ***midline*** of this function is the line $y = 1$ since the maximum output for the function is 4 while the minimum output is -2, and $\frac{4+(-2)}{2} = 1$.

The ***amplitude*** of this function is 3 units.

EXAMPLE 4: The Amusement Park has a Ferris wheel 200 feet in diameter. The wheel rotates at a constant rate and completes a rotation once every 40 minutes. Let $h(t)$ represent the height in feet of a Ferris wheel passenger t minutes after boarding the wheel at ground level. Sketch a graph of $y = h(t)$.



CLICK HERE for a video of this example.

SOLUTION:

Since the Ferris wheel completes a rotation once every 40 minutes, the values of the height function $y = h(t)$ will repeat every 40 minutes so the period of $y = h(t)$ is 40 minutes.

Since the wheel is rotating at a constant rate, if it takes 40 minutes to complete a full rotation, it will take 20 minutes to travel half-way around the wheel. So at $t = 0$, the passenger will be at ground level (i.e., $h(0) = 0$); then at $t = 20$ the passenger will be at the top of the wheel (i.e., $h(20) = 200$), then at $t = 40$ the passenger will be at ground level again (i.e., $h(40) = 0$), etc. Furthermore, since it takes 20 minutes to travel from ground level to the top of the wheel, we can infer that it takes 10 minutes to travel half the distance from the ground to the top of the wheel, or 100 feet, so at $h(10) = 100$. Similarly, it takes 10 minutes to travel from the top of the wheel half down to a height of 100 feet, so $h(30) = 100$.

We can summarize this information in the table below, and then plot the ordered pairs $(t, h(t))$ on the coordinate plane in Figure 1. (Note that the colors in the table correspond to the colors of the points in Figure 1.)

t (minutes)	0	10	20	30	40	50	60	70	80
$h(t)$ (feet)	0	100	200	100	0	100	200	100	0

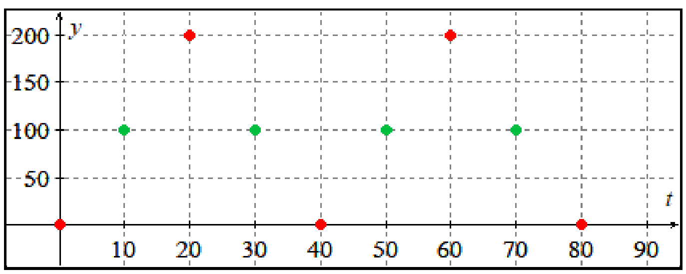


Figure 1: Some points on the graph of $y = h(t)$.

Now we need to connect the dots. To determine how the dots should be connected, let's imagine that we are the Ferris wheel's passengers. (To help you get a good mental image of the trip around the wheel, just imagine traveling around a circle; see Figure 2, below.)

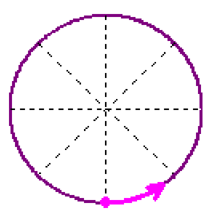


Figure 2: A very simple Ferris wheel.

When we first begin to travel around the wheel (starting at ground level, i.e., at the 6 o'clock position on the wheel), at first we don't gain much elevation. After a short period (near the tip of the red arrow in Figure 2), we begin to gain elevation more and more quickly. One-quarter of the way around the wheel we'll be gaining elevation most quickly (since the wheel is vertical here). As we near the top of the wheel it gets flatter and flatter so we'll begin to gain less and less elevation until we reach the top of the wheel. This tells us that our graph of $y = h(t)$ should be steep near the green dots less and less steep as it approaches the red dots. Let's use this information to connect the dots on our graph:

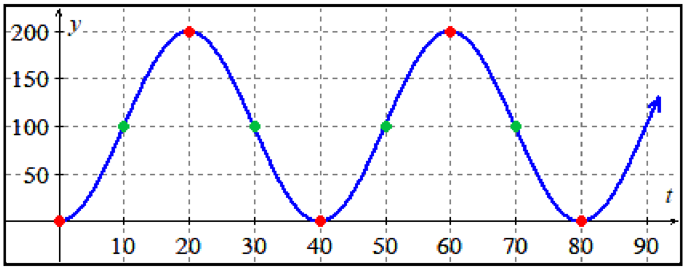


Figure 3: A graph of $y = h(t)$.

Section I: The Trigonometric Functions

Chapter 3, Part 1: Intro to the Trigonometric Functions

In Example 4 in Section I: Chapter 2, we observed that a circle rotating about its center (i.e., a Ferris wheel) lends itself naturally to the study of periodic functions. In fact, the two most important *trigonometric functions* are defined in terms of a circle – specifically a **unit circle**:

DEFINITION: A **unit circle** is a circle with a radius of 1 unit.

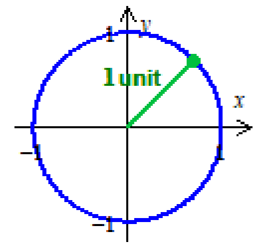


Figure 1: A unit circle.

Now we can define the **sine** and **cosine** functions. (There are four other trigonometric functions that can be defined in terms of the sine and cosine functions so first we'll get familiar with sine and cosine and then later in this chapter we'll define the four other trigonometric functions.)

DEFINITION: The **sine function**, denoted $\sin(\theta)$, associates each angle θ with the vertical coordinate (i.e., the y-coordinate) of the point P specified by θ on the circumference of a unit circle.

The **cosine function**, denoted $\cos(\theta)$, associates each angle θ with the horizontal coordinate (i.e., the x-coordinate) of the point P specified by θ on the circumference of a unit circle.

So the point P in Figure 2 has coordinates $(x, y) = (\cos(\theta), \sin(\theta))$.

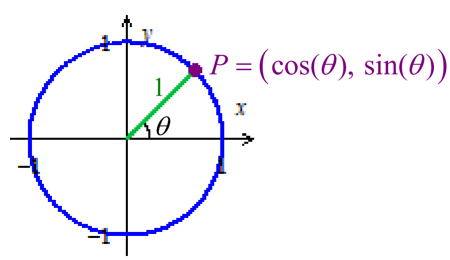


Figure 2

CLICK HERE for a video presentation of the definitions of the **sine** and **cosine** functions.

EXAMPLE 1: The angle θ specifies the point $P = \left(-\frac{3}{5}, \frac{4}{5}\right)$ on the circumference of a unit circle; see Figure 3. Find $\sin(\theta)$ and $\cos(\theta)$.

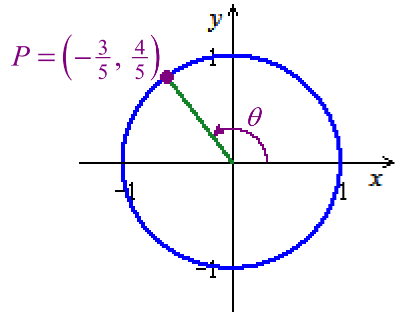


Figure 3

SOLUTION:

$$\sin(\theta) = \tfrac{4}{5} \quad \text{and} \quad \cos(\theta) = -\tfrac{3}{5}$$

In order to enhance our understanding of the sine and cosine functions, we should determine some particular values for the functions and sketch their graphs. But before we confront these details, let's determine the signs (positive or negative) of the sine and cosine functions in the four different quadrants of the coordinate plane. (To review the quadrants of the coordinate plane, see Section I: Chapter 1, Figure 5.)

- When the terminal side of angle θ is in Quadrant I, both the x- and y-coordinates of point P are positive. Therefore, **if θ is in Quadrant I, $\cos(\theta) > 0$ and $\sin(\theta) > 0$**.
- When the terminal side of angle θ is in Quadrant II, the y-coordinate of point P is positive but the x-coordinate is negative. Therefore, **if θ is in Quadrant II, $\cos(\theta) < 0$ and $\sin(\theta) > 0$**.
- When the terminal side of angle θ is in Quadrant III, both the x- and y-coordinates of point P are negative. Therefore, **if θ is in Quadrant III, $\cos(\theta) < 0$ and $\sin(\theta) < 0$**.
- When the terminal side of angle θ is in Quadrant IV, the x-coordinate of point P is positive but the y-coordinate is negative. Therefore, **if θ is in Quadrant IV, $\cos(\theta) > 0$ and $\sin(\theta) < 0$**.

Let's summarize in Figure 4 what we've determined about the signs of the sine and cosine functions in the different quadrants:

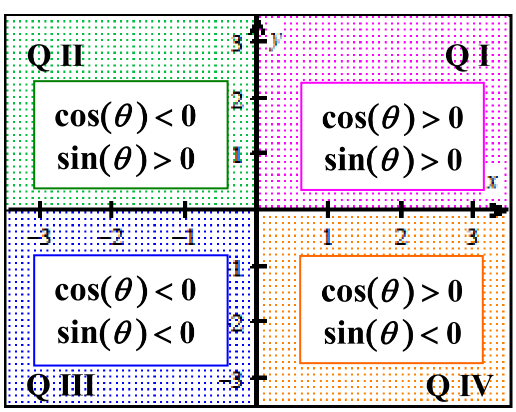


Figure 4: The signs of sine and cosine in the four quadrants.

Now let's find the sine and cosine of a few particular angles. Recall that the sine and cosine functions represent the coordinates of points on a unit circle, and the easiest points for us to find on the unit circle are points where the circumference of the circle intersects the coordinate axes; let's start by finding the corresponding sine and cosine values. Keep in mind that **cosine** represents the **x-coordinate** and **sine** represents the **y-coordinate**.

[Note that this information is discussed in the videos linked from the next page.]

- The angle $\theta = 90^\circ$, i.e., $\theta = \frac{\pi}{2}$ radians, specifies the point $(0, 1)$ on the circumference of a unit circle; see Figure 5a.

  Thus, $\cos\left(\frac{\pi}{2}\right) = 0$ and $\sin\left(\frac{\pi}{2}\right) = 1$.

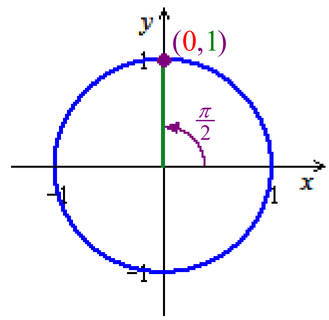


Figure 5a

- The angle $\theta = 180^\circ$, i.e., $\theta = \pi$ radians, specifies the point $(-1, 0)$ on the circumference of a unit circle; see Figure 5b.

  Thus, $\cos(\pi) = -1$ and $\sin(\pi) = 0$.

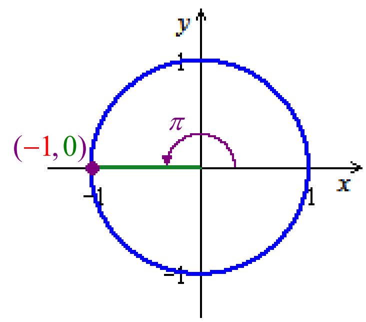


Figure 5b

- The angle $\theta = 270^\circ$, i.e. $\theta = \frac{3\pi}{2}$ radians, specifies the point $(0, -1)$ on the circumference of a unit circle; see Figure 5c.

  Thus, $\cos\left(\frac{3\pi}{2}\right) = 0$ and $\sin\left(\frac{3\pi}{2}\right) = -1$.

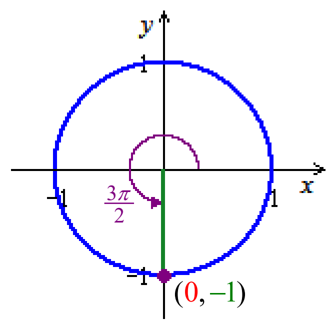


Figure 5c

- The angle $\theta = 360^\circ$, i.e., $\theta = 2\pi$ radians, specifies the point $(1, 0)$ on the circumference of a unit circle; see Figure 5d.

  Thus, $\cos(2\pi) = 1$ and $\sin(2\pi) = 0$.

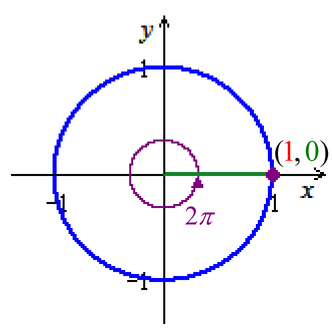


Figure 5d

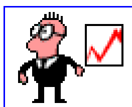
Notice that angles of measure 2π radians (i.e., $\theta = 360^\circ$) and 0 radians specify the same point: $(1, 0)$. Thus, the sine and cosine values for 2π radians and 0 radians are the same, i.e.,

$$\cos(2\pi) = \cos(0) = 1 \quad \text{and} \quad \sin(2\pi) = \sin(0) = 0.$$

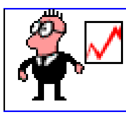
Since ANY angle θ and $\theta + 2\pi$ specify the *same* point on the unit circle, the sine and cosine values of θ and $\theta + 2\pi$ are the same; therefore, the period of the sine and cosine functions is 2π radians.

For all θ, $\sin(\theta) = \sin(\theta + 2\pi)$ and $\cos(\theta) = \cos(\theta + 2\pi)$ so the **period** of both $s(\theta) = \sin(\theta)$ and $c(\theta) = \cos(\theta)$ is 2π radians (i.e., 360°).

Now we'll sketch graphs of the sine and cosine functions.



CLICK HERE for a video that shows how to graph the **sine function**.



CLICK HERE for a video that shows how to graph the **cosine function**.

Let's start by organizing the function values we determined above in a table:

θ (degrees)	0°	90°	180°	270°	360°	450°	540°	630°	720°
θ (radians)	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π	$\frac{7\pi}{2}$	4π
$y = \cos(\theta)$	1	0	-1	0	1	0	-1	0	1
$y = \sin(\theta)$	0	1	0	-1	0	1	0	-1	0

In Figure 6a and 6b we've plotted the information in the table on two coordinate planes.

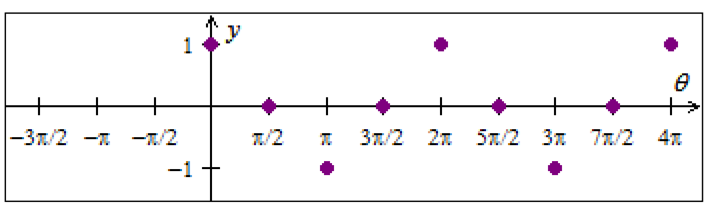


Figure 6a: Some points on $y = \cos(\theta)$.

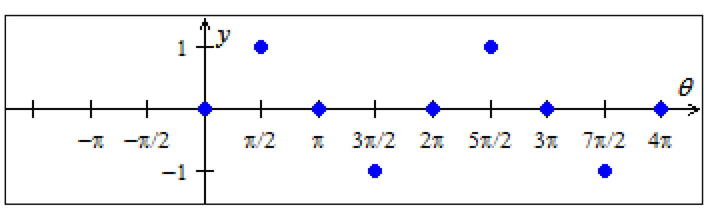


Figure 6b: Some points on $y = \sin(\theta)$.

In the next chapters we'll find many more values of sine and cosine, but we can go ahead and connect the dots in these graphs to obtain reasonable sketches of the graphs of the sine and cosine functions in Figures 7a and 7b. (We can use what we observed in Example 4 from Section I: Chapter 2 when we studied the Ferris wheel.)

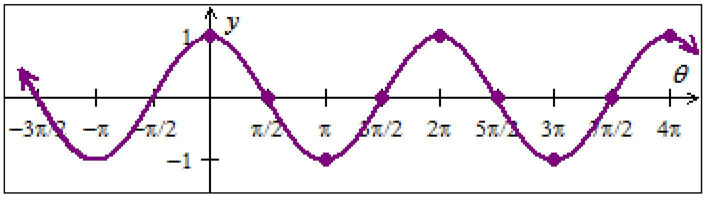


Figure 7a: The graph of $y = \cos(\theta)$.

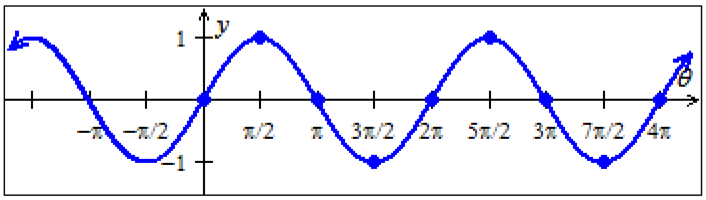


Figure 7b: The graph of $y = \sin(\theta)$.

KEY POINT: To get these graphs on your calculator, be sure to change the **angle mode** of to **radians**. If you want to get graphs of sine and cosine in **degree mode**, be sure the **window** has a horizontal interval like $[-300, 1440]$ since the period of the function is $360°$.

Notice that the graphs of $y = \cos(\theta)$ and $y = \sin(\theta)$ are very similar. In fact, if we shift $y = \sin(\theta)$ to the left $\frac{\pi}{2}$ units, we'll obtain the graph of $y = \cos(\theta)$. Using what we learned about graph transformation in MTH 111, this means that $\mathbf{\cos(\theta) = \sin\left(\theta + \frac{\pi}{2}\right)}$. Similarly, if we shift $y = \cos(\theta)$ to the right $\frac{\pi}{2}$ units, we'll obtain the graph of $y = \sin(\theta)$. Using what we know about graph transformation, this means that $\mathbf{\sin(\theta) = \cos\left(\theta - \frac{\pi}{2}\right)}$.

The two bold equations above are called **identities** since the left and right sides of the equations are *always* identical, no matter what value of θ is used.

DEFINITION: An **identity** is an equation that is true for all values in the domains of the involved expressions.

Earlier in this chapter we observed a couple of identities but didn't call them identities. The equations

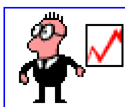
$$\sin(\theta) = \sin(\theta + 2\pi) \quad \text{and} \quad \cos(\theta) = \cos(\theta + 2\pi)$$

are identities since they are true for *all* values of θ. We can use the definitions and graphs of sine and cosine to recognize a few other important identities.

SOME IMPORTANT TRIG IDENTITIES

- $\cos(\theta) = \cos(\theta + 2\pi)$
- $\sin(\theta) = \sin(\theta + 2\pi)$
- $\sin(\theta) = \cos\left(\theta - \frac{\pi}{2}\right)$
- $\cos(\theta) = \sin\left(\theta + \frac{\pi}{2}\right)$
- $\cos(-\theta) = \cos(\theta)$
- $\sin(-\theta) = -\sin(\theta)$
- $\cos(\theta) = \cos(2\pi - \theta)$
- $\sin(\theta) = \sin(\pi - \theta)$

Do your best to convince yourself that each of following identities is true using what you learned in MTH 111 about graph transformations and symmetry, and using what you know about the definitions of sine and cosine – see the video below



CLICK HERE for a video where these trig identities are explored.

We'll study a few more important identities at the end of this chapter and we'll study *proving trigonometric identities* in Section II: Chapter 3.

We can generalize the definitions of the sine and cosine functions so that they are applicable to circles of any size, rather than only for unit circles.

DEFINITION: If the point $T = (x, y)$ is specified by the angle θ on the circumference of a circle of radius r then

$$\cos(\theta) = \frac{x}{r} \text{ and } \sin(\theta) = \frac{y}{r}.$$

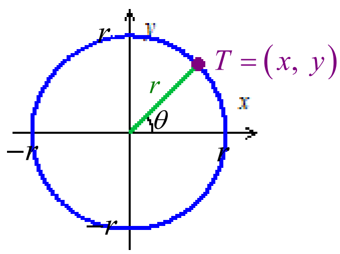


Figure 8

Notice that if $r = 1$ then this definition $\cos(\theta)$ and $\sin(\theta)$ is equivalent we saw at the beginning of this chapter:

$$\cos(\theta) = \frac{x}{r} = \frac{x}{1} = x \quad \text{and} \quad \sin(\theta) = \frac{y}{r} = \frac{y}{1} = y$$

If we solve the equations $\cos(\theta) = \frac{x}{r}$ and $\sin(\theta) = \frac{y}{r}$ for x and y, respectively, we can obtain the coordinates of a point on the circumference of a circle of any r:

$$\cos(\theta) = \frac{x}{r} \Rightarrow x = r\cos(\theta) \quad \text{and} \quad \sin(\theta) = \frac{y}{r} \Rightarrow y = r\sin(\theta)$$

If the point $T = (x, y)$ is specified by the angle θ on the circumference of a circle of radius, r, then

$$x = r\cos(\theta) \quad \text{and} \quad y = r\sin(\theta).$$

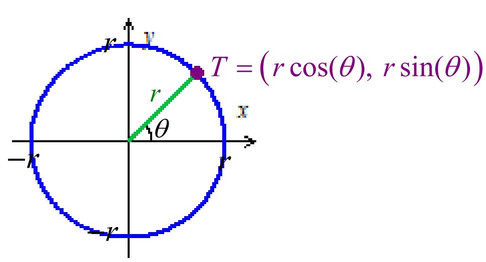


Figure 9

EXAMPLE 2: A circle with a radius of 6 units is given in Figure 10. The point Q is specified by the angle α. Use the sine and cosine function to express the exact coordinates of point Q.

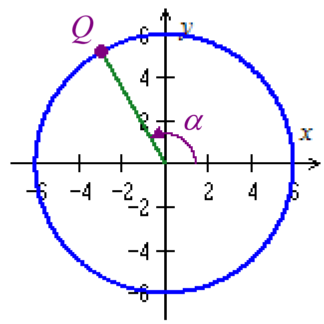


Figure 10

SOLUTION:

The point Q is specified by α on the circumference of a circle of radius 6 units. Thus,

$$Q = \left(6\cos(\alpha),\ 6\sin(\alpha)\right)$$

Recall that the cosine and sine functions represent the horizontal and vertical coordinates of a point on the circumference of a unit circle; see Figure 11. This situation creates a right triangle with hypotenuse of length 1 unit and side-lengths of $\cos(\theta)$ and $\sin(\theta)$ units; see Figure 12.

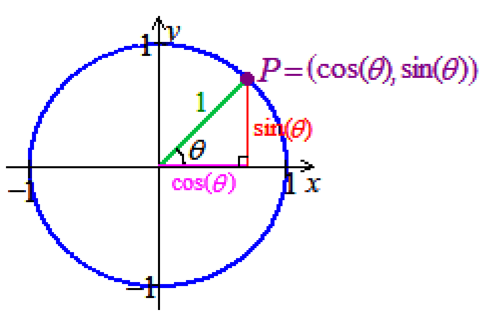


Figure 11

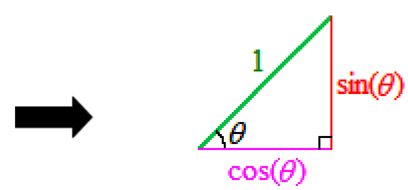


Figure 12

Now we can apply the famous *Pythagorean Theorem* to this right triangle – first let's review the Pythagorean Theorem:

THE PYTHAGOREAN THEOREM:

If the sides of a right triangle (i.e., a triangle with a 90° angle) are labeled like the one given in Figure 13, then $a^2 + b^2 = c^2$.

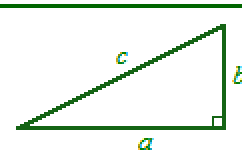


Figure 13

Applying the Pythagorean Theorem to the right triangle in Figure 12 we obtain what is called the **Pythagorean Identity**:

$$\left(\sin(\theta)\right)^2 + \left(\cos(\theta)\right)^2 = 1$$

When we use exponents with trigonometric functions, we can use unusual notation. Instead of using parentheses around the entire expression, we can put the exponent between the letters that name the function and the input for the function. Thus, we can write an expression like "$\left(\sin(\theta)\right)^2$" as "$\sin^2(\theta)$". We can use this notation to express the Pythagorean Identity:

THE PYTHAGOREAN IDENTITY:

For all $\theta \in \mathbb{R}$, $\sin^2(\theta) + \cos^2(\theta) = 1$.

EXAMPLE 3: If $\sin(A) = \frac{1}{3}$ and $\frac{\pi}{2} < A < \pi$ (i.e., A is in Quadrant II), find $\cos(A)$.

SOLUTION:

Since the Pythagorean Identity gives us an equation involving sine and cosine, we can use it to find one of the values when we know the other value. In this case, we know the value of $\sin(A)$, so we can use the Pythagorean Identity to find $\cos(A)$:

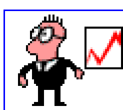
$$\sin^2(A) + \cos^2(A) = 1$$

$$\Rightarrow \quad \left(\tfrac{1}{3}\right)^2 + \cos^2(A) = 1$$

$$\Rightarrow \quad \cos^2(A) = 1 - \left(\tfrac{1}{3}\right)^2$$

$$\Rightarrow \quad \cos(A) = -\sqrt{\tfrac{8}{9}}$$ (we choose the negative square root since cosine is negative in Quadrant II.)

$$\Rightarrow \quad \cos(A) = -\frac{2\sqrt{2}}{3}$$



CLICK HERE for a video of this example.

Section I: The Trigonometric Functions

Chapter 3, Part 2: Intro to the Trigonometric Functions

As mentioned in Chapter 3, Part 1, there are four other trigonometric functions besides sine and cosine. These four functions are defined in terms of the sine and cosine functions:

DEFINITIONS:

The **tangent function**, denoted $\tan(\theta)$, is defined by $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$.

The **cotangent function**, denoted $\cot(\theta)$, is defined by $\cot(\theta) = \dfrac{1}{\tan(\theta)}$.

Consequently, $\cot(\theta) = \dfrac{\cos(\theta)}{\sin(\theta)}$.

The **secant function**, denoted $\sec(\theta)$, is defined by $\sec(\theta) = \dfrac{1}{\cos(\theta)}$.

The **cosecant function**, denoted $\csc(\theta)$, is defined by $\csc(\theta) = \dfrac{1}{\sin(\theta)}$.

Since all three of these functions are defined in terms of the sine and cosine function, it's reasonable to consider the sine and cosine to be more important. The tangent function is the ratio of the sine and cosine functions so it provides truly different information than does sine or cosine, so the tangent function is also important. But cotangent, secant, and cosecant are just reciprocals of tangent, cosine, and sine so they don't provide any new information (instead, they provide the same information in a different way) so, arguably, they are less important functions. In Chapter 5 we'll look at graphs of these "other trig functions" but in this chapter we'll focus on getting familiar with their definitions and, in the process, review the sine and cosine values we observed in Chapter 3, Part 1.

EXAMPLE 1: Find $\tan(\theta)$, $\sec(\theta)$, $\csc(\theta)$, and $\cot(\theta)$ if...

a. ...$\theta = \pi$.

b. ...$\theta = \frac{3\pi}{2}$.

c. ...$\theta = 2\pi$.

SOLUTION:

a. $\tan(\pi) = \dfrac{\sin(\pi)}{\cos(\pi)}$ (using the definition of tangent)

$= \dfrac{0}{-1}$ (since $\sin(\pi) = 0$ and $\cos(\pi) = -1$; see Part 1 of Chapter 3)

$= 0$

$\cot(\pi) = \dfrac{\cos(\pi)}{\sin(\pi)}$ (using the definition of cotantent)

Recall from Part 1 of Chapter 3 that $\sin(\pi) = 0$ so $\cot(\pi)$ involves division by 0, so $\cot(\pi)$ is *undefined*.

$\sec(\pi) = \dfrac{1}{\cos(\pi)}$ (using the definition of secant)

$= \dfrac{1}{-1}$ (since $\cos(\pi) = -1$; see Part 1 of Chapter 3)

$= -1$

$\csc(\pi) = \dfrac{1}{\sin(\pi)}$ (using the definition of cosecant)

Since $\sin(\pi) = 0$, $\csc(\pi)$ involves division by 0 so $\csc(\pi)$ is *undefined*.

b. $\tan\left(\frac{3\pi}{2}\right) = \dfrac{\sin\left(\frac{3\pi}{2}\right)}{\cos\left(\frac{3\pi}{2}\right)}$ (using the definition of tangent)

Recall from Part 1 of Chapter 3 that $\cos\left(\frac{3\pi}{2}\right) = 0$ so $\tan\left(\frac{3\pi}{2}\right)$ involves division by 0, so $\tan\left(\frac{3\pi}{2}\right)$ is *undefined*.

$\cot\left(\frac{3\pi}{2}\right) = \dfrac{\cos\left(\frac{3\pi}{2}\right)}{\sin\left(\frac{3\pi}{2}\right)}$ (using the definition of cotangent)

$= \dfrac{0}{-1}$ (since $\cos\left(\frac{3\pi}{2}\right) = 0$ and $\sin\left(\frac{3\pi}{2}\right) = -1$; see Part 1 of Chapter 3)

$= 0$

$\sec\left(\frac{3\pi}{2}\right) = \dfrac{1}{\cos\left(\frac{3\pi}{2}\right)}$ (using the definition of secant)

Since $\cos\left(\frac{3\pi}{2}\right) = 0$, $\sec\left(\frac{3\pi}{2}\right)$ involves division by 0 so $\sec\left(\frac{3\pi}{2}\right)$ is *undefined*.

$\csc\left(\frac{3\pi}{2}\right) = \dfrac{1}{\sin\left(\frac{3\pi}{2}\right)}$ (using the definition of cosecant)

$= \dfrac{1}{-1}$ (since $\sin\left(\frac{3\pi}{2}\right) = -1$; see Part 1 of Chapter 3)

$= -1$

c. $\tan(2\pi) = \dfrac{\sin(2\pi)}{\cos(2\pi)}$ (using the definition of tangent)

$= \dfrac{0}{1}$ (since $\sin(2\pi) = 0$ and $\cos(2\pi) = 1$; see Part 1 of Chapter 3)

$= 0$

$\cot(2\pi) = \dfrac{\cos(2\pi)}{\sin(2\pi)}$ (using the definition of cotangent)

Recall from Part 1 of Chapter 3 that $\sin(2\pi) = 0$ so $\csc(2\pi)$ involves division by 0, so $\cot(2\pi)$ is *undefined*.

$\sec(2\pi) = \dfrac{1}{\cos(2\pi)}$ (using the definition of secant)

$= \dfrac{1}{1}$ (since $\cos(2\pi) = 1$; see Part 1 of Chapter 3)

$= 1$

$\csc(2\pi) = \dfrac{1}{\sin(2\pi)}$ (using the definition of cosecant)

Since $\sin(2\pi) = 0$, $\cot(2\pi)$ involves division by 0 so $\csc(2\pi)$ is *undefined*.

Other Pythagorean Identities

Recall the Pythagorean Identity from Chapter 3, Part 1: $\sin^2(\theta)+\cos^2(\theta)=1$. There are two other identities that can be obtained from the Pythagorean Identity.

One of these identities can be found by dividing both sides of the Pythagorean identity by $\cos^2(\theta)$:

$$\begin{aligned} &\sin^2(\theta)+\cos^2(\theta)=1 \\ \Rightarrow\ &\frac{\sin^2(\theta)}{\cos^2(\theta)}+\frac{\cos^2(\theta)}{\cos^2(\theta)}=\frac{1}{\cos^2(\theta)} \\ \Rightarrow\ &\tan^2(\theta)+1=\sec^2(\theta) \end{aligned}$$

Alternatively, we can divide both sides of the Pythagorean identity by $\sin^2(\theta)$ and find another identity:

$$\begin{aligned} &\sin^2(\theta)+\cos^2(\theta)=1 \\ \Rightarrow\ &\frac{\sin^2(\theta)}{\sin^2(\theta)}+\frac{\cos^2(\theta)}{\sin^2(\theta)}=\frac{1}{\sin^2(\theta)} \\ \Rightarrow\ &1+\cot^2(\theta)=\csc^2(\theta) \end{aligned}$$

This gives us three identities that are considered “The Pythagorean Identities”.

The Pythagorean Identities

$$\sin^2(\theta)+\cos^2(\theta)=1$$

$$\tan^2(\theta)+1=\sec^2(\theta)$$

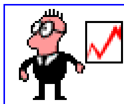
$$1+\cot^2(\theta)=\csc^2(\theta)$$

Section I: The Trigonometric Functions

Chapter 3, Part 3: Intro to the Trigonometric Functions

Now let's determine the sine and cosine of some important angles, namely, 30°, 45°, and 60° (i.e., $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$ radians). We focus on these angles since we can use some basic geometry to easily find their sine and cosine values – but we cannot easily find the sine and cosine of most other angles. During this course, we'll refer to these angles (and multiples of these angels) as *friendly angles* since their sine and cosine values are easy to find. You ***need*** to learn (or "memorize") these values.

Let's start by finding the sine and cosine of 30° (i.e., $\frac{\pi}{6}$ radians).



CLICK HERE for a video that shows how to find the sine and cosine of 30°.

Since we want to find $\sin(30^\circ)$ and $\cos(30^\circ)$, we need to find the horizontal and vertical coordinates of the point P on the circumference of the unit circle specified by the angle 30°; see Figure 1.

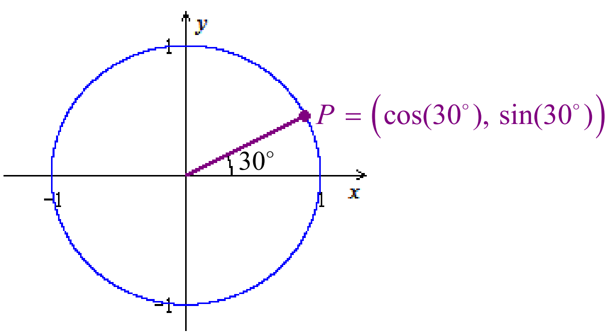


Figure 1

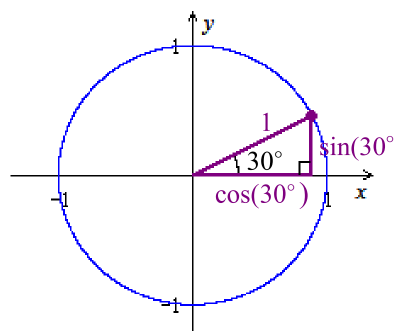


Figure 2

Notice that we can use the 30° angle and the terminal side of the 30° angle (i.e., the radius of the unit circle) in Figure 1 to construct a right-triangle with a 1 unit long hypotenuse and side-lengths that are the horizontal and vertical coordinates of the point P; see Figure 2. If we can find the side-lengths of the triangle in Figure 2, we will have found $\sin(30^\circ)$ and $\cos(30^\circ)$. In Figure 3, we've magnified the triangle from Figure 2. Notice that, since the sum of the angles in a triangle is always 180° and since this triangle already has a 30° angle and a 90° angle, the third angle must be 60°.

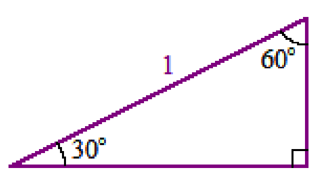


Figure 3

In order to find the lengths of the sides of this triangle, in Figure 4 we've placed a "mirror image" of the triangle under the triangle to form a larger triangle.

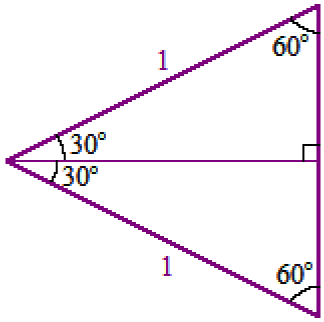


Figure 4

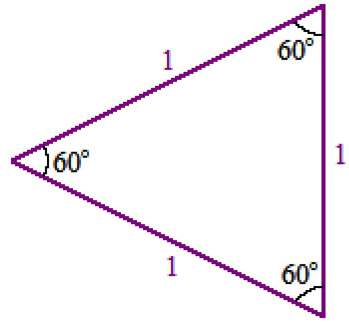


Figure 5

Notice that this larger triangle is equiangular since all three of its angles have the same measure: 60°; see Figure 5. As you may have studied previously in a geometry course, equiangular triangles are also equilateral, i.e., they contain three equal sides. Since two of the sides are each 1 unit long, the other side must also be 1 unit long.

Since the sides of the triangle are each 1 unit long, if we cut one of the sides in half, each part will be $\frac{1}{2}$ of a unit. Notice how, in Figure 4, the horizontal segment that creates the two 30° angles cuts the side opposite these angles in half; see Figure 6.

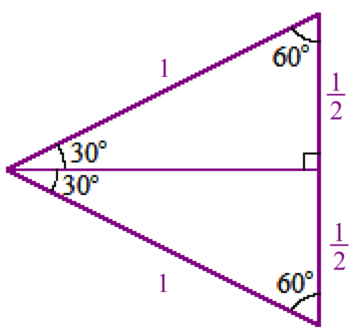


Figure 6

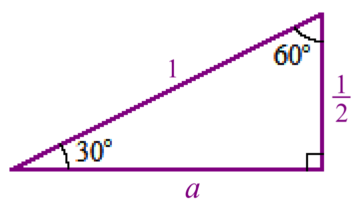


Figure 7

Now we return to the original right triangle that we were looking at in Figure 3, and label in Figure 7 all of the information we've found so far. We've labeled the unknown side a. We can find a using the Pythagorean Theorem:

THE PYTHAGOREAN THEOREM:

If the sides of a right triangle (i.e., a triangle with a 90° angle) are labeled like the one given in Figure 8, then $a^2 + b^2 = c^2$.

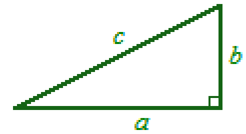


Figure 8

Using the *Pythagorean Theorem* (see the **green box** above) we can find a:

$$a^2 + \left(\tfrac{1}{2}\right)^2 = 1^2$$
$$\Rightarrow \quad a^2 + \tfrac{1}{4} = 1$$
$$\Rightarrow \quad a^2 = \tfrac{3}{4}$$
$$\Rightarrow \quad a = \tfrac{\sqrt{3}}{2}$$

In Figure 9, we've labeled all of the side-lengths of this triangle.

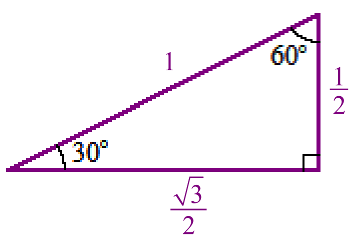


Figure 9

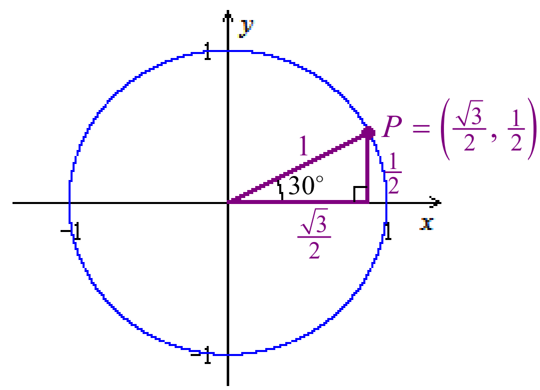
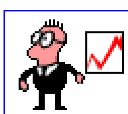


Figure 10

Now we can put the triangle from Figure 9 back inside the unit circle; see Figure 10.

Recall that the coordinates of P are $\left(\cos(30^\circ), \sin(30^\circ)\right)$, and now we know that the actual coordinates of P are $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, so we can conclude that $\mathbf{\cos(30^\circ) = \frac{\sqrt{3}}{2}}$ and $\mathbf{\sin(30^\circ) = \frac{1}{2}}$.

Now let's find the sine and cosine values for 60° (i.e., $\frac{\pi}{3}$ radians).



CLICK HERE for a video that shows how to find the sine and cosine of 60°.

Since we want to find $\sin(60^\circ)$ and $\cos(60^\circ)$, we need to find the horizontal and vertical coordinates of the point Q on the circumference of the unit circle specified by the angle 60°; see Figure 11.

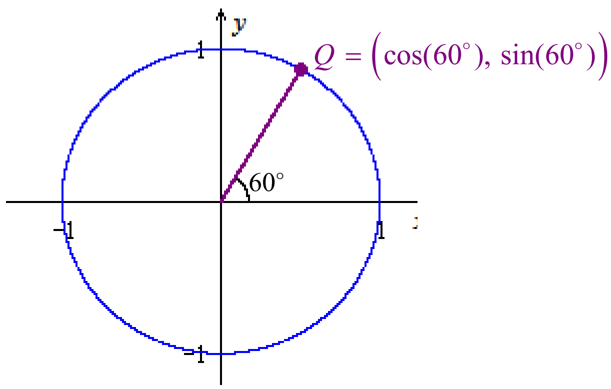


Figure 11

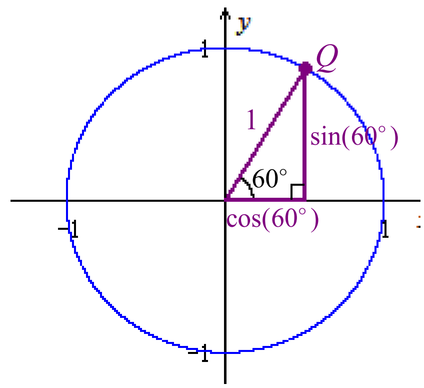


Figure 12

As we did with 30°, we can use the 60° angle in Figure 11 to construct a right-triangle that we can use to find the coordinates of the point Q; see Figure 12.

In Figure 13, we've magnified the triangle from Figure 12. Notice that, since the sum of the angles in a triangle is always 180° and since this triangle already has a 60° angle and a 90° angle, the third angle must be 30°.

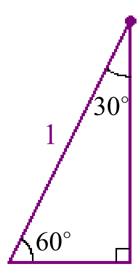


Figure 13

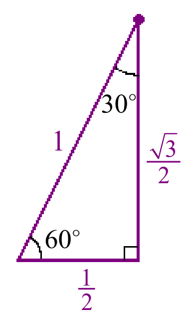


Figure 14

Notice that this is essentially the same triangle that we studied in Figure 8, just with a different orientation. So we can use the triangle in Figure 8 to obtain the lengths of the sides of this triangle; see Figure 14.

Now we can put this triangle back inside the unit circle; see Figure 15.

Recall that the coordinates of Q are $\left(\cos(60^\circ),\ \sin(60^\circ)\right)$, and now we know that the actual coordinates of Q are $\left(\frac{1}{2},\ \frac{\sqrt{3}}{2}\right)$, so we can conclude that $\mathbf{\cos(60^\circ) = \frac{1}{2}}$ and $\mathbf{\sin(60^\circ) = \frac{\sqrt{3}}{2}}$.

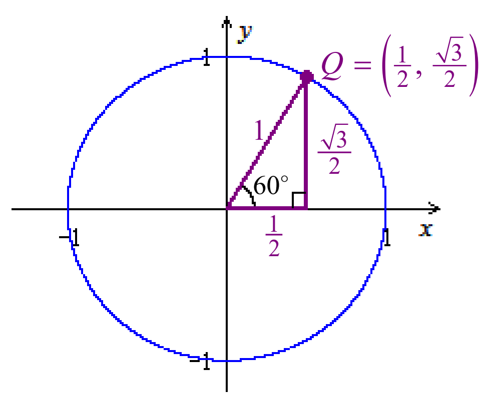
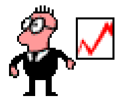


Figure 15

Now let's find the sine and cosine values for 45° (i.e., $\frac{\pi}{4}$ radians).



CLICK HERE for a video that shows how to find the sine and cosine of 45°.

Since we want to find $\sin(45^\circ)$ and $\cos(45^\circ)$, we need to find the horizontal and vertical coordinates of the point M on the circumference of the unit circle specified by the angle 45°; see Figure 16.

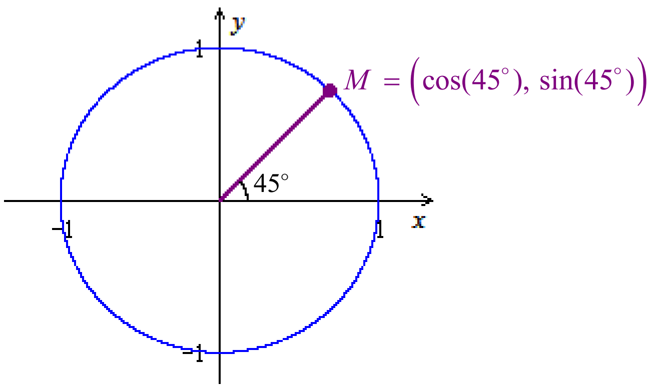


Figure 16

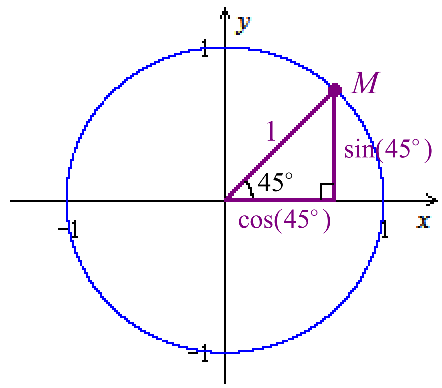


Figure 17

As we did with 30° and 60°, we can use the 45° angle in Figure 16 to construct a right-triangle that we can use to find the coordinates of the point M; see Figure 17. In Figure 18, we've magnified the triangle from Figure 17.

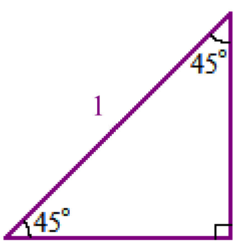


Figure 18

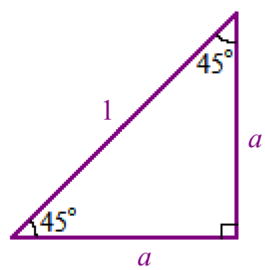


Figure 19

Notice that, since the sum of the angles in a triangle is always 180° and since this triangle already has a 45° angle and a 90° angle, the third angle must also be 45°. This is an *isosceles* triangle since two of the angles are equal; the sides opposite the equal angles must also be of equal length. (This is a property of isosceles triangles.) In Figure 19, we've used the same symbol, a, to label the lengths of these two sides.

We can now use the Pythagorean Theorem to find a:

$$
\begin{aligned}
& a^2 + a^2 = 1^2 \\
\Rightarrow \quad & 2a^2 = 1 \\
\Rightarrow \quad & a^2 = \tfrac{1}{2} \\
\Rightarrow \quad & a = \sqrt{\tfrac{1}{2}} = \tfrac{1}{\sqrt{2}}
\end{aligned}
$$

In Figure 20, we've labeled the side-lengths of this triangle:

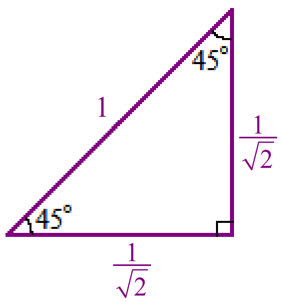


Figure 20

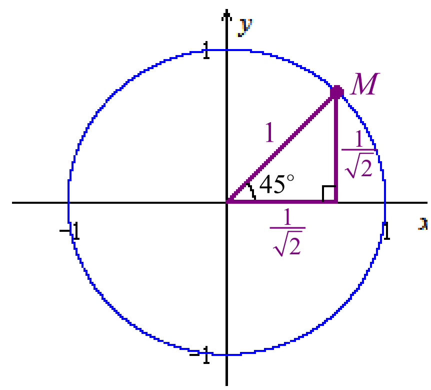


Figure 21

Now we can put this triangle back inside the unit circle; see Figure 21.

Recall that the coordinates of M are $\left(\cos(45^\circ), \sin(45^\circ)\right)$, and now we know that the actual coordinates of M are $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, so we can conclude that $\mathbf{\cos(45^\circ) = \frac{1}{\sqrt{2}}}$ and $\mathbf{\sin(45^\circ) = \frac{1}{\sqrt{2}}}$.

Note that although there is nothing wrong with the expression $\frac{1}{\sqrt{2}}$, in the "old days" (i.e., the pre-calculator era), people didn't like to have irrational numbers (i.e., numbers like $\sqrt{2}$) in the denominator of fractions since the tables they used to help them approximate values didn't contain approximations for expressions with irrational numbers in the denominator. As a result, there is a procedure known as "rationalizing the denominator" which allows us to get rid of the radicals in the denominator. (You should have studied this in an Intermediate Algebra course.) When you study trigonometry, you often see the rationalized form of the number so it's worth taking note of the rationalized form of $\frac{1}{\sqrt{2}}$:

$$
\begin{aligned}
\frac{1}{\sqrt{2}} &= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} \\
&= \frac{\sqrt{2}}{2}
\end{aligned}
$$

Let's put the values for sine and cosine that we've found so far in a table and label the relevant coordinates on a unit circle. **It is important that you learn these values**. In Part 4 of Chapter 3, we'll discuss a strategy for remembering these values.

θ (degrees)	30°	45°	60°
θ (radians)	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
$\cos(\theta)$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\sin(\theta)$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$

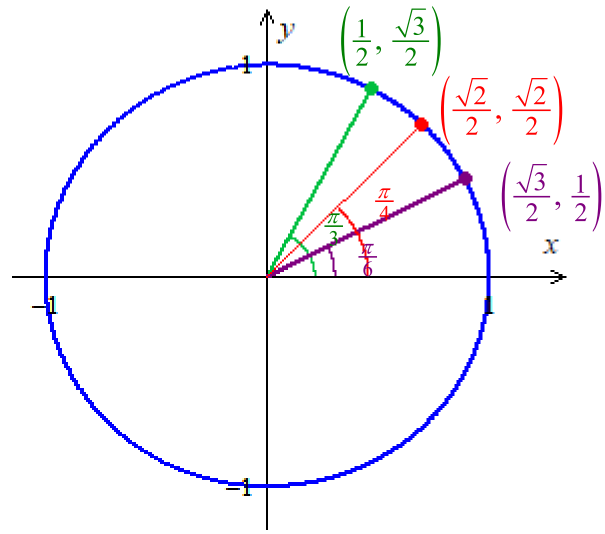


Figure 22

Recall from Chapter 3: Part 2 the following definitions of the "other trig functions."

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \qquad \cot(\theta) = \frac{1}{\tan(\theta)}$$

$$\sec(\theta) = \frac{1}{\cos(\theta)} \qquad \csc(\theta) = \frac{1}{\sin(\theta)}$$

In Examples 1–3 we'll find the of tangent, secant, cosecant, and cotangent of $\frac{\pi}{6}$, $\frac{\pi}{4}$ and $\frac{\pi}{3}$.

EXAMPLE 1: Find $\tan\left(\frac{\pi}{6}\right)$, $\sec\left(\frac{\pi}{6}\right)$, $\csc\left(\frac{\pi}{6}\right)$, and $\cot\left(\frac{\pi}{6}\right)$.

SOLUTION:

$$\tan\left(\frac{\pi}{6}\right) = \frac{\sin\left(\frac{\pi}{6}\right)}{\cos\left(\frac{\pi}{6}\right)}$$ (using the definition of tangent)

$$= \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}$$ (since $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$ and $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$)

$$= \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$ (either form of this number is acceptable)

$$\sec\left(\frac{\pi}{6}\right) = \frac{1}{\cos\left(\frac{\pi}{6}\right)}$$ (using the definition of secant)

$$= \frac{1}{\frac{\sqrt{3}}{2}}$$ (since $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$)

$$= \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$ (either form of this number is acceptable)

$$\csc\left(\frac{\pi}{6}\right) = \frac{1}{\sin\left(\frac{\pi}{6}\right)}$$ (using the definition of cosecant)

$$= \frac{1}{\frac{1}{2}}$$ (since $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$)

$$= 2$$

$$\cot\left(\frac{\pi}{6}\right) = \frac{\cos\left(\frac{\pi}{6}\right)}{\sin\left(\frac{\pi}{6}\right)}$$ (using the definition of cotangent)

$$= \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}$$ (since $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$ and $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$)

$$= \sqrt{3}$$

EXAMPLE 2: Find $\tan\left(\frac{\pi}{4}\right)$, $\sec\left(\frac{\pi}{4}\right)$, $\csc\left(\frac{\pi}{4}\right)$, and $\cot\left(\frac{\pi}{4}\right)$.

SOLUTION:

$$\tan\left(\frac{\pi}{4}\right) = \frac{\sin\left(\frac{\pi}{4}\right)}{\cos\left(\frac{\pi}{4}\right)}$$ (using the definition of tangent)

$$= \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}$$ (since $\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ and $\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$)

$$= 1$$

$$\sec\left(\frac{\pi}{4}\right) = \frac{1}{\cos\left(\frac{\pi}{4}\right)}$$ (using the definition of secant)

$$= \frac{1}{\frac{\sqrt{2}}{2}}$$ (since $\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$)

$$= \frac{2}{\sqrt{2}} = \sqrt{2}$$ (either form of this number is acceptable)

$$\csc\left(\frac{\pi}{4}\right) = \frac{1}{\sin\left(\frac{\pi}{4}\right)}$$ (using the definition of cosecant)

$$= \frac{1}{\frac{\sqrt{2}}{2}}$$ (since $\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$)

$$= \frac{2}{\sqrt{2}} = \sqrt{2}$$ (either form of this number is acceptable)

$$\cot\left(\frac{\pi}{4}\right) = \frac{\cos\left(\frac{\pi}{4}\right)}{\sin\left(\frac{\pi}{4}\right)}$$ (using the definition of cotangent)

$$= \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}$$ (since $\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ and $\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$)

$$= 1$$

EXAMPLE 3: Find $\tan\left(\frac{\pi}{3}\right)$, $\sec\left(\frac{\pi}{3}\right)$, $\csc\left(\frac{\pi}{3}\right)$, and $\cot\left(\frac{\pi}{3}\right)$.

SOLUTION:

$$\tan\left(\frac{\pi}{3}\right) = \frac{\sin\left(\frac{\pi}{3}\right)}{\cos\left(\frac{\pi}{3}\right)}$$ (using the definition of tangent)

$$= \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}$$ (since $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$ and $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$)

$$= \sqrt{3}$$

$$\sec\left(\frac{\pi}{3}\right) = \frac{1}{\cos\left(\frac{\pi}{3}\right)}$$ (using the definition of secant)

$$= \frac{1}{\frac{1}{2}}$$ (since $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$)

$$= 2$$

$$\csc\left(\frac{\pi}{3}\right) = \frac{1}{\sin\left(\frac{\pi}{3}\right)}$$ (using the definition of cosecant)

$$= \frac{1}{\frac{\sqrt{3}}{2}}$$ (since $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$)

$$= \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$ (either form of this number is acceptable)

$$\cot\left(\frac{\pi}{3}\right) = \frac{\cos\left(\frac{\pi}{3}\right)}{\sin\left(\frac{\pi}{3}\right)}$$ (using the definition of cotangent)

$$= \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}$$ (since $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$ and $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$)

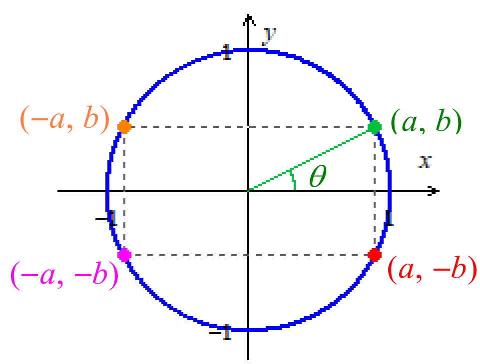
$$= \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$ (either form of this number is acceptable)

Section I: The Trigonometric Functions

Chapter 3, Part 4: Intro to the Trigonometric Functions

Recall that the sine and cosine function represent the coordinates of points in the circumference of a unit circle. In Part 3 of Chapter 3, we found the sine and cosine values for 30°, 45°, and 60° (i.e., for $\frac{\pi}{6}$, $\frac{\pi}{4}$ and $\frac{\pi}{3}$) by finding the coordinates of the points on the circumference of the unit circle specified by these angles. The points we found were all in Quadrant I but, since a circle is symmetric about both the x and y axes, we can reflect these points about the coordinate axes to determine the coordinates of corresponding points in the other quadrants. This means that we can use the sine and cosine values of $\frac{\pi}{6}$, $\frac{\pi}{4}$ and $\frac{\pi}{3}$ to find the sine and cosine values of corresponding angles in the other quadrants.

Because of the symmetry of a circle, we can take a point in Quadrant I and reflect it about the x-axis, the y-axis, and about both axes in order to obtain corresponding points, one in each of the three other quadrants; the absolute value of the coordinates of all four of these points is the same, i.e., they only differ by their signs. In Figure 1, we've plotted the point (a, b) specified by angle θ in Quadrant I along with the corresponding points in the other quadrants.



Notice that if you construct a segment between the origin and each of these four points, then the acute angle between this segment and the x-axis is the same angle, θ; see Figure 2.

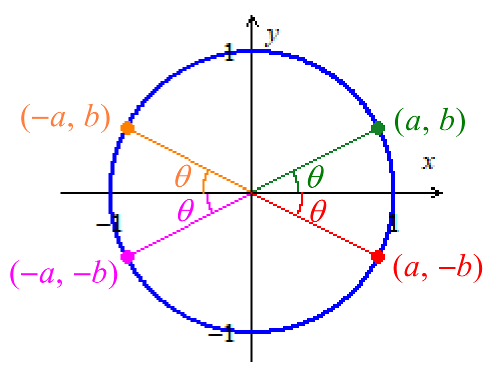


Figure 2

Although all four of the points in Figure 2 are specified by a different angle, all four of the angles share the same *reference angle*, θ.

DEFINITION: The **reference angle** for an angle is the acute (i.e., smaller than 90°) angle between the terminal side of the angle and the x-axis.

EXAMPLE 1: **a.** Find the reference angle for 150°.

b. Find the reference angle for $\frac{5\pi}{4}$.

SOLUTION:

a. The reference angle is for 150° is 30°; see Figure 3.

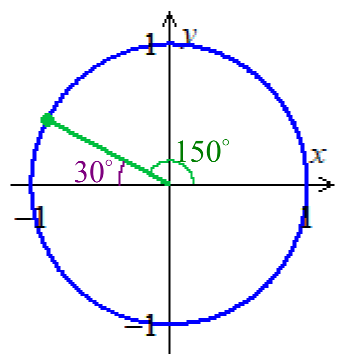


Figure 3

b. The reference angle for $\frac{5\pi}{4}$ is $\frac{\pi}{4}$; see Figure 4.

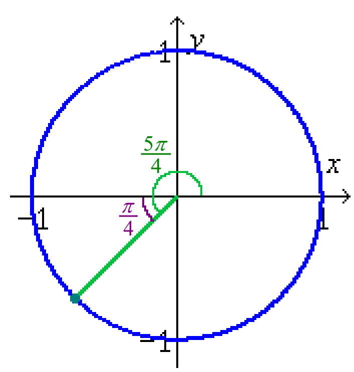
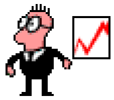


Figure 4

Now let's discuss how we can use reference angles to determine the sine and cosine of any integer multiple of $\frac{\pi}{6}$, $\frac{\pi}{4}$ and $\frac{\pi}{3}$.



CLICK HERE for a video that discusses working with multiples of $\frac{\pi}{6}$, $\frac{\pi}{4}$ and $\frac{\pi}{3}$.

In Part 3 of Chapter 3, we determined the sine and cosine values of $\frac{\pi}{6}$, $\frac{\pi}{4}$ and $\frac{\pi}{3}$ which gave us the exact the coordinates of the points on the unit circle specified by these angles; see Figure 5.

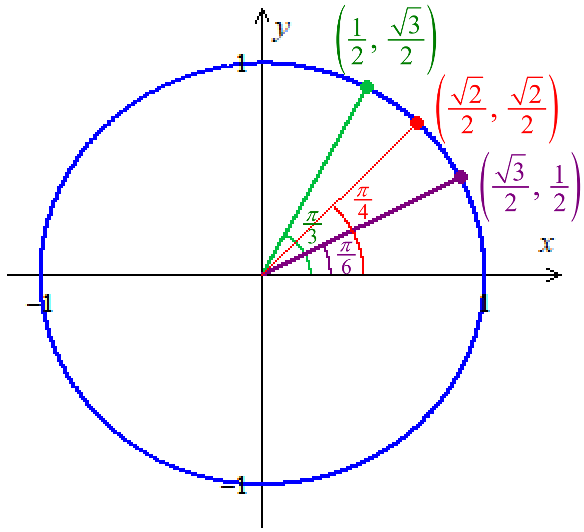


Figure 5

We can use the information in Figure 5 to find the sine and cosine of any angle that has $\frac{\pi}{6}$, $\frac{\pi}{4}$, or $\frac{\pi}{3}$ as its reference angle.

First let's focus on angles with reference angle $\frac{\pi}{4}$.

Notice that both the horizontal and vertical coordinates of the point on the unit circle specified by $\frac{\pi}{4}$ are both $\frac{\sqrt{2}}{2}$. Of course, this means that $\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ and $\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$, so whenever we are working with $\frac{\pi}{4}$, we should remember that we are going to use the number $\frac{\sqrt{2}}{2}$. Let's consider an example:

EXAMPLE 2a: Find $\sin\left(\frac{5\pi}{4}\right)$ and $\cos\left(\frac{5\pi}{4}\right)$.

SOLUTION:

As we observed in Example 1, the reference angle for $\frac{5\pi}{4}$ is $\frac{\pi}{4}$ so we know that the absolute value of $\sin\left(\frac{5\pi}{4}\right)$ will be the same as $\sin\left(\frac{\pi}{4}\right)$ and the absolute value of $\cos\left(\frac{5\pi}{4}\right)$ will be the same as $\cos\left(\frac{\pi}{4}\right)$ but, since $\frac{5\pi}{4}$ is in the third quadrant, both its sine and cosine values will be *negative*. We know that $\frac{\pi}{4}$ has a sine and cosine value of $\frac{\sqrt{2}}{2}$, so we can conclude that

$$\sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2} \quad \text{and} \quad \cos\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}.$$

Figure 6 shows this information communicated as the coordinate of the point specified by $\frac{5\pi}{4}$ on the circumference of a unit circle.

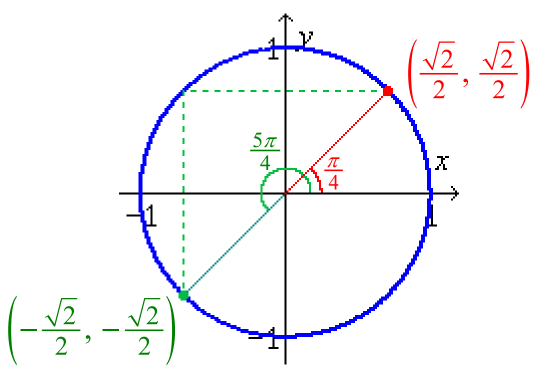


Figure 6

EXAMPLE 2b: Use Example 2a to find $\tan\left(\frac{5\pi}{4}\right)$, $\cot\left(\frac{5\pi}{4}\right)$, $\sec\left(\frac{5\pi}{4}\right)$, $\csc\left(\frac{5\pi}{4}\right)$.

SOLUTION:

$$\tan\left(\frac{5\pi}{4}\right) = \frac{\sin\left(\frac{5\pi}{4}\right)}{\cos\left(\frac{5\pi}{4}\right)} = \frac{-\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}} = 1$$

$$\cot\left(\frac{5\pi}{4}\right) = \frac{1}{\tan\left(\frac{5\pi}{4}\right)} = \frac{1}{1} = 1$$

$$\sec\left(\frac{5\pi}{4}\right) = \frac{1}{\cos\left(\frac{5\pi}{4}\right)} = \frac{1}{-\frac{\sqrt{2}}{2}} = -\frac{2}{\sqrt{2}}$$

$$\csc\left(\frac{5\pi}{4}\right) = \frac{1}{\sin\left(\frac{5\pi}{4}\right)} = \frac{1}{-\frac{\sqrt{2}}{2}} = -\frac{2}{\sqrt{2}}$$

Now let's focus on angles with a reference angle of either $\frac{\pi}{6}$ or $\frac{\pi}{3}$. Recall from Figure 5 that $\frac{\pi}{6}$ specifies the point $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ on the unit circle and that specifies $\frac{\pi}{3}$ the point $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ on the unit circle. Thus, the horizontal and vertical coordinates of the points specified by $\frac{\pi}{6}$ or $\frac{\pi}{3}$ are either $\frac{1}{2}$ or $\frac{\sqrt{3}}{2}$ (these are the *only* options), so whenever we are working with $\frac{\pi}{6}$ or $\frac{\pi}{3}$, we should remember that we are going to use either $\frac{1}{2}$ or $\frac{\sqrt{3}}{2}$. But we need a way to decide which of these two numbers we need to use.

Notice that $\frac{1}{2} < \frac{\sqrt{3}}{2}$ and that $\frac{\pi}{6} < \frac{\pi}{3}$, and observe that when the horizontal coordinate is larger than the vertical coordinate, i.e., if the ordered pair is $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, then the point is close to the x-axis and the angle that specifies the point is a small angle, i.e., $\frac{\pi}{6}$. Similarly, observe that when the horizontal coordinate is smaller than the vertical coordinate, i.e., if the ordered pair is $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, then the point is further above the x-axis and the angle that specifies the point is a large angle, i.e., $\frac{\pi}{3}$. So, when the angle is smaller there hasn't been much rotation so the horizontal coordinate is larger and the vertical coordinate is smaller, but when the angle is larger, there has been substantial rotation so the vertical coordinate is larger and the horizontal coordinate is smaller. (Spend some time with this paragraph until it makes sense.)

Let's use this way of thinking to evaluate a few expressions.

EXAMPLE 3: Find $\sin\left(\frac{\pi}{3}\right)$ and $\cos\left(\frac{\pi}{6}\right)$.

- To find $\sin\left(\frac{\pi}{3}\right)$, first take note that the function is sine, so it's a *vertical* coordinate that we're looking for. Next, consider the angle, $\frac{\pi}{3}$. This is the angle that, along with $\frac{\pi}{6}$, has sine and cosine values of $\frac{1}{2}$ or $\frac{\sqrt{3}}{2}$, so we know that we have to chose one of these for our sine value. Since $\frac{\pi}{3}$ is larger than $\frac{\pi}{6}$, it specifies a point on the unit circle with a larger vertical coordinate so the sine value must be the larger of our two choices so we can conclude that $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$.

- To find $\cos\left(\frac{\pi}{6}\right)$, first take note that the function is cosine, so it's a *horizontal* coordinate that we're looking for. Next, consider the angle, $\frac{\pi}{6}$, and note that, along with $\frac{\pi}{3}$, it has sine and cosine values of $\frac{1}{2}$ or $\frac{\sqrt{3}}{2}$, so we know that we have to chose one of these for our cosine value. Since $\frac{\pi}{6}$ is smaller than $\frac{\pi}{3}$, it specifies a point on the unit circle with a larger horizontal coordinate and smaller vertical coordinate, so the cosine value must be the larger of our two choices so we can conclude that $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$.

Now we can use what we know about $\frac{\pi}{6}$ and $\frac{\pi}{3}$ to find the sine and cosine of angles in other quadrants that have $\frac{\pi}{6}$ or $\frac{\pi}{3}$ as their reference angle.

EXAMPLE 4: Find $\sin\left(\frac{5\pi}{3}\right)$.

SOLUTION:

To find $\sin\left(\frac{5\pi}{3}\right)$, first take note that the function is sine, so it's a *vertical* coordinate that we're looking for. Next, consider the angle, $\frac{5\pi}{3}$; it's in Quadrant IV and vertical coordinates are negative in Quadrant IV, so we know that our sine value is negative. Since $\frac{5\pi}{3} = 2\pi - \frac{\pi}{3}$, we see that the reference angle for $\frac{5\pi}{3}$ is $\frac{\pi}{3}$, so the absolute value of our sine value must be either $\frac{1}{2}$ or $\frac{\sqrt{3}}{2}$. (In the discussion above, we noticed that these are our only choices when we're working with $\frac{\pi}{3}$.) Since $\frac{\pi}{3}$ is larger than $\frac{\pi}{6}$ we know that $\frac{\pi}{3}$ specifies a point on the unit circle with a larger vertical coordinate, so we know that we'll need to use the larger of $\frac{1}{2}$ and $\frac{\sqrt{3}}{2}$ for our sine value, so we can conclude that $\sin\left(\frac{5\pi}{3}\right) = -\frac{\sqrt{3}}{2}$; see Figure 7.

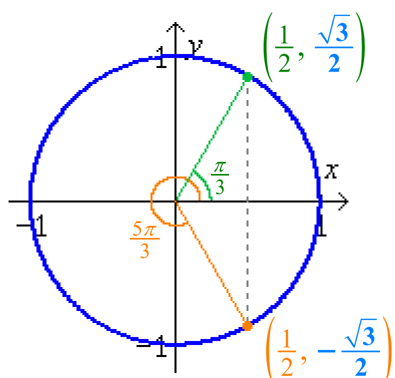


Figure 7

EXAMPLE 5: Find $\cos\left(\frac{5\pi}{6}\right)$.

SOLUTION:

To find $\cos\left(\frac{5\pi}{6}\right)$, first take note that the function is cosine, so it's a *horizontal* coordinate that we're looking for. Next, consider the angle, $\frac{5\pi}{6}$; it's in Quadrant II and horizontal coordinates are negative in Quadrant II, so we know that our cosine value is negative. Since $\frac{5\pi}{6} = \pi - \frac{\pi}{6}$, we see that the reference angle for $\frac{5\pi}{6}$ is $\frac{\pi}{6}$, so the absolute value of our cosine value must be either $\frac{1}{2}$ or $\frac{\sqrt{3}}{2}$. Since $\frac{\pi}{6}$ is smaller than $\frac{\pi}{3}$, we know that $\frac{\pi}{6}$ specifies a point on the unit circle with a larger horizontal coordinate, so we know that we'll need to use the larger of $\frac{1}{2}$ and $\frac{\sqrt{3}}{2}$ for our cosine value, so we can conclude that $\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$; see Figure 8.

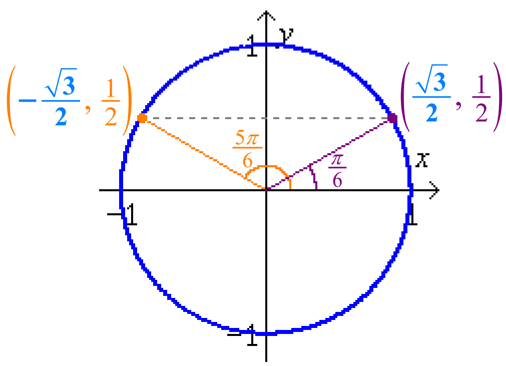


Figure 8

EXAMPLE 6: Find $\cos\left(\frac{4\pi}{3}\right)$.

SOLUTION:

To find $\cos\left(\frac{4\pi}{3}\right)$, first take note that the function is cosine, so it's a *horizontal* coordinate that we're looking for. Next, consider the angle, $\frac{4\pi}{3}$; it's in Quadrant III and horizontal coordinates are negative in Quadrant III, so we know that our cosine value is negative. Since $\frac{4\pi}{3} = \pi + \frac{\pi}{3}$, we see that the reference angle for $\frac{4\pi}{3}$ is $\frac{\pi}{3}$, so the absolute value of our cosine value must be either $\frac{1}{2}$ or $\frac{\sqrt{3}}{2}$. Since $\frac{\pi}{3}$ is larger than $\frac{\pi}{6}$, we know that $\frac{\pi}{3}$

specifies a point on the unit circle with a smaller horizontal coordinate, so we know that we'll need to use the smaller of $\frac{1}{2}$ and $\frac{\sqrt{3}}{2}$ for our cosine value, so we can conclude that $\cos\left(\frac{4\pi}{3}\right) = -\frac{1}{2}$; see Figure 9.

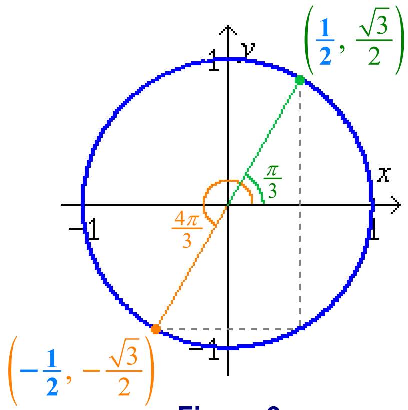


Figure 9

EXAMPLE 7a: Find $\cos(150^\circ)$ and $\sin(150^\circ)$.

SOLUTION:

As shown in Figure 10, the reference angle is for 150° is 30° so the sine and cosine values for 150° are the same as the sine and cosine values of 30° except, since 150° is in Quadrant II, the cosine value is negative. Thus,

$$\cos(150^\circ) = -\cos(30^\circ) = -\frac{\sqrt{3}}{2}$$

and

$$\sin(150^\circ) = \sin(30^\circ) = \frac{1}{2}$$

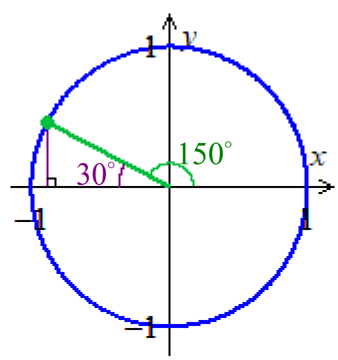


Figure 10

EXAMPLE 7b: Use Example 7a to find $\tan(150^\circ)$, $\cot(150^\circ)$, $\sec(150^\circ)$, $\csc(150^\circ)$.

SOLUTION:

$$\tan(150^\circ) = \frac{\sin(150^\circ)}{\cos(150^\circ)} = \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}}$$

$$\cot(150^\circ) = \frac{1}{\tan(150^\circ)} = \frac{1}{-\frac{1}{\sqrt{3}}} = -\sqrt{3}$$

$$\sec(150^\circ) = \frac{1}{\cos(150^\circ)} = \frac{1}{-\frac{\sqrt{3}}{2}} = -\frac{2}{\sqrt{3}}$$

$$\csc(150^\circ) = \frac{1}{\sin(150^\circ)} = \frac{1}{\frac{1}{2}} = 2$$

EXAMPLE 8a: Find $\cos(405^\circ)$ and $\sin(405^\circ)$.

SOLUTION:

Since $405^\circ = 360^\circ + 45^\circ$, 405° is co-terminal with 45°; see Figure 11.

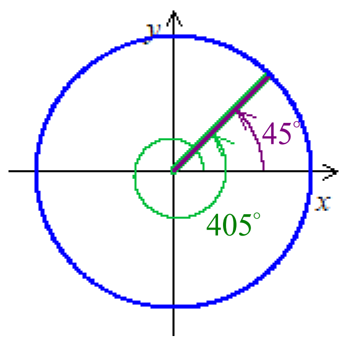


Figure 11

Therefore, the reference angle for 405° is 45°, and the sine and cosine values for 405° are the same as the sine and cosine of 45°. Thus,

$$\cos(405^\circ) = \cos(45^\circ) = \frac{\sqrt{2}}{2}$$

and

$$\sin(405^\circ) = \sin(45^\circ) = \frac{\sqrt{2}}{2}$$

EXAMPLE 8b: Use Example 8a to find $\tan(405^\circ)$, $\cot(405^\circ)$, $\sec(405^\circ)$, $\csc(405^\circ)$.

SOLUTION:

$$\tan(405^\circ) = \frac{\sin(405^\circ)}{\cos(405^\circ)} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$$

$$\cot(405^\circ) = \frac{1}{\tan(405^\circ)} = \frac{1}{1} = 1$$

$$\sec(405^\circ) = \frac{1}{\cos(405^\circ)} = \frac{1}{\frac{\sqrt{2}}{2}} = \frac{2}{\sqrt{2}}$$

$$\csc(405^\circ) = \frac{1}{\sin(405^\circ)} = \frac{1}{\frac{\sqrt{2}}{2}} = \frac{2}{\sqrt{2}}$$

EXAMPLE 9: A circle with a radius of 6 units is given in Figure 12. The point Q is specified by the angle $\frac{2\pi}{3}$. Use the sine and cosine function to find the exact coordinates of point Q.

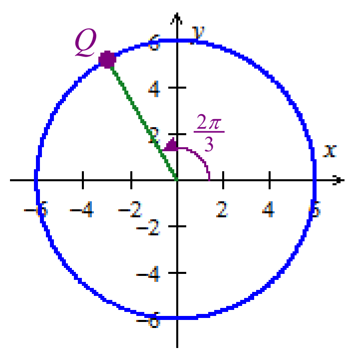


Figure 12

SOLUTION:

The point Q is specified by $\frac{2\pi}{3}$ on the circumference of a circle of radius 6 units. Thus,

$$\begin{aligned} Q &= \left(6\cos\left(\tfrac{2\pi}{3}\right),\ 6\sin\left(\tfrac{2\pi}{3}\right)\right) \\ &= \left(6\cdot\left(-\tfrac{1}{2}\right),\ 6\cdot\left(\tfrac{\sqrt{3}}{2}\right)\right) \\ &= \left(-3,\ 3\sqrt{3}\right) \end{aligned}$$

Section I: The Trigonometric Functions

Chapter 4: Graphing Sinusoidal Functions

DEFINITION: A **sinusoidal function** is function of the form

$$y = A\sin\big(\omega(t-h)\big) + k \quad \text{or} \quad y = A\cos\big(\omega(t-h)\big) + k,$$

where $A, \omega, h, k \in \mathbb{R}$.

Based what we know about graph transformations (which are studied in the previous course), we should recognize that a sinusoidal function is a transformation of $y = \sin(t)$ or $y = \cos(t)$. Consequently, sinusoidal functions are waves with the same curvy shape as the graphs of sine and cosine but with different periods, midlines, and/or amplitudes.

Below is a summary of what we studied about graph transformations in the previous course. We'll use this information in order to graph sinusoidal functions.

SUMMARY OF GRAPH TRANSFORMATIONS

Suppose that f and g are functions such that $g(t) = A\cdot f\big(\omega(t-h)\big) + k$ and $A, \omega, h, k \in \mathbb{R}$. In order to transform the graph of the function f into the graph of g…

1st: horizontally stretch/compress the graph of f by a factor of $\frac{1}{|\omega|}$ and, if $\omega < 0$, reflect it about the y-axis.

2nd: shift the graph horizontally h units (shift right if h is positive and left if h is negative).

3rd: vertically stretch/compress the graph by a factor of $|A|$ and, if $A < 0$, reflect it about the t-axis.

4th: shift the graph vertically k units (shift up if k is positive and down if k is negative).

(The *order* in which these transformations are performed *matters*.)

Examples 1 – 4 (below) will provide a review of the graph transformations as well as an investigation of the effect of the constants A, ω, h, and k on the period, midline, amplitude, and horizontal shift of a sinusoidal function. You may want to follow along by graphing the functions on your graphing calculator. Don't forget to change the **mode** of the calculator to the **radian** setting under the heading **angle**.

EXAMPLE 1: Describe how we can transform the graph of $f(t) = \sin(t)$ into the graph of $g(t) = 2\sin(t) - 3$. State the period, midline, and amplitude of $y = g(t)$.

SOLUTION:

Our goal is to use Examples 1 – 4 to determine how the constants A, ω, h, and k affect the period, midline, amplitude, and horizontal shift of a sinusoidal function so let's start by observing what the values of A, ω, h, and k are in $g(t) = 2\sin(t) - 3$. It should be clear that function g is a sinusoidal function of the form $y = A\sin\big(\omega(t-h)\big) + k$ where $A = 2$, $\omega = 1$, $h = 0$, and $k = -3$.

After inspecting the rules for the functions f and g, we should notice that we could construct the function $g(t) = 2\sin(t) - 3$ by multiplying the outputs of the function $f(t) = \sin(t)$ by 2 and then subtracting 3 from the result. We can express this algebraically with the equation below:

$$g(t) = 2f(t) - 3$$

Based on what we know about graph transformations, we can conclude that we can obtain graph of g by starting with the graph of f and first stretching it vertically by a factor of 2 and then shifting it down 3 units. Since $f(t) = \sin(t)$ has amplitude 1 unit, if we stretch it vertically by a factor of 2 then we'll double the amplitude, so we should expect that the amplitude of g to be 2 units. Also, since $f(t) = \sin(t)$ has midline $y = 0$, when we shift it down 3 units to draw the graph of g, the resulting midline will be $y = -3$. (Note that since graphing g required no horizontal transformations of $f(t) = \sin(t)$, the graph of g must have the same period as the graph of $f(t) = \sin(t)$: 2π units.) Let's summarize what we've learned about $g(t) = 2\sin(t) - 3$:

period: 2π units

midline: $y = -3$

amplitude: 2 units

horizontal shift: 0 units

The graphs of $f(t) = \sin(t)$ and $g(t) = 2\sin(t) - 3$ are given in Figure 1 below.

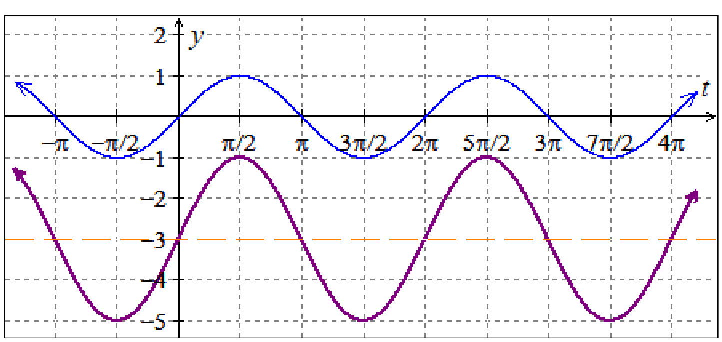


Figure 1: The graphs of $f(t) = \sin(t)$ and $g(t) = 2\sin(t) - 3$.

EXAMPLE 2: Describe how we can transform the graph of $f(t) = \sin(t)$ into the graph of $n(t) = \sin\left(t + \frac{\pi}{4}\right)$; state the period, midline, and amplitude of $y = n(t)$.

SOLUTION:

Notice that the function n is a sinusoidal function of the form $y = A\sin\left(\omega\left(t - h\right)\right) + k$ where $A = 1$, $\omega = 1$, $h = -\frac{\pi}{4}$, and $k = 0$.

After inspecting the rules for the functions f and n, it should be clear that we can write n in terms of f as follows: $n(t) = f\left(t + \frac{\pi}{4}\right)$. Based on what we know about graph transformations, we can conclude that we can obtain graph of n by starting with the graph of f and shifting it left $\frac{\pi}{4}$ units. Since a horizontal shift won't affect the period, midline, or amplitude, we should expect that the period, midline, and amplitude of $n(t) = \sin\left(t + \frac{\pi}{4}\right)$ are the same as $f(t) = \sin(t)$:

period: 2π units

midline: $y = 0$

amplitude: 1 unit

horizontal shift: $-\frac{\pi}{4}$ units

The graphs of $f(t) = \sin(t)$ and $n(t) = \sin\left(t + \frac{\pi}{4}\right)$ are given in Figure 2.

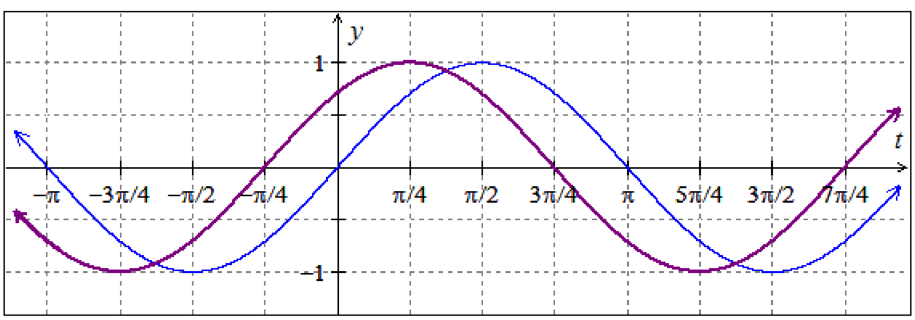


Figure 2: The graphs of $f(t) = \sin(t)$ (blue) and $n(t) = \sin\left(t + \frac{\pi}{4}\right)$ (purple).

EXAMPLE 3: Describe how we can transform the graph of $p(t) = \cos(t)$ into the graph of $q(t) = -\cos(2t)$ and find the period, midline, and amplitude of $y = q(t)$.

SOLUTION:

Notice that the function q is a sinusoidal function of the form $y = A\cos\big(\omega\big(t - h\big)\big) + k$ where $A = -1$, $\omega = 2$, $h = 0$, and $k = 0$.

After inspecting the rules for the functions p and q, it should be clear that we can write q in terms of p as follows: $q(t) = -p(2t)$. Based on what we know about graph transformations, we can conclude that we can obtain graph of q by starting with the graph of p and first stretching it horizontally by a factor of $\frac{1}{2}$ (i.e., compressing the graph by a factor of 2) and then reflecting it about the t-axis. Since $p(t) = \cos(t)$ has period 2π units, if we compress the graph by a factor of 2 then the period will be shrunk to π units. Since we aren't stretching the graph of p vertically, we should expect that the amplitude of q is the same as the amplitude of p: 1 unit. Also, since we aren't shifting the graph of p vertically, we should expect that the midline of q is the same as the midline of p: $y = 0$. Let's summarize what we've learned about $q(t) = -\cos(2t)$:

period: π units

midline: $y = 0$

amplitude: 1 unit

horizontal shift: 0 units

The graphs of $p(t) = \cos(t)$ and $q(t) = -\cos(2t)$ are given in Figure 3.

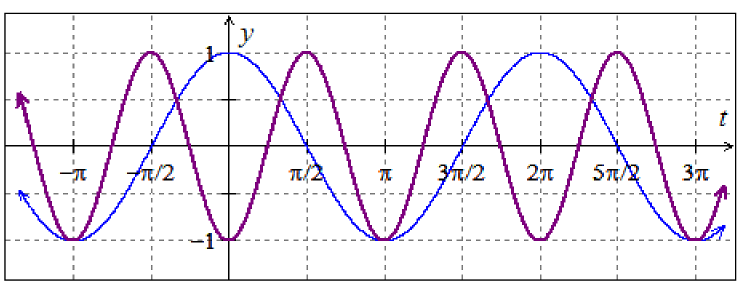


Figure 3: The graphs of $p(t) = \cos(t)$ (blue) and $q(t) = -\cos(2t)$ (purple).

Notice that the graph of $q(t) = -\cos(2t)$ completes two periods in the interval $[0, 2\pi]$. In general, the number ω in a sinusoidal function of the form $y = A\sin\big(\omega\big(t - h\big)\big) + k$ or $y = A\cos\big(\omega\big(t - h\big)\big) + k$ represents the number of periods (or "cycles") that the function completes on an interval of length 2π. This number ω is called the **angular frequency** of a sinusoidal function.

When we use sinusoidal functions to represent real-life situations, we often take the input variable to be a unit of *time*. Suppose that in the function $q(t) = -\cos(2t)$, t represents seconds. Since the input of the cosine function *must* be radians, the units of $\omega = 2$ must be "radians per second". This way,

$$2\,\frac{\text{radians}}{\cancel{\text{second}}}\cdot t\ \cancel{\text{seconds}} = 2t\ \text{radians},$$

which has the appropriate units for the input of the cosine function. So if t represents seconds, the **angular frequency** of $q(t) = -\cos(2t)$ is "2 radians per second".

Another way to obtain the unit of the angular frequency is to use what we noticed above: the number 2 in $q(t) = -\cos(2t)$ represents the number of cycles that the function completes on an interval of length 2π. Since a cycle is equivalent to a complete rotation around a circle, or 2π radians, two cycles is equivalent to 4π radians. If the input variable t represents seconds, then the angular frequency is

$$\frac{4\pi\ \text{radians}}{2\pi\ \text{seconds}} = 2\ \text{rad/sec.}$$

EXAMPLE 4: Describe how we can transform the graph of $p(t) = \cos(t)$ into the graph $m(t) = 3\cos\left(\frac{1}{2}\left(t - \frac{\pi}{3}\right)\right) + 5$. State the period, midline, and amplitude of $y = m(t)$.

SOLUTION:

Notice that the function w is a sinusoidal function of the form $y = A\cos\left(\omega\left(t - h\right)\right) + k$ where $A = 3$, $\omega = \frac{1}{2}$, $h = \frac{\pi}{3}$, and $k = 5$. After inspecting the rules for the functions p and m, it should be clear that we can write m in terms of p as follows: $m(t) = 3p\left(\frac{1}{2}\left(t - \frac{\pi}{3}\right)\right) + 5$. Based on what we know about graph transformations, we can conclude that we can obtain graph of m by starting with the graph of p and first stretching it horizontally by a factor of 2, then shifting it right $\frac{\pi}{3}$ units, then stretching it vertically by a factor 3, and finally shifting it up 5 units. Since $p(t) = \cos(t)$ has period 2π units, if we stretch the graph by a factor of 2 then the period will be stretched to 4π units. Similarly, if we stretch the graph of $p(t) = \cos(t)$ vertically by a factor of 3 then we'll triple the amplitude, so we should expect the amplitude of m to be 3 units. Also, since $p(t) = \cos(t)$ has midline $y = 0$, when we shift it up 5 units to draw the graph of m, the resulting midline will be $y = 5$. Since we are shifting the graph right $\frac{\pi}{3}$ units, the horizontal shift is $\frac{\pi}{3}$ units. Let's summarize what we've learned about $y = m(t)$:

period: 4π units

midline: $y = 5$

amplitude: 3 units

horizontal shift: $\frac{\pi}{3}$ units

The graphs of $p(t) = \cos(t)$ and $m(t) = 3\cos\left(\frac{1}{2}\left(t - \frac{\pi}{3}\right)\right) + 5$ are given in Figure 4.

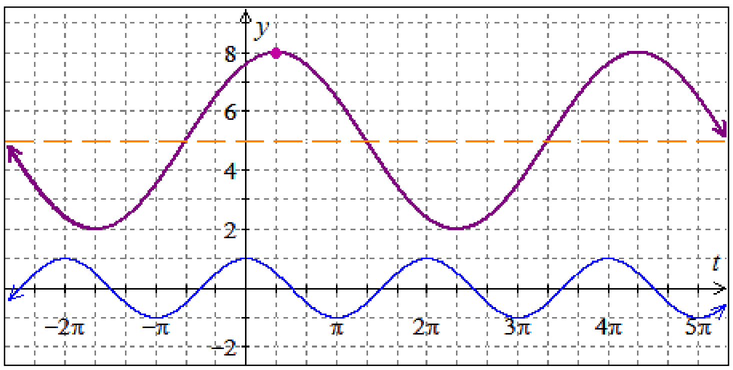


Figure 4: The graphs of $p(t) = \cos(t)$ (blue) and $m(t) = 3\cos\left(\frac{1}{2}\left(t - \frac{\pi}{3}\right)\right) + 5$ (purple).

Notice that the graph of $m(t) = 3\cos\left(\frac{1}{2}\left(t - \frac{\pi}{3}\right)\right) + 5$ completes one-half of a period (or "cycle") in the interval $[0, 2\pi]$. If we let the input variable, t, represent seconds, then $m(t) = 3\cos\left(\frac{1}{2}\left(t - \frac{\pi}{3}\right)\right) + 5$ completes one-half of a cycle every 2π seconds. Since one-half of a cycle is equivalent to half of a rotation around a circle, or π radians, then the **angular frequency** of the function m is

$$\frac{\pi \text{ radians}}{2\pi \text{ seconds}} = \frac{1}{2} \text{ rad/sec.}$$

Based on what we learned in the examples above, we can summarize the affect of the constants A, ω, h, and k on the period, midline, amplitude, and horizontal shift of functions of the form $y = A\sin\left(\omega\left(t-h\right)\right)+k$ and $y = A\cos\left(\omega\left(t-h\right)\right)+k$.

SUMMARY: Graphs of Sinusoidal Functions

The graphs of the sinusoidal functions

$$y = A\sin\left(\omega\left(t-h\right)\right)+k \quad \text{and} \quad y = A\cos\left(\omega\left(t-h\right)\right)+k$$

(where $A, \omega, h, k \in \mathbb{R}$) have the following properties:

period: $\frac{2\pi}{|\omega|}$ units

midline: $y = k$

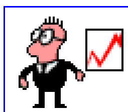
amplitude: $|A|$ units

horizontal shift: h units

angular frequency: ω radians per unit of t

EXAMPLE 5: Sketch a graph of $f(t) = 2\sin\left(\pi t - \frac{\pi}{4}\right) - 3$.

SOLUTION:



CLICK HERE to see a video of this example.

In order to use what we've just studied about functions of the form $y = A\sin\left(\omega\left(t-h\right)\right)+k$, we need to write the given function in this form, i.e., we need to factor π (which is playing the role of "ω") out of the input expression "$\pi t - \frac{\pi}{4}$":

$$\begin{aligned} f(t) &= 2\sin\left(\pi t - \tfrac{\pi}{4}\right) - 3 \\ &= 2\sin\left(\pi\left(t - \tfrac{1}{4}\right)\right) - 3 \end{aligned}$$

It should be clear that $f(t) = 2\sin\left(\pi\left(t - \frac{1}{4}\right)\right) - 3$ is a sinusoidal function of the form $y = A\sin(\omega(t-h)) + k$ where $A = 2$, $\omega = \pi$, $h = \frac{1}{4}$, and $k = -3$. Using what we found above, we can find the period, midline, amplitude, and horizontal shift of $y = f(t)$:

period: $\frac{2\pi}{|\omega|} = \frac{2\pi}{|\pi|} = 2$ units

midline: $y = -3$

amplitude: $|2| = 2$ units

horizontal shift: $\frac{1}{4}$ of a unit

We can use this information to sketch a graph of $f(t) = 2\sin\left(\pi\left(t - \frac{1}{4}\right)\right) - 3$; see Fig. 5. (Note that the horizontal shift tells us where to "start" our usual sine wave.)

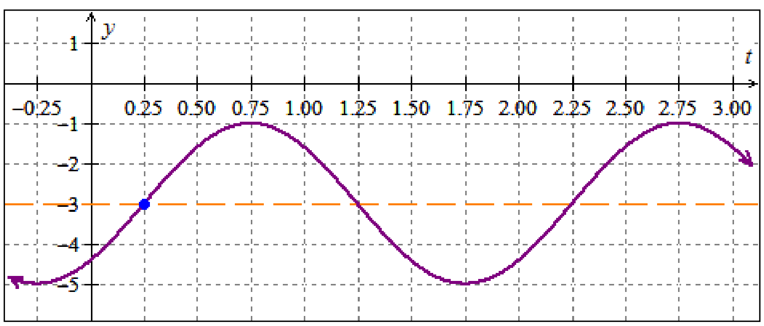


Figure 5: The graph of $f(t) = 2\sin\left(\pi\left(t - \frac{1}{4}\right)\right) - 3$. The blue point represents where we "start" our sine wave since the horizontal shift is $\frac{1}{4}$ of a unit.

Note that, according to what we discussed in Examples 3 and 4, if we let t represent seconds then we could state that the **angular frequency** of $f(t) = 2\sin\left(\pi\left(t - \frac{1}{4}\right)\right) - 3$ is π radians per second. Since π radians represents one-half of a rotation around a circle, the angular frequency "π radians per second t," is equivalent to one-half of a cycle per second. Notice that our graph in Figure 5 shows a function that completes one-half of a period in one unit of t!

EXAMPLE 6: Find two different algebraic rules (one involving sine and one involving cosine) for the function $y = g(t)$ graphed in Figure 6.

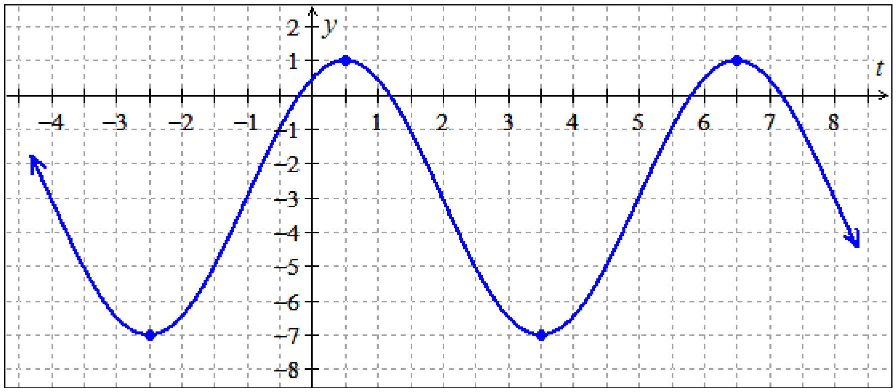
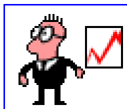


Figure 6: The graph of $y = g(t)$.

SOLUTION:



CLICK HERE to see a video of this example.

First let's write a rule involving sine, so our rule will have the form $g(t) = A\sin\big(\omega(t-h)\big) + k$ and we need to determine the values of A, ω, h, and k.

- The midline is the line midway between the function's maximum and minimum output values. The function's maximum output value is 1 and its minimum output value is -7. Since -3 is the average of these values, the midline is $y = -3$ so $k = -3$.
- The amplitude is the distance between the function's maximum output value, 1, and its midline $y = -3$, which is 4 units. Therefore, $|A| = 4$.
- The function completes one period between $t = -1$ and $t = 5$. Thus, the period of the function is $5 - (-1) = 6$. To find ω we need to solve $6 = 2\pi \cdot \frac{1}{\omega}$:

$$6 = 2\pi \cdot \frac{1}{\omega}$$

$$\Rightarrow \quad \omega = 2\pi \cdot \frac{1}{6}$$

$$\Rightarrow \quad \omega = \frac{\pi}{3}$$

- We know that, near y-axis, the graph of $y = \sin(t)$ is increasing and passes through its midline: since we want to use sine as our 'root' function, we need to look for a spot in the graph of $y = g(t)$ where it shows this behavior. One such spot is at $t = -1$ so we can view the graph of $y = g(t)$ as a sine wave shifted left 1 unit and use $h = -1$.

Therefore, an algebraic rule for g is $g(t) = 4\sin\left(\frac{\pi}{3}\left(t - (-1)\right)\right) - 3$, which we can simplify as $g(t) = 4\sin\left(\frac{\pi}{3}\left(t + 1\right)\right) - 3$. (Note that $g(t) = 4\sin\left(\frac{\pi}{3}\left(t - 5\right)\right) - 3$ is another possibility.)

Now we'll write a rule involving cosine, so our rule will have the form $g(t) = A\sin\left(\omega\left(t - h\right)\right) + k$. Since the amplitude, period, and midline aren't dependent on whether we use sine or cosine in our algebraic rule, we can use the same values for A, ω, and k that we used above. So we only need to determine an appropriate horizontal shift, h, that works for cosine. We know that the graph of $y = \cos(t)$ reaches its maximum value at the y-axis: since we want to use cosine as our 'root' function, we need to look for a spot in the graph of $y = g(t)$ where it reaches its maximum. One such spot is at $t = \frac{1}{2}$ so we can view the graph of $y = g(t)$ as a cosine wave shifted right $\frac{1}{2}$ of a unit and use $h = \frac{1}{2}$. Therefore, an algebraic rule g is $g(t) = 4\cos\left(\frac{\pi}{3}\left(t - \frac{1}{2}\right)\right) - 3$. (Note that $g(t) = 4\cos\left(\frac{\pi}{3}\left(t - 6.5\right)\right) - 3$ is another possibility.)

Section I: The Trigonometric Functions

Chapter 5: Graphs of the Other Trigonometric Functions

Recall from Part 2 of Chapter 3 the following definitions of the "other trigonometric functions":

DEFINITIONS: The **tangent function**, denoted $\tan(\theta)$, is defined by $\tan(\theta)=\dfrac{\sin(\theta)}{\cos(\theta)}$.

The **cotangent function**, denoted $\cot(\theta)$, is defined by $\cot(\theta)=\dfrac{1}{\tan(\theta)}$.

Consequently, $\cot(\theta)=\dfrac{\cos(\theta)}{\sin(\theta)}$.

The **secant function**, denoted $\sec(\theta)$, is defined by $\sec(\theta)=\dfrac{1}{\cos(\theta)}$.

The **cosecant function**, denoted $\csc(\theta)$, is defined by $\csc(\theta)=\dfrac{1}{\sin(\theta)}$.

In this chapter, we'll graph all four of these functions. [In §5.5 of our textbook, transformations of these functions are studied but you are only responsible for understanding the non-transformed versions of these functions.]

EXAMPLE 1: Graph the *tangent* function, $y=\tan(t)$.

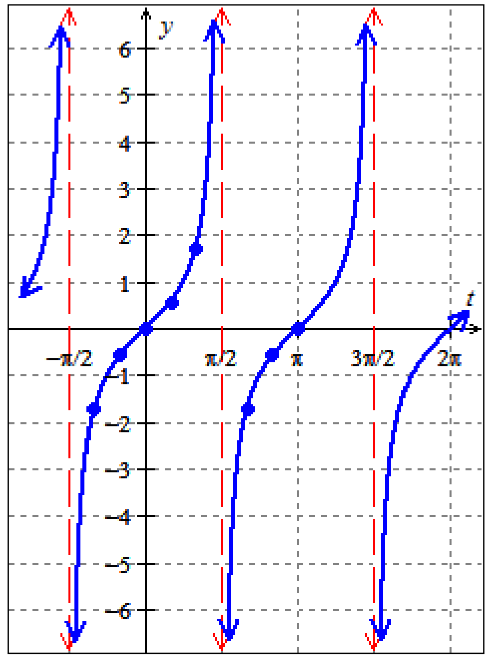
SOLUTION:

We can use the fact that

$$\tan(t)=\frac{\sin(t)}{\cos(t)}$$

to calculate values of tangent, and we can use these values to form ordered pairs that we can plot on a coordinate plane. See the table and graph in Figure 1 on the next page:

t	$\tan(t)$	$(t, \tan(t))$
$-\frac{\pi}{2}$	unde-fined	no point
$-\frac{\pi}{3}$	$-\sqrt{3}$	$\left(-\frac{\pi}{3}, -\sqrt{3}\right)$
$-\frac{\pi}{6}$	$-\frac{1}{\sqrt{3}}$	$\left(-\frac{\pi}{6}, -\frac{1}{\sqrt{3}}\right)$
0	0	$(0, 0)$
$\frac{\pi}{6}$	$\frac{1}{\sqrt{3}}$	$\left(\frac{\pi}{6}, \frac{1}{\sqrt{3}}\right)$
$\frac{\pi}{3}$	$\sqrt{3}$	$\left(\frac{\pi}{3}, \sqrt{3}\right)$
$\frac{\pi}{2}$	unde-fined	no point
$\frac{2\pi}{3}$	$-\sqrt{3}$	$\left(\frac{2\pi}{3}, -\sqrt{3}\right)$
$\frac{5\pi}{6}$	$-\frac{1}{\sqrt{3}}$	$\left(\frac{5\pi}{6}, -\frac{1}{\sqrt{3}}\right)$
π	0	$(\pi, 0)$



Graph of $y = \tan(t)$.

Figure 1

Note that the period of the tangent function is π units, unlike the sine and cosine functions whose periods are both 2π units.

Also note that $\tan(t)$ is undefined for $t = -\frac{\pi}{2}$, $t = \frac{\pi}{2}$, $t = \frac{3\pi}{2}$, etc., since $\cos(t)$ equals 0 at these values and division by 0 is undefined. As you can see in the graph, $y = \tan(t)$ has a *vertical asymptote* at $t = -\frac{\pi}{2}$, $t = \frac{\pi}{2}$, $t = \frac{3\pi}{2}$, etc., i.e., $y = \tan(t)$ has a vertical asymptote at each t-value that makes $\cos(t) = 0$. The fact that there are vertical asymptotes tells us that there are no maximum or minimum outputs for the tangent function. Since the definitions of *midline* and *amplitude* involve maximum and minimum outputs, the tangent function has no midline or amplitude. Still, the t-axis (i.e., the line $y = 0$) behaves like a midline for $y = \tan(t)$.

EXAMPLE 2: Graph the *secant* function, $y = \sec(t)$.

SOLUTION:

We can use the fact that

$$\sec(t) = \frac{1}{\cos(t)}$$

to help us graph $y = \sec(t)$. In Figure 2, we've graphed $y = \sec(t)$ on a coordinate plane that also shows the graph of $y = \cos(t)$. Notice that $y = \sec(t)$ has vertical asymptotes at all t-values for which $\cos(t) = 0$.

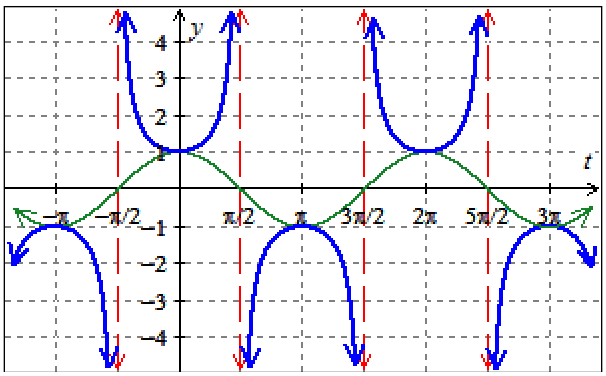


Figure 2: The graph of $y = \sec(t)$ (in blue) and $y = \cos(t)$ (in green)

EXAMPLE 3: Graph the *cosecant* function, $y = \csc(t)$.

SOLUTION:

We can use the fact that

$$\csc(t) = \frac{1}{\sin(t)}$$

to help us graph $y = \csc(t)$. In Figure 3, we've graphed $y = \csc(t)$ on a coordinate plane that also shows the graph of $y = \sin(t)$. Notice that $y = \csc(t)$ has vertical asymptotes at all t-values for which $\sin(t) = 0$.

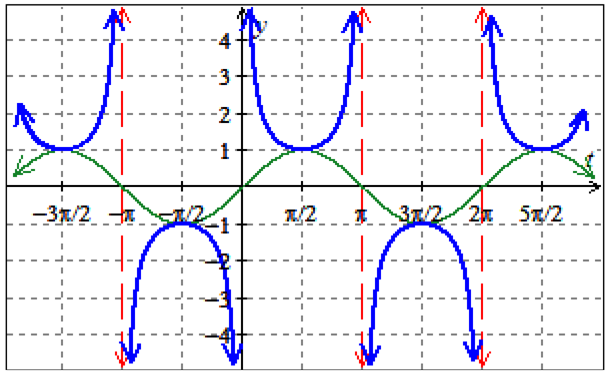


Figure 3: The graph of $y = \csc(t)$ (in blue) and $y = \sin(t)$ (in green)

EXAMPLE 4: Graph the *cotangent* function, $y = \cot(t)$.

SOLUTION:

We can use the fact that

$$\cot(t) = \frac{1}{\tan(t)}$$

to help us graph $y = \cot(t)$. In Figure 4, we've graphed $y = \cot(t)$ on a coordinate plane; notice that $y = \cot(t)$ has vertical asymptotes at all t-values for which $\tan(t) = 0$.

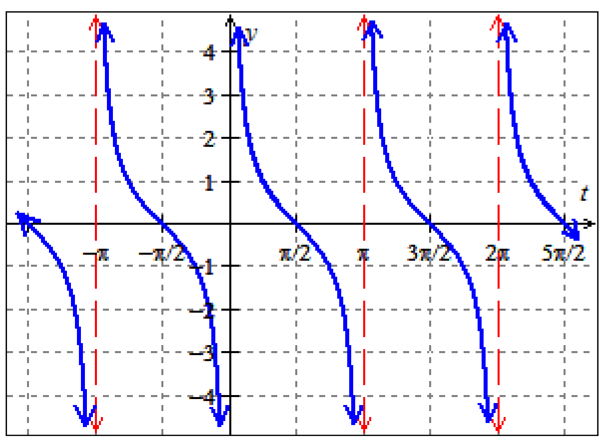


Figure 4: The graph of $y = \cot(t)$.

Section I: The Trigonometric Functions

Chapter 6: Inverse Trig Functions

As we studied in MTH 111, the inverse of a function reverses the roles of the inputs and the outputs. (For more information on inverse functions, check out these MTH 111 lecture notes.) For example, if f and f^{-1} are inverses of one another and if $f(a)=b$, then $f^{-1}(b)=a$. Inverse functions are extremely valuable since they "undo" one another and allow us to solve equations. For example, we can solve the equation $x^3=10$ by using the inverse of the cubing function, namely the cube-root function, to "undo" the cubing involved in the equation:

$$\begin{aligned} x^3 &= 10 \\ \Rightarrow \sqrt[3]{x^3} &= \sqrt[3]{10} \\ \Rightarrow \quad x &= \sqrt[3]{10} \end{aligned}$$

As we studied in MTH 111, the cubing function has an inverse function because each output value corresponds to exactly one input value (e.g., the only number whose cube is 8 is 2). This means that the cubing function is **one-to-one**, and it's only one-to-one functions whose inverses are also functions.

Unfortunately, the trig functions aren't one-to-one so, in their natural form, they don't have inverse functions. For example, consider the output $\frac{1}{2}$ for the sine function: this output corresponds to the inputs $\frac{\pi}{6}$, $\frac{5\pi}{6}$, $\frac{13\pi}{6}$, $\frac{17\pi}{6}$, etc.; see Figure 1.

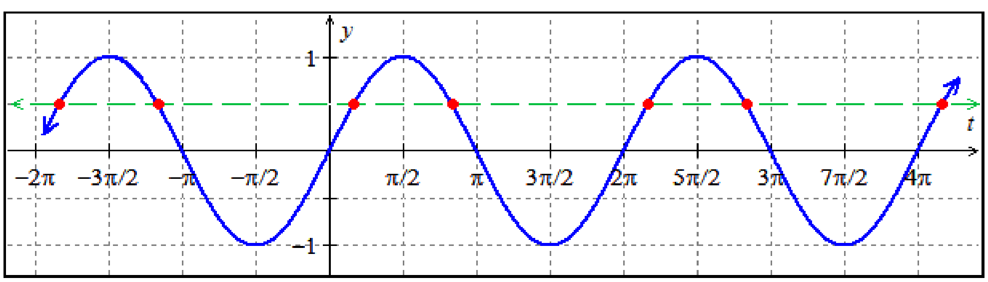


Figure 1: The graph of $y=\sin(t)$ with red dots on the line $y=\frac{1}{2}$ that represent some of the locations where the sine function reaches the output $\frac{1}{2}$. In order to be a one-to-one function that has an inverse, the graph can only reach each output value *once*.

Since inverse functions can be so valuable, we *really* want inverse trig functions, so we need to **restrict the domains** of the functions to intervals on which they are one-to-one, and then we can construct inverse functions. Let's start by constructing the inverse of the sine function.

In order to construct the inverse of the sine function, we need to restrict the domain to an interval on which the function is one-to-one, and we need to choose an interval of the domain that utilizes the entire range of the sine function, $[-1, 1]$. Following tradition, we well choose the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. In Figure 2, this interval of the sine function is highlighted; notice that this on this interval, the function is one-to-one and has the same range as the sine function.

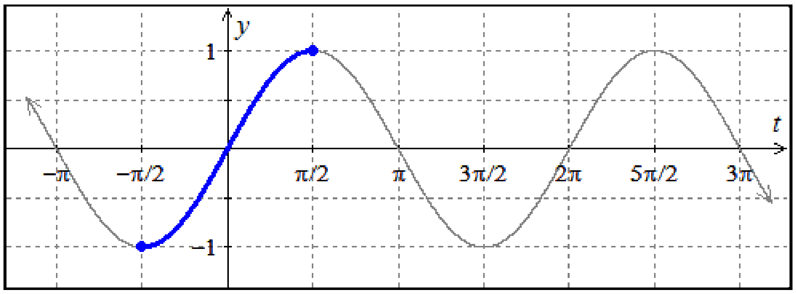


Figure 2: The interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ of the graph of $y = \sin(t)$; on this interval, the sine function is one-to-one and has the same range as the entire sine function.

Recall that when we construct the inverse of a function we need to reverse the rolls of the inputs and the outputs, so that the inputs for the original function become the outputs for the inverse function, and the outputs for the original become the inputs for the inverse.

DEFINITION: The **inverse sine function**, denoted $y = \sin^{-1}(t)$, is defined by the following:

If $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ and $\sin(y) = t$, then $y = \sin^{-1}(t)$.

By construction, the range of $y = \sin^{-1}(t)$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, and the domain is the same as the range of the sine function: $[-1, 1]$. Note that the inverse sine function is often called the **arcsine function** and denoted $y = \arcsin(t)$

Now let's construct the inverse of the cosine function. Like the sine function, the cosine function isn't one-to-one so we'll need to restrict its domain to construct the inverse cosine function. As we can see in Figure 3, the cosine function is one-to-one on the interval $[0, \pi]$ and, on this interval, the graph utilizes the entire range of the cosine function, $[-1, 1]$. So we can define the inverse cosine function on this portion of the cosine function.

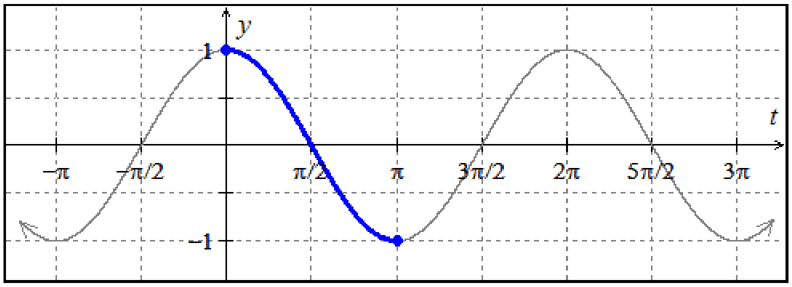


Figure 3: The interval $[0, \pi]$ of the graph of $y = \cos(t)$; on this interval, the cosine function is one-to-one and has the same range as the entire cosine function.

Recall that when we construct the inverse of a function we need to reverse the rolls of the inputs and the outputs, so that the inputs for the original function become the outputs for the inverse function, and the outputs for the original become the inputs for the inverse.

DEFINITION: The **inverse cosine function**, denoted $y = \cos^{-1}(t)$, is defined by the following:

$$\text{If } 0 \le y \le \pi \text{ and } \cos(y) = t \text{, then } y = \cos^{-1}(t).$$

By construction, the range of $y = \cos^{-1}(t)$ is $[0, \pi]$, and the domain is the same as the range of the cosine function: $[-1, 1]$. Note that the inverse cosine function is often called the **arccosine function** and denoted $y = \arccos(t)$.

Key Point:

As we've discussed in Part 1 of Chapter 3, we can denote powers of trigonometric functions by putting the exponent between the function name and the input variable; for example, $(\sin(t))^2 = \sin^2(t)$. The definition above implies that inverse function notation looks like the sine function raised to the -1 power (i.e., the reciprocal of the sine function), but the reciprocal of a function isn't the same as its inverse! In order to avoid ambiguous notation, the notation $\sin^{-1}(t)$ *always* refers to the inverse function. If you want to denote the reciprocal of the sine function, you need to use the notation "$(\sin(t))^{-1}$":

$$(\sin(t))^{-1} = \frac{1}{\sin(t)} = \csc(t) \quad \text{but} \quad \csc(t) \neq \sin^{-1}(x)\,!$$

Now let's define the inverse tangent function. Recall that the tangent function is one-to-one on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$; since the period of tangent is π units, this interval represents a complete period of tangent. In order to construct the inverse tangent function, we restrict the tangent function to the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

DEFINITION: The **inverse tangent function**, denoted $y = \tan^{-1}(t)$, is defined by the following:

If $-\frac{\pi}{2} < y < \frac{\pi}{2}$ and $\tan(y) = t$, then $y = \tan^{-1}(t)$.

By construction, the range of $y = \tan^{-1}(t)$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and the domain is the same as the range of the tangent function: $\mathbb{R}$. Note that the inverse tangent function is often called the **arctangent function** and denoted $y = \arctan(t)$.

EXAMPLE 1: **a.** Evaluate $\sin^{-1}\left(-\frac{1}{2}\right)$.

b. Evaluate $\cos^{-1}(0)$.

c. Evaluate $\tan^{-1}(1)$.

SOLUTION:

a. To evaluate $\sin^{-1}\left(-\frac{1}{2}\right)$, we need to find a value, p, such that $-\frac{\pi}{2} \le p \le \frac{\pi}{2}$ and $\sin(p) = -\frac{1}{2}$. Our experience tells us that $p = -\frac{\pi}{6}$. Thus, $\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$.

b. To evaluate $\cos^{-1}(0)$, we need to find a value, p, such that $0 \le p \le \pi$ and $\cos(p) = 0$. Our experience tells us that $p = \frac{\pi}{2}$. Thus, $\cos^{-1}(0) = \frac{\pi}{2}$.

c. To evaluate $\tan^{-1}(1)$, we need to find a value, p, such that $-\frac{\pi}{2} < p < \frac{\pi}{2}$ and $\tan(p) = 1$. Our experience tells us that $p = \frac{\pi}{4}$. Thus, $\tan^{-1}(1) = \frac{\pi}{4}$.

EXAMPLE 2: a. Evaluate $\sin\left(\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)$.

b. Evaluate $\cos\left(\cos^{-1}\left(\frac{1}{2}\right)\right)$.

c. Evaluate $\tan\left(\tan^{-1}\left(-\sqrt{3}\right)\right)$.

SOLUTION:

a. To evaluate $\sin\left(\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)$, we need to first evaluate find $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$, so we need to find a value, p, such that $-\frac{\pi}{2} \le p \le \frac{\pi}{2}$ and $\sin(p) = \frac{\sqrt{3}}{2}$. Our experience tells us that $p = \frac{\pi}{3}$. Thus, $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$. Now we can evaluate $\sin\left(\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)$:

$$\begin{aligned}\sin\left(\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right) &= \sin\left(\frac{\pi}{3}\right)\\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

b. To evaluate $\cos\left(\cos^{-1}\left(\frac{1}{2}\right)\right)$, we need to first evaluate find $\cos^{-1}\left(\frac{1}{2}\right)$, so we need to find a value, p, such that $0 \le p \le \pi$ and $\cos(p) = \frac{1}{2}$. Our experience tells us that $p = \frac{\pi}{3}$. Thus, $\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$. Now we can evaluate $\cos\left(\cos^{-1}\left(\frac{1}{2}\right)\right)$:

$$\begin{aligned}\cos\left(\cos^{-1}\left(\frac{1}{2}\right)\right) &= \cos\left(\frac{\pi}{3}\right)\\ &= \frac{1}{2}\end{aligned}$$

c. To evaluate $\tan\left(\tan^{-1}\left(-\sqrt{3}\right)\right)$, we need to first evaluate find $\tan^{-1}\left(-\sqrt{3}\right)$, so we need to find a value, p, such that $-\frac{\pi}{2} < p < \frac{\pi}{2}$ and $\tan(p) = -\sqrt{3}$. Our experience tells us that $p = -\frac{\pi}{3}$. Thus, $\tan^{-1}\left(-\sqrt{3}\right) = -\frac{\pi}{3}$. Now we can evaluate $\tan\left(\tan^{-1}\left(-\sqrt{3}\right)\right)$:

$$\begin{aligned}\tan\left(\tan^{-1}\left(-\sqrt{3}\right)\right) &= \tan\left(-\frac{\pi}{3}\right)\\ &= -\sqrt{3}\end{aligned}$$

Notice that the answers to all three parts of Example 2 are exactly what we should have expected the answers to be since inverse functions "undo each other." But we have to be careful since the inverse sine, cosine, and tangent functions are NOT the inverses of the complete sine, cosine, and tangent functions. The next example should help explain why we need to be careful with the inverse trigonometric functions.

EXAMPLE 3: **a.** Evaluate $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$.

b. Evaluate $\cos^{-1}(\cos(2\pi))$

c. Evaluate $\tan^{-1}\left(\tan\left(\frac{5\pi}{4}\right)\right)$

SOLUTION:

a. Based on what we noticed in the last example and what we know about how inverse functions "undo each other," we might assume that $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$ is equal to $\frac{2\pi}{3}$ but this isn't true. (It can't possibly be true since the answer to this question is an output for the inverse sine function and $\frac{2\pi}{3}$ isn't in the range of this function, $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.)

$$\sin^{-1}\left(\sin\left(\tfrac{2\pi}{3}\right)\right) = \sin^{-1}\left(\tfrac{\sqrt{3}}{2}\right) \quad \text{since } \sin\left(\tfrac{2\pi}{3}\right) = \tfrac{\sqrt{3}}{2}.$$

$$= \tfrac{\pi}{3} \quad \text{since } \sin\left(\tfrac{\pi}{3}\right) = \tfrac{\sqrt{3}}{2} \text{ and } -\tfrac{\pi}{2} \le \tfrac{\pi}{3} \le \tfrac{\pi}{2}.$$

So $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$ is equal to $\frac{\pi}{3}$, not $\frac{2\pi}{3}$, since $\frac{2\pi}{3}$ isn't in the range of $y = \sin^{-1}(t)$.

b. Since inverse functions "undo each other," we might assume that $\cos^{-1}(\cos(2\pi))$ is equal to 2π, but this is NOT true. (Notice that it can't possibly be true since the answer to this question is an output for the inverse cosine function and 2π isn't in the range of this function, $[0, \pi]$.)

$$\cos^{-1}(\cos(2\pi)) = \cos^{-1}(1) \quad \text{since } \cos(2\pi) = 1.$$

$$= 0 \quad \text{since } \cos(0) = 1 \text{ and } 0 \le 0 \le \pi.$$

So $\cos^{-1}(\cos(2\pi))$ is equal to 0, not 2π, since 2π isn't in the range of $y = \cos^{-1}(t)$.

c. Since inverse functions "undo each other," we might assume that $\tan^{-1}\left(\tan\left(\frac{5\pi}{4}\right)\right)$ is equal to $\frac{5\pi}{4}$, but this is NOT true. (Notice that it can't possibly be true since the answer to this question is an output for the inverse tangent function and $\frac{5\pi}{4}$ isn't in the range of this function, $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.)

$$\tan^{-1}\left(\tan\left(\frac{5\pi}{4}\right)\right) = \tan^{-1}(1) \quad \text{since } \tan\left(\frac{5\pi}{4}\right) = 1.$$

$$= \frac{\pi}{4} \quad \text{since } \tan\left(\frac{\pi}{4}\right) = 1 \text{ and } -\frac{\pi}{2} < \frac{\pi}{4} < \frac{\pi}{2}.$$

So $\tan^{-1}\left(\tan\left(\frac{5\pi}{4}\right)\right)$ is equal to $\frac{\pi}{4}$, not $\frac{5\pi}{4}$, since $\frac{5\pi}{4}$ isn't in the range of $y = \tan^{-1}(t)$.

Section I: The Trigonometric Functions

Chapter 7: Solving Trig Equations

Let's start by solving a couple of equations that involve the sine function.

EXAMPLE 1a: Solve the equation $\sin(t) = \frac{1}{2}$.

SOLUTION:

The inverse functions we constructed in Chapter 6 can be used to solve equations like $\sin(t) = \frac{1}{2}$ but the fraction $\frac{1}{2}$ is a "friendly" sine value so we don't need to use the inverse sine function: our experience with the sine function tells us that that $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$, so we know that $t = \frac{\pi}{6}$ is a solution to $\sin(t) = \frac{1}{2}$. We also know that the sine function is periodic with period 2π, so its values repeat every 2π units, so angles like

$$t = \frac{\pi}{6} + 2\pi = \frac{13\pi}{6} \quad \textit{and} \quad t = \frac{\pi}{6} + 4\pi = \frac{25\pi}{6} \quad \textit{and} \quad t = \frac{\pi}{6} - 2\pi = -\frac{11\pi}{6}$$

are also solutions. We can represent multiples of the period with the expression $2k\pi$ where k is any integer, i.e., $k \in \mathbb{Z}$, so we can represent all of the solutions that are "related" to $\frac{\pi}{6}$ with the expression $\frac{\pi}{6} + 2k\pi$, $k \in \mathbb{Z}$. This expression represents *infinitely many solutions*, but it still doesn't represent all of the solutions; see Figure 1.

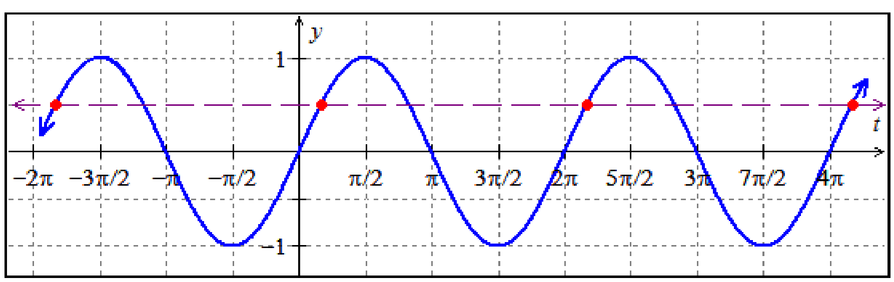


Figure 1: The graph of $y = \sin(t)$ and the line $y = \frac{1}{2}$. The red dots represent points with horizontal coordinates of the form $t = \frac{\pi}{6} + 2k\pi$, $k \in \mathbb{Z}$. The other instances where the blue graph intersects the line $y = \frac{1}{2}$ are also solutions to the equation $\sin(t) = \frac{1}{2}$ but they are NOT represented by $t = \frac{\pi}{6} + 2k\pi$.

Notice that one of the solutions that we are missing is just as close to π as our original solution, $t=\frac{\pi}{6}$, is to 0. Recall the identity $\sin(t)=\sin(\pi-t)$ that we first noticed in Part 1 of Chapter 3: this identity tells us that the angles t and $\pi - t$ always have the same sine value. This means that whenever we've found a solution, t, to an equation involving sine, we can find another solution by computing $\pi - t$. Now let's apply this observation to find the rest of the solutions to $\sin(t) = \frac{1}{2}$: since we know that $t=\frac{\pi}{6}$ is a solution to $\sin(t) = \frac{1}{2}$, we know that $t = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$ is another solution. And now we can again utilize the fact that the period of the sine function is 2π so we can express the rest of the solutions with $t = \frac{5\pi}{6} + 2k\pi$, $k \in \mathbb{Z}$; in Figure 2, these solutions are colored green. So the complete solution to the equation $\sin(t) = \frac{1}{2}$ is:

$$t = \frac{\pi}{6} + 2k\pi \quad \text{or} \quad t = \frac{5\pi}{6} + 2k\pi \quad \text{for all } k \in \mathbb{Z}.$$

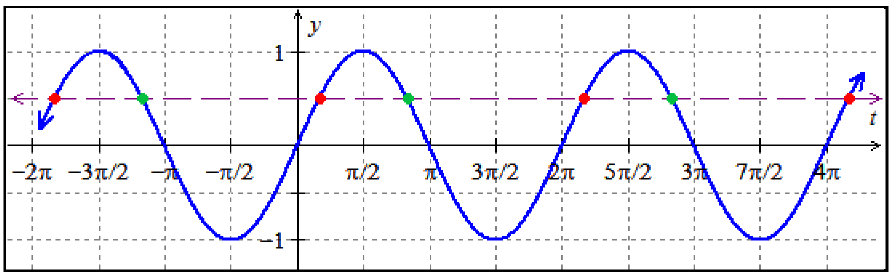


Figure 2: The red dots represent points with horizontal coordinates of the form $t = \frac{\pi}{6} + 2k\pi$, $k \in \mathbb{Z}$, while the green dots represent points with horizontal coordinates of the form $t = \frac{5\pi}{6} + 2k\pi$, $k \in \mathbb{Z}$. The red and green dots together represent all of the solutions to $\sin(t) = \frac{1}{2}$.

EXAMPLE 1b: Solve the equation $\sin(t) = -0.555$.

SOLUTION:

Unlike Example 1a where the equation involved a "friendly" sine value, -0.555 isn't a "friendly" sine value: we don't know what input for the sine function is related to the output -0.555, so we need to utilize the inverse sine function that we constructed in Chapter 6 in order to solve the equation:

$$\sin(t) = -0.555$$

$$\Rightarrow \sin^{-1}(\sin(t)) = \sin^{-1}(-0.555)$$ (apply the inverse sine function to both sides of the equation)

$$\Rightarrow t = \sin^{-1}(-0.555) \approx -0.588$$

(Note that you can utilize a calculator to obtain an approximation for $\sin^{-1}(-0.555)$ by accessing a button labeled "$\sin^{-1}$".)

Although we've found *one* solution to the equation, **we aren't done yet!** The inverse sine inverse only gives us one value, but we know that the periodic nature of the sine function suggests that there are *infinitely many solutions* to an equation like this; see Figure 3.

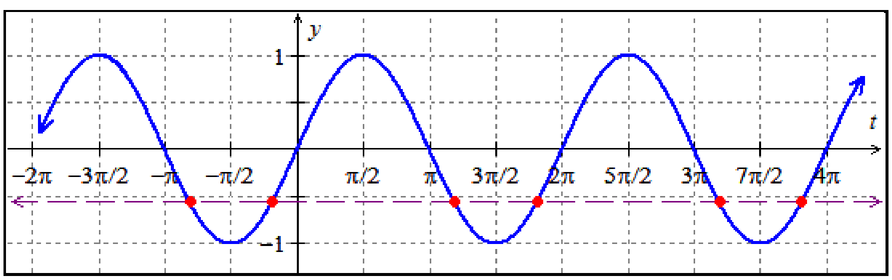


Figure 3: The graph of $y = \sin(t)$ intersecting the line $y = -0.555$ many, many times. Each point of intersection represents a solution to $\sin(t) = -0.555$.

We can find all of the solutions by using the solution that we found using the inverse sine function along with the fact that the sine function has period 2π: since the sine function has period 2π units, we know that the outputs repeat every 2π units. So if $t \approx -0.588$ is a solution, the values represented by $t \approx -0.588 + 2k\pi$, $k \in \mathbb{Z}$ must also be solutions. This gives us LOTS of solutions, but we are still missing *half* of them. (Recall we had the same problem in Example 1a.) In order to get the rest of the solutions, can use the identity $\sin(t) = \sin(\pi - t)$, and subtract our original solution $(t \approx -0.588)$ from π: $t \approx \pi - (-0.588) + 2k\pi$, $k \in \mathbb{Z}$. Therefore, the complete solution to the equation is:

$$t \approx -0.588 + 2k\pi \text{ or } t \approx \pi + 0.588 + 2k\pi \text{ for all } k \in \mathbb{Z}.$$

EXAMPLE 2a: Solve the equation $\cos(t) = -\frac{\sqrt{3}}{2}$.

SOLUTION:

Like Example 1a, $-\frac{\sqrt{3}}{2}$ is a “friendly” cosine value so we can use our knowledge about the cosine function, rather than the inverse cosine function, to solve the equation. Our experience with the cosine function tells us that that $\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$, so we know that $t = \frac{5\pi}{6}$ is a solution to $\cos(t) = -\frac{\sqrt{3}}{2}$. We also know that the cosine function is periodic with period 2π, so its values repeat every 2π units, so angles like

$$t = \frac{5\pi}{6} + 2\pi = \frac{17\pi}{6} \quad \textit{and} \quad t = \frac{5\pi}{6} + 4\pi = \frac{29\pi}{6} \quad \textit{and} \quad t = \frac{5\pi}{6} - 2\pi = -\frac{7\pi}{6}$$

are also solutions. We can represent all of the solutions that are “related” to $\frac{5\pi}{6}$ with the expression $\frac{5\pi}{6} + 2k\pi$, $k \in \mathbb{Z}$. This expression represents *infinitely many solutions*, but it still doesn’t represent all of the solutions; see Figure 4.

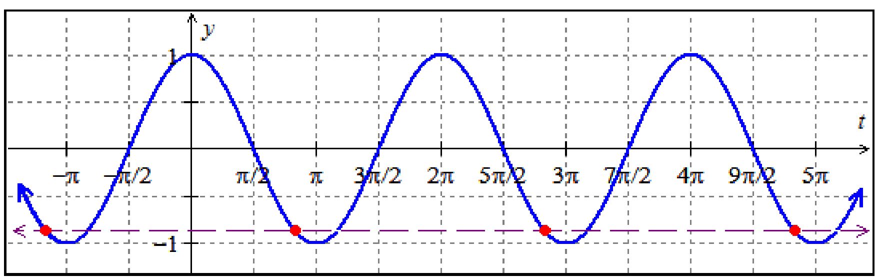


Figure 4: The graph of $y = \cos(t)$ and the line $y = -\frac{\sqrt{3}}{2}$. The red dots represent points with horizontal coordinates of the form $t = \frac{5\pi}{6} + 2k\pi$, $k \in \mathbb{Z}$. The other instances where the blue graph intersects the line $y = -\frac{\sqrt{3}}{2}$ are also solutions to the equation $\cos(t) = -\frac{\sqrt{3}}{2}$ but they are NOT represented by $t = \frac{5\pi}{6} + 2k\pi$.

Recall the identity $\cos(t) = \cos(-t)$ that we noticed in Part 1 of Chapter 3: this identity tells us that the angles t and $-t$ always have the same cosine value. This means that whenever we’ve found a solution, t, to an equation involving cosine, we can find another solution by computing $-t$. Now let’s apply this observation to find the rest of the solutions to $\cos(t) = -\frac{\sqrt{3}}{2}$: since we know that $t = \frac{5\pi}{6}$ is a solution to $\cos(t) = -\frac{\sqrt{3}}{2}$, we know that $t = -\frac{5\pi}{6}$ is another solution. Now we can again utilize the fact that the period of cosine is

2π so we can express the rest of the solutions with $t = -\frac{5\pi}{6} + 2k\pi$, $k \in \mathbb{Z}$; in Figure 5, these solutions are colored green. So the complete solution to the equation $\cos(t) = -\frac{\sqrt{3}}{2}$ is:

$$t = \frac{5\pi}{6} + 2k\pi \text{ or } t = -\frac{5\pi}{6} + 2k\pi \text{ for all } k \in \mathbb{Z}.$$

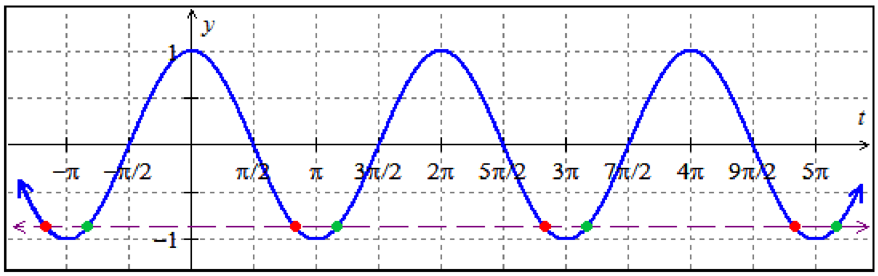


Figure 5: The red dots represent points with horizontal coordinates of the form $t = \frac{5\pi}{6} + 2k\pi$, $k \in \mathbb{Z}$, while the green dots represent points with horizontal coordinates of the form $t = -\frac{5\pi}{6} + 2k\pi$, $k \in \mathbb{Z}$. The red and green dots together represent all of the solutions to $\cos(t) = -\frac{\sqrt{3}}{2}$.

EXAMPLE 2b: Solve the equation $\cos(t) = 0.4$.

SOLUTION:

Like in Example 1b, 0.4 isn't a "friendly" cosine value so we need to utilize the inverse cosine function that we constructed in Chapter 6 in order to solve the equation:

$$\cos(t) = 0.4$$

$$\Rightarrow \cos^{-1}(\cos(t)) = \cos^{-1}(0.4)$$ (apply the inverse cosine function to both sides of the equation)

$$\Rightarrow \quad t = \cos^{-1}(0.4) \approx 1.16$$

(Note that you can utilize a calculator to obtain an approximation for $\cos^{-1}(0.4)$ by accessing a button labeled "$\cos^{-1}$".)

Although we have found a solution to the given equation, **we aren't done yet!** The inverse cosine function only gives us *one* value but we know that the periodic nature of the cosine function suggests that there are *infinitely many solutions* to an equation like this; see Figure 6.

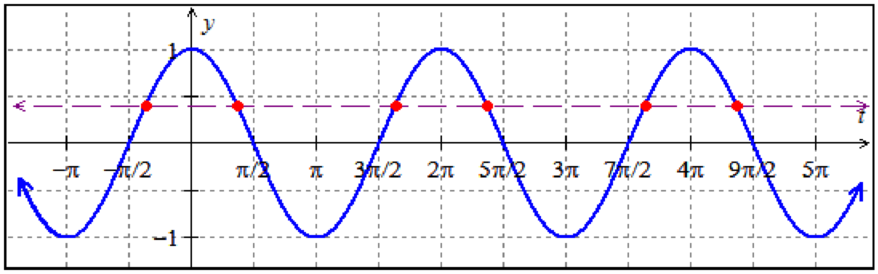


Figure 6: The graph of $y = \cos(t)$ intersecting the line $y = 0.4$ many, many times. Each point of intersection represents a solution to $\cos(t) = 0.4$.

We can find all of the solutions by using the solution that we found using the inverse cosine function along with the fact that the cosine function has period 2π: since the cosine function has period 2π units, we know that the outputs repeat every 2π units. So if $t \approx 1.16$ is a solution, the values represented by $t \approx 1.16 + 2k\pi$, $k \in \mathbb{Z}$ must also be solutions. This gives us LOTS of solutions, but we are still missing *half* of them. (Recall we had the same problem in Example 2a.) In order to get the rest of the solutions, we can use the identity $\cos(t) = \cos(-t)$ and take the opposite of original solution to find a second "family" of solutions: $t \approx -1.16 + 2k\pi$, $k \in \mathbb{Z}$. Therefore, the complete solution to the equation $\cos(t) = 0.4$ is:

$$t \approx 1.16 + 2k\pi \text{ or } t \approx -1.16 + 2k\pi \text{ for all } k \in \mathbb{Z}.$$

EXAMPLE 3a: Solve the equation $2\cos(t) = -1$.

SOLUTION:

CLICK HERE to see a video of this example.

$$2\cos(t) = -1$$

$$\Rightarrow \quad \cos(t) = -\tfrac{1}{2}$$

$$\Rightarrow \quad \cos^{-1}\big(\cos(t)\big) = \cos^{-1}\left(-\tfrac{1}{2}\right)$$

(this step is optional since $-\frac{1}{2}$ is a "friendly" cosine value)

$$\Rightarrow t = \tfrac{2\pi}{3} + 2k\pi \text{ or } t = -\tfrac{2\pi}{3} + 2k\pi \text{ for } k \in \mathbb{Z}$$

(since $\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$ and $\cos(t) = \cos(-t)$)

[Since $-\frac{1}{2}$ is a "friendly" cosine value, we didn't need to use the inverse cosine function as we did in the third step – the inverse trig functions are available when solving equations but we don't need to use them if the values are friendly.]

EXAMPLE 3b: Solve the equation $8\cos(x) + 9 = 10$.

SOLUTION:

$$8\cos(x) + 9 = 10$$

$$\Rightarrow \quad 8\cos(x) = 1$$

$$\Rightarrow \quad \cos(x) = \tfrac{1}{8}$$

$$\Rightarrow \quad \cos^{-1}\big(\cos(x)\big) = \cos^{-1}\left(\tfrac{1}{8}\right)$$

$$\Rightarrow t = \cos^{-1}\left(\tfrac{1}{8}\right) + 2k\pi \text{ or } t = -\cos^{-1}\left(\tfrac{1}{8}\right) + 2k\pi \text{ for } k \in \mathbb{Z}$$

$$\Rightarrow t \approx 1.445 + 2k\pi \quad \text{or } t \approx -1.445 + 2k\pi \text{ for } k \in \mathbb{Z}$$

(since $\cos^{-1}\left(\frac{1}{8}\right) \approx 1.445$)

[Since $\frac{1}{8}$ isn't a "friendly" cosine value, we need to use the inverse cosine function as we've done in the fourth step. Also note that in the last step we've approximated the solutions: this requires a calculator, so it's not something that we would need to do on no-calculator exams.]

EXAMPLE 4a: Solve the equation $4\sin(\theta) + \sqrt{3} = -\sqrt{3}$.

SOLUTION:

CLICK HERE to see a video of this example.

$$4\sin(\theta) + \sqrt{3} = -\sqrt{3}$$

$$\Rightarrow \quad 4\sin(\theta) = -2\sqrt{3}$$

$$\Rightarrow \quad \sin(\theta) = -\frac{2\sqrt{3}}{4} = -\frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad \sin^{-1}(\sin(\theta)) = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$$ (this step is optional since $-\frac{\sqrt{3}}{2}$ is a "friendly" sine value)

$$\Rightarrow \theta = -\frac{\pi}{3} + 2k\pi \text{ or } \theta = \pi - \left(-\frac{\pi}{3}\right) + 2k\pi \text{ for } k \in \mathbb{Z}$$ (since $\sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$ and $\sin(t) = \sin(\pi - t)$)

$$\Rightarrow \theta = -\frac{\pi}{3} + 2k\pi \text{ or } \theta = \frac{4\pi}{3} + 2k\pi \text{ for } k \in \mathbb{Z}$$

[Since $-\frac{\sqrt{3}}{2}$ is a “friendly” value, we didn’t need to employ the inverse sine function as we did in the fourth step – the inverse trig functions are always available when solving trig equations but we don’t need to use them when the values are friendly.]

EXAMPLE 4b: Solve the equation $13\sin(t) = 6$.

SOLUTION:

$$13\sin(t) = 6$$

$$\Rightarrow \quad \sin(t) = \frac{6}{13}$$

$$\Rightarrow \quad \sin^{-1}(\sin(t)) = \sin^{-1}\left(\frac{6}{13}\right)$$

$$\Rightarrow t = \sin^{-1}\left(\frac{6}{13}\right) + 2k\pi \text{ or } t = \pi - \sin^{-1}\left(\frac{6}{13}\right) + 2k\pi \text{ for all } k \in \mathbb{Z}$$

$$\Rightarrow t \approx 0.48 + 2k\pi \quad \text{or} \quad t \approx \pi - 0.48 + 2k\pi \text{ for all } k \in \mathbb{Z}$$ (since $\sin^{-1}\left(\frac{6}{13}\right) \approx 0.48$)

$$\Rightarrow t \approx 0.48 + 2k\pi \quad \text{or} \quad t \approx 2.66 + 2k\pi \text{ for all } k \in \mathbb{Z}$$

[Since $\frac{6}{13}$ isn’t a “friendly” sine value, we need to employ the inverse sine function as we’ve done in the third step. Also note that in the second-to-last step we’ve approximated the solutions: this requires a calculator, so it’s not something that we need to do on a no-calculator activity.]

EXAMPLE 5a: Solve the equation $\sqrt{3}\tan(x) = 1$.

SOLUTION:

$$\sqrt{3}\tan(x) = 1$$

$$\Rightarrow \quad \tan(x) = \tfrac{1}{\sqrt{3}}$$

$$\Rightarrow \quad \tan^{-1}\left(\tan(x)\right) = \tan^{-1}\left(\tfrac{1}{\sqrt{3}}\right)$$

(this step is optional since $\frac{1}{\sqrt{3}}$ is a "friendly" tangent value)

$$\Rightarrow \quad x = \tfrac{\pi}{6} + k\pi \quad \text{for all } k \in \mathbb{Z}$$

(since $\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$)

[Notice that we add $k\pi$ (rather than $2k\pi$) to our solutions since, unlike sine and cosine, the period of tangent is π units. Also, there's only one "family" of solutions since tangent only reaches each output value once in each period.]

EXAMPLE 5b: Solve the equation $5\tan(\theta) - 10 = -6$.

SOLUTION:

$$5\tan(\theta) - 10 = -6$$

$$\Rightarrow \quad 5\tan(\theta) = 4$$

$$\Rightarrow \quad \tan(\theta) = \tfrac{4}{5}$$

$$\Rightarrow \quad \tan^{-1}\left(\tan(\theta)\right) = \tan^{-1}\left(\tfrac{4}{5}\right)$$

$$\Rightarrow \quad \theta = \tan^{-1}\left(\tfrac{4}{5}\right) + k\pi \quad \text{for all } k \in \mathbb{Z}$$

[Since $\frac{4}{5}$ isn't a "friendly" tangent value, we need to use the inverse tangent function as we've done in the fourth step. As mentioned above in Example 5a, we add $k\pi$ (rather than $2k\pi$) to our solutions since the period of tangent is π units and there's only one "family" of solutions since tangent only reaches each output value once in each period.]

EXAMPLE 6: **a.** Find all of the solutions to the equation $6\sin(2x) = 3\sqrt{2}$.

b. Find the solutions to $6\sin(2x) = 3\sqrt{2}$ that are in the interval $[0, 2\pi]$.

SOLUTION:

a. Notice that the trigonometric function involved in the given equation is $\sin(2x)$, and recall that $\sin(2x)$ has period π units, i.e., the values for $\sin(2x)$ repeat every π units. This means that once we find a solution to the given equation we'll be able to add to it any integer multiple of π and obtain another solution. Thus, we should expect the phrase "$k\pi$ for all $k \in \mathbb{Z}$" to be involved in our solutions. You'll see in the work below that we add $2k\pi$ to our solutions after applying the inverse sine function since the period of the sine function is 2π units. In the last step, we finish solving for x and obtain the desired period-shift of $k\pi$ units.

$$6\sin(2x) = 3\sqrt{2}$$

$$\Rightarrow \quad \sin(2x) = \frac{\sqrt{2}}{2}$$

$$\Rightarrow \quad \sin^{-1}\left(\sin(2x)\right) = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$$

$$\Rightarrow \quad 2x = \tfrac{\pi}{4} + 2k\pi \quad \text{or} \quad 2x = \pi - \tfrac{\pi}{4} + 2k\pi \quad \text{for all } k \in \mathbb{Z}$$

$$\Rightarrow \quad x = \tfrac{1}{2}\left(\tfrac{\pi}{4} + 2k\pi\right) \quad \text{or} \quad x = \tfrac{1}{2}\left(\tfrac{3\pi}{4} + 2k\pi\right) \quad \text{for all } k \in \mathbb{Z}$$

$$\Rightarrow \quad x = \tfrac{\pi}{8} + k\pi \quad \text{or} \quad x = \tfrac{3\pi}{8} + k\pi \quad \text{for all } k \in \mathbb{Z}$$

b. Now we need to substitute specific values of k into the solutions we found in part (a) and determine which solutions are in the interval $[0, 2\pi]$.

$$\boldsymbol{k = -1}: \quad x = \tfrac{\pi}{8} + (\mathbf{-1})\cdot\pi = -\tfrac{7\pi}{8} \quad \text{or} \quad x = \tfrac{3\pi}{8} + (\mathbf{-1})\cdot\pi = -\tfrac{5\pi}{8}$$

Both of these values are negative so they aren't in the interval $[0, 2\pi]$. Smaller values of k will produce even smaller values of x so we don't need to try smaller values of k.

$$\boldsymbol{k = 0}: \quad x = \tfrac{\pi}{8} + \mathbf{0}\cdot\pi = \tfrac{\pi}{8} \quad \text{or} \quad x = \tfrac{3\pi}{8} + \mathbf{0}\cdot\pi = \tfrac{3\pi}{8}$$

Both of these values are in the interval $[0, 2\pi]$.

$\boldsymbol{k = 1}$: $x = \frac{\pi}{8} + \mathbf{1} \cdot \pi$ or $x = \frac{3\pi}{8} + \mathbf{1} \cdot \pi$

$= \frac{9\pi}{8}$ $= \frac{11\pi}{8}$

Both of these values are in the interval $[0, 2\pi]$.

$\boldsymbol{k = 2}$: $x = \frac{\pi}{8} + \mathbf{2} \cdot \pi$ or $x = \frac{3\pi}{8} + \mathbf{2} \cdot \pi$

$= \frac{17\pi}{8}$ $= \frac{19\pi}{8}$

Since both of these values are greater than 2π, they aren't in the interval $[0, 2\pi]$. Certainly larger values of k will produce even larger values of x so we don't need to try larger values of k.

Therefore, the solution set to the equation $6\sin(2x) = 3\sqrt{2}$ on the interval $[0, 2\pi]$ is $\left\{\frac{\pi}{8}, \frac{3\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}\right\}$.

EXAMPLE 7: **a.** Find all of the solutions to the equation $2\cos(3t) = -1$.

b. Find the solutions in the interval $[0, 2\pi]$ to the equation $2\cos(3t) = -1$.

SOLUTION:

a. Notice that the trigonometric function involved in the given equation is $\cos(3t)$, and recall that $\cos(3t)$ has period $\frac{2\pi}{3}$ units, i.e., the values for $\cos(3t)$ repeat every $\frac{2\pi}{3}$ units. This means that once we find a solution to the given equation we'll be able to add to it any integer multiple of $\frac{2\pi}{3}$ and obtain another solution. Therefore, we should expect the phrase "$\frac{2k\pi}{3}$ for all $k \in \mathbb{Z}$" to be involved in our solutions. You'll see in the work below that we add $2k\pi$ to our solutions after applying the inverse cosine function since the period of the cosine function is 2π units. In the last step, we finish solving for t and obtain the desired period-shift of $\frac{2k\pi}{3}$ units.

$$
\begin{aligned}
& 2\cos(3t) = -1 \\
\Rightarrow\quad & \cos(3t) = -\tfrac{1}{2} \\
\Rightarrow\quad & \cos^{-1}\left(\cos(3t)\right) = \cos^{-1}\left(-\tfrac{1}{2}\right) \\
\Rightarrow\quad & 3t = \tfrac{2\pi}{3} + 2k\pi \quad \text{or} \quad 3t = -\tfrac{2\pi}{3} + 2k\pi \quad \text{for all } k \in \mathbb{Z} \\
\Rightarrow\quad & t = \tfrac{1}{3}\left(\tfrac{2\pi}{3} + 2k\pi\right) \quad \text{or} \quad t = \tfrac{1}{3}\left(-\tfrac{2\pi}{3} + 2k\pi\right) \quad \text{for all } k \in \mathbb{Z} \\
\Rightarrow\quad & t = \tfrac{2\pi}{9} + \tfrac{2k\pi}{3} \quad \text{or} \quad t = -\tfrac{2\pi}{9} + \tfrac{2k\pi}{3} \quad \text{for all } k \in \mathbb{Z}
\end{aligned}
$$

b. Now we need to substitute specific values of k into the solutions we found in part (a) and determine which solutions are in the interval $[0, 2\pi]$.

$\boldsymbol{k = -1}$: $t = \frac{2\pi}{9} + \frac{2(\mathbf{-1})\pi}{3} = -\frac{4\pi}{9}$ or $t = -\frac{2\pi}{9} + \frac{2(\mathbf{-1})\pi}{3} = -\frac{8\pi}{9}$

Both of these values are negative so they aren't in the interval $[0, 2\pi]$. Smaller values of k will produce even smaller values of t so we don't need to try smaller values of k.

$\boldsymbol{k = 0}$: $t = \frac{2\pi}{9} + \frac{2(\mathbf{0})\pi}{3} = \frac{2\pi}{9}$ or $t = -\frac{2\pi}{9} + \frac{2(\mathbf{0})\pi}{3} = -\frac{2\pi}{9}$

Only $\frac{2\pi}{9}$ is in the interval $[0, 2\pi]$ since $-\frac{2\pi}{9}$ is negative.

$\boldsymbol{k = 1}$: $t = \frac{2\pi}{9} + \frac{2(\mathbf{1})\pi}{3} = \frac{8\pi}{9}$ or $t = -\frac{2\pi}{9} + \frac{2(\mathbf{1})\pi}{3} = \frac{4\pi}{9}$

Both of these values are in the interval $[0, 2\pi]$.

$\boldsymbol{k = 2}$: $t = \frac{2\pi}{9} + \frac{2(\mathbf{2})\pi}{3} = \frac{14\pi}{9}$ or $t = -\frac{2\pi}{9} + \frac{2(\mathbf{2})\pi}{3} = \frac{10\pi}{9}$

Both of these values are in the interval $[0, 2\pi]$.

$\boldsymbol{k = 3}$: $t = \frac{2\pi}{9} + \frac{2(\mathbf{3})\pi}{3} = \frac{20\pi}{9}$ or $t = -\frac{2\pi}{9} + \frac{2(\mathbf{3})\pi}{3} = \frac{16\pi}{9}$

Only $\frac{16\pi}{9}$ is in the interval $[0, 2\pi]$ since $\frac{20\pi}{9}$ is greater than 2π.

$\boldsymbol{k = 4}$: $t = \frac{2\pi}{9} + \frac{2(\mathbf{4})\pi}{3} = \frac{26\pi}{9}$ or $t = -\frac{2\pi}{9} + \frac{2(\mathbf{4})\pi}{3} = \frac{22\pi}{9}$

Since both of these values are greater than 2π, they aren't in the interval $[0, 2\pi]$. Certainly larger values of k will produce even larger values of t so we don't need to try larger values of k.

Therefore, the solution set to the equation $6\sin(2x) = 3\sqrt{2}$ on the interval $[0, 2\pi]$ is $\left\{\frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}, \frac{10\pi}{9}, \frac{14\pi}{9}, \frac{16\pi}{9}\right\}$.

EXAMPLE 8: Find the solutions to the equation $3\cos(2x) - 2 = 0$ on the interval $[-\pi, \pi]$.

SOLUTION:

$$3\cos(2x) - 2 = 0$$

$$\Rightarrow \quad \cos(2x) = \frac{2}{3}$$

$$\Rightarrow \quad \cos^{-1}\big(\cos(2x)\big) = \cos^{-1}\left(\frac{2}{3}\right)$$

$\Rightarrow 2x = \cos^{-1}\left(\frac{2}{3}\right) + 2k\pi$ or $2x = -\cos^{-1}\left(\frac{2}{3}\right) + 2k\pi$ for all $k \in \mathbb{Z}$

$\Rightarrow x = \frac{1}{2}\left(\cos^{-1}\left(\frac{2}{3}\right) + 2k\pi\right)$ or $x = \frac{1}{2}\left(-\cos^{-1}\left(\frac{2}{3}\right) + 2k\pi\right)$ for all $k \in \mathbb{Z}$

$\Rightarrow x = \frac{1}{2}\cos^{-1}\left(\frac{2}{3}\right) + k\pi$ or $x = -\frac{1}{2}\cos^{-1}\left(\frac{2}{3}\right) + k\pi$ for all $k \in \mathbb{Z}$

Notice that we were asked to find the solutions in the interval $[-\pi, \pi]$, so we need to find which of the infinitely many solutions we've found are on the interval. It might help if we approximate the values we found above:

$$x = \tfrac{1}{2}\cos^{-1}\left(\tfrac{2}{3}\right) + k\pi \approx 0.42 + k\pi$$

or

$$x = -\tfrac{1}{2}\cos^{-1}\left(\tfrac{2}{3}\right) + k\pi \approx -0.42 + k\pi \quad \text{for all } k \in \mathbb{Z}$$

We know that $\pi \approx 3.14$ so we need to find values that satisfy the equation as above and are between -3.14 and 3.14.

$\boldsymbol{k = -1}$: $x \approx 0.42 + (\mathbf{-1})\cdot\pi \approx -2.72$ or $x \approx -0.42 + (\mathbf{-1})\cdot\pi \approx -3.56$

Only -2.72 is in the interval $[-\pi, \pi]$.

$\boldsymbol{k = 0}$: $x \approx 0.42 + \mathbf{0}\cdot\pi \approx 0.42$ or $x \approx -0.42 + \mathbf{0}\cdot\pi \approx -0.42$

Both of these values are in the interval $[-\pi, \pi]$.

$\boldsymbol{k = 1}$: $x \approx 0.42 + \mathbf{1}\cdot\pi \approx 3.56$ or $x \approx -0.42 + \mathbf{1}\cdot\pi \approx 2.72$

Only 2.72 is in the interval $[-\pi, \pi]$.

$\boldsymbol{k = 2}$: $x \approx 0.42 + \mathbf{2}\cdot\pi \approx 6.7$ or $x \approx -0.42 + \mathbf{2}\cdot\pi \approx 5.86$

Neither of these values is in the interval $[-\pi, \pi]$.

We could try more values of k but we can tell from the work we've done thus far that no other values of k will give us solutions that are on the interval $[-\pi, \pi]$. Thus, the solution set to the equation $3\cos(2x) - 2 = 0$ on the interval $[-\pi, \pi]$ is $\{-2.72, -0.42, 0.42, 2.72\}$.

Section I: The Trigonometric Functions

Chapter 8: Right Triangle Trigonometry

As we saw in Part 1 of Chapter 3, when we put an angle in standard position in a unit circle, we create a right triangle with side lengths $\cos(\theta)$, $\sin(\theta)$, and 1; see the left side of Figure 1. If we put the same angle in standard position in a circle of a different radius, r, we generate a *similar triangle*; see the right side of Figure 1.

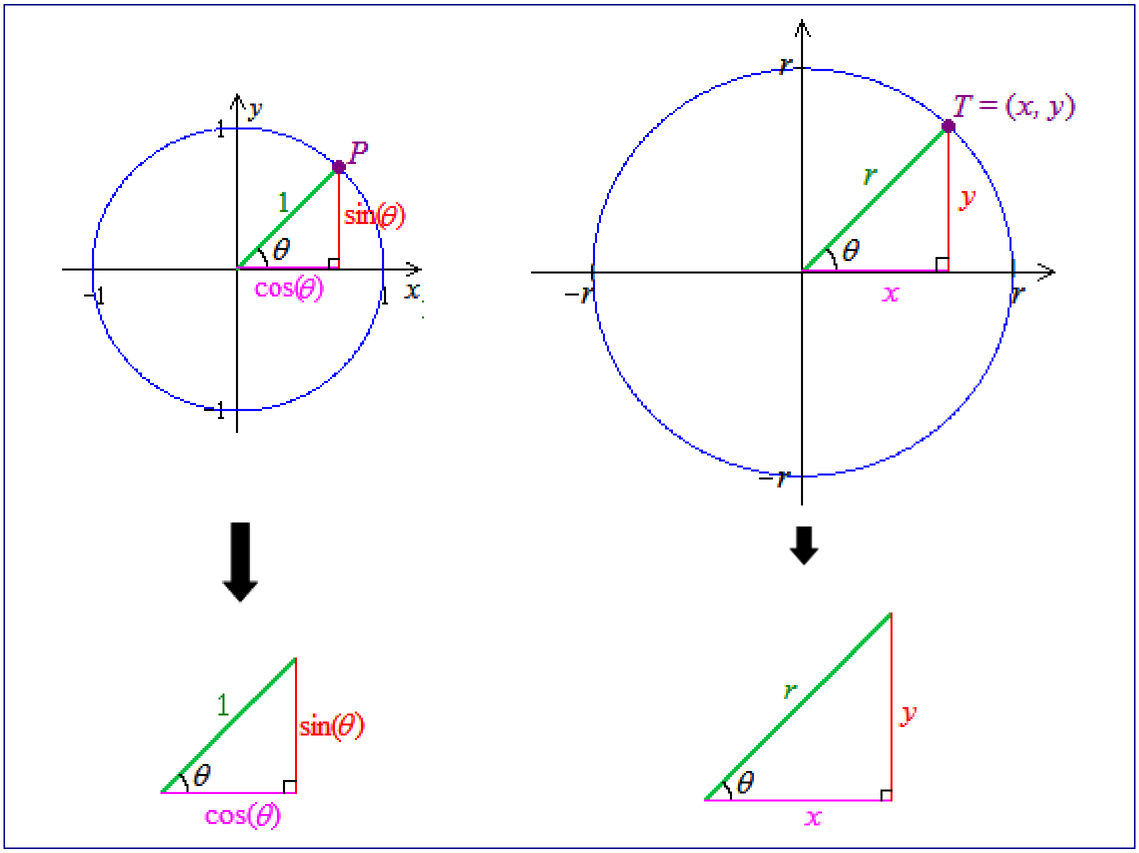


Figure 1: The angle θ in both a unit circle and in a circle of radius r create a pair of similar right triangles.

As you may recall from a high school geometry class, when figures are similar like the two triangles we created in Figure 1, ratios of corresponding components of the triangles must be equal. So we can obtain following ratios:

$$\frac{\cos(\theta)}{1} = \frac{x}{r} \quad \text{and} \quad \frac{\sin(\theta)}{1} = \frac{y}{r}$$

Solving these ratios for $\cos(\theta)$ and $\sin(\theta)$, respectively, gives us the following:

$$\cos(\theta) = \frac{x}{r} \quad \text{and} \quad \sin(\theta) = \frac{y}{r}$$

To help us remember these ratios, it's best to imagine yourself standing at angle θ looking into the triangle. Then the side labeled "y" is on the **opposite** side of the triangle while the side labeled "x" is **adjacent** to you. We use these descriptions (as well as the fact that the side labeled "r" is the **hypotenuse** of the triangle) to refer to the sides of the triangle in Fig. 2.

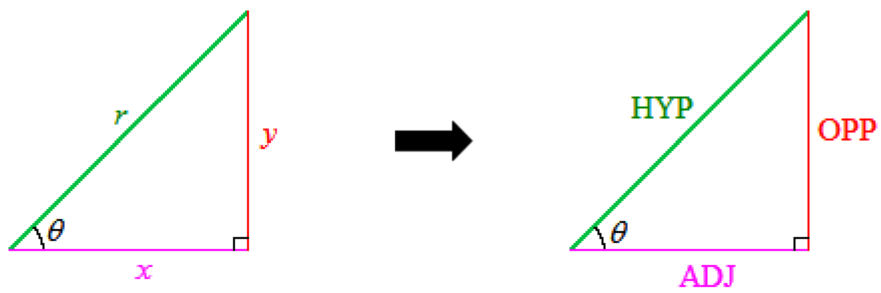


Figure 2: We use the terms **opposite** (or **OPP**), **adjacent** (or **ADJ**), and **hypotenuse** (or **HYP**) to refer to the sides of a right triangle.

DEFINITION: If θ is the angle given in the right triangles in Figure 2 (above), then

$$\sin(\theta) = \frac{y}{r} = \frac{\text{OPP}}{\text{HYP}} \quad \text{and} \quad \cos(\theta) = \frac{x}{r} = \frac{\text{ADJ}}{\text{HYP}}.$$

Consequently, the other trigonometric functions can be defined as follows:

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{\frac{\text{OPP}}{\text{HYP}}}{\frac{\text{ADJ}}{\text{HYP}}} = \frac{\text{OPP}}{\text{ADJ}} \qquad \cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)} = \frac{\frac{\text{ADJ}}{\text{HYP}}}{\frac{\text{OPP}}{\text{HYP}}} = \frac{\text{ADJ}}{\text{OPP}}$$

$$\sec(\theta) = \frac{1}{\cos(\theta)} = \frac{1}{\frac{\text{ADJ}}{\text{HYP}}} = \frac{\text{HYP}}{\text{ADJ}} \qquad \csc(\theta) = \frac{1}{\sin(\theta)} = \frac{1}{\frac{\text{OPP}}{\text{HYP}}} = \frac{\text{HYP}}{\text{OPP}}$$

We can use these ratios along with the Pythagorean Theorem (see below) to learn a great deal about right triangles.

THE PYTHAGOREAN THEOREM:

If the sides of a right triangle (i.e., a triangle with a 90° angle) are labeled like the one given in Figure 3, then $a^2 + b^2 = c^2$.

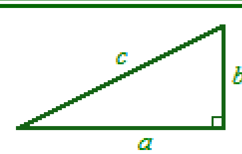


Figure 3

EXAMPLE 1: Find the value for all six trigonometric functions of the angle α given in the right triangle in Figure 4. (The triangle may not be drawn to scale.)

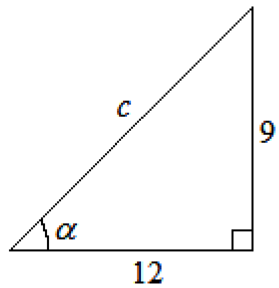


Figure 4

SOLUTION:

First we need to use the Pythagorean Theorem to find the length of the hypotenuse c.

$$\begin{aligned} &(12)^2 + (9)^2 = c^2 \\ \Rightarrow\ & 144 + 81 = c^2 \\ \Rightarrow\ & c^2 = 225 \\ \Rightarrow\ & c = 15 \end{aligned}$$

We can use this value to label our triangle:

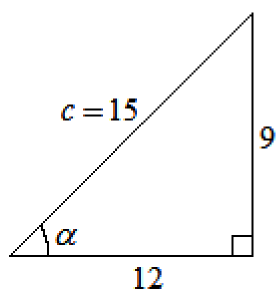


Figure 5

Thus,

$$\sin(\alpha) = \frac{\text{OPP}}{\text{HYP}} = \frac{9}{15} = \frac{3}{5} \qquad \cos(\alpha) = \frac{\text{ADJ}}{\text{HYP}} = \frac{12}{15} = \frac{4}{5}$$

$$\tan(\alpha) = \frac{\text{OPP}}{\text{ADJ}} = \frac{9}{12} = \frac{3}{4} \qquad \cot(\alpha) = \frac{\text{ADJ}}{\text{OPP}} = \frac{12}{9} = \frac{4}{3}$$

$$\sec(\alpha) = \frac{\text{HYP}}{\text{ADJ}} = \frac{15}{12} = \frac{5}{4} \qquad \csc(\alpha) = \frac{\text{HYP}}{\text{OPP}} = \frac{15}{9} = \frac{5}{3}$$

EXAMPLE 2: Find the value for all six trigonometric functions of the angle β given in the right triangle in Figure 6. (The triangle may not be drawn to scale.)

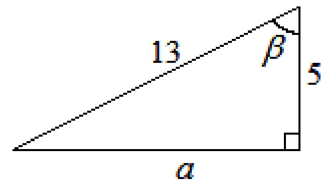


Figure 6

SOLUTION:

CLICK HERE to see a video of this example.

First we need to use the Pythagorean Theorem to find the length of the side labeled a.

$$\begin{aligned} & a^2 + (5)^2 = (13)^2 \\ \Rightarrow\quad & a^2 + 25 = 169 \\ \Rightarrow\quad & a^2 = 144 \\ \Rightarrow\quad & a = 12 \end{aligned}$$

We can use this value to label our triangle:

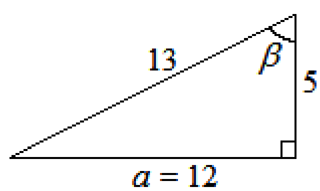


Figure 7

To determine the sine and cosine values of angle β, imagine standing at angle β and looking into the triangle. Then,

$$\sin(\alpha) = \frac{\text{OPP}}{\text{HYP}} = \frac{12}{13} \qquad \cos(\alpha) = \frac{\text{ADJ}}{\text{HYP}} = \frac{5}{13}$$

$$\tan(\alpha) = \frac{\text{OPP}}{\text{ADJ}} = \frac{12}{5} \qquad \cot(\alpha) = \frac{\text{ADJ}}{\text{OPP}} = \frac{5}{12}$$

$$\sec(\alpha) = \frac{\text{HYP}}{\text{ADJ}} = \frac{13}{5} \qquad \csc(\alpha) = \frac{\text{HYP}}{\text{OPP}} = \frac{13}{12}$$

We can use the trigonometric functions, along with the Pythagorean Theorem to **solve a right triangle**, i.e., find the missing side-lengths and missing angle-measures for a triangle.

EXAMPLE 3: Solve the triangle in Figure 8 by finding c, α, and β. (The triangle may not be drawn to scale.)

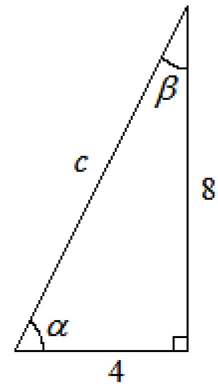


Figure 8

SOLUTION:

We can use the Pythagorean Theorem to find c.

$$(4)^2 + (8)^2 = c^2$$
$$\Rightarrow \quad 16 + 64 = c^2$$
$$\Rightarrow \quad c^2 = 80$$
$$\Rightarrow \quad c = 4\sqrt{5}$$

Now we can use the tangent function to find α. Note that we choose to use tangent, not sine or cosine, to find the α since it allows us to use the given info, rather than info that we've found. (If we made a mistake finding c, we don't want to compound that mistake but using the incorrect value to find other values.)

$$\tan(\alpha) = \frac{8}{4}$$
$$\Rightarrow \quad \tan(\alpha) = 2$$
$$\Rightarrow \quad \alpha = \tan^{-1}(2)$$
$$\Rightarrow \quad \alpha \approx 63.43^\circ$$

Note also that we could have just as easily found β first, instead of α. No matter which angle we find first, we can easily find the last angle by using the fact that the sum of the angles in a triangle is 180°:

$$\alpha + \beta + 90^\circ = 180^\circ$$
$$\Rightarrow \quad 63.43^\circ + \beta + 90^\circ \approx 180^\circ$$
$$\Rightarrow \quad \beta \approx 180^\circ - 90^\circ - 63.43^\circ$$
$$\Rightarrow \quad \beta \approx 26.57^\circ$$

Let's summarize our findings: $c = 4\sqrt{5}$, $\alpha \approx 63.43^\circ$, and $\beta \approx 26.57^\circ$.

EXAMPLE 4: Solve the triangle in Figure 9 by finding b, α, and β. (The triangle may not be drawn to scale.)

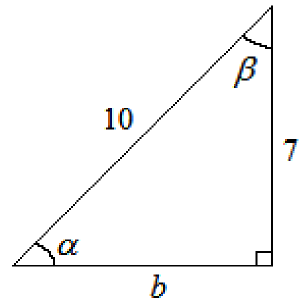


Figure 9

SOLUTION:

CLICK HERE to see a video of this example.

We can use the Pythagorean Theorem to find b.

$$7^2 + b^2 = 10^2$$
$$\Rightarrow 49 + b^2 = 100$$
$$\Rightarrow b^2 = 51$$
$$\Rightarrow b = \sqrt{51}$$

Now we can use the sine function to find α:

$$\sin(\alpha) = \frac{7}{10}$$
$$\Rightarrow \alpha = \sin^{-1}\left(\frac{7}{10}\right)$$
$$\Rightarrow \alpha \approx 44.43^\circ$$

(Although it would be just as easy to use tangent or cosine to find α, to we choose to use sine since it allows us to use the given info, rather than info that we've found, in order to avoid the possibility of compounding our mistakes.)

Now we can use the fact that the sum of the angles in a triangle is 180°:

$$\alpha + \beta + 90^\circ = 180^\circ$$
$$\Rightarrow 44.43^\circ + \beta + 90^\circ \approx 180^\circ$$
$$\Rightarrow \beta \approx 180^\circ - 90^\circ - 44.43^\circ$$
$$\Rightarrow \beta \approx 45.57^\circ$$

Let's summarize our findings: $b = \sqrt{51}$, $\alpha \approx 44.43^\circ$, and $\beta \approx 45.57^\circ$.

EXAMPLE 5: Solve the right triangle given in Figure 10 by finding A, b, and c. (The triangle may not be drawn to scale.)

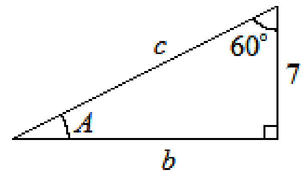


Figure 10

SOLUTION:

First, notice that angle A must measure 30° since the sum of the angles in a triangle is 180° and our triangle already has angles measuring 60° and 90°.

The only thing we know about the sides of the triangle is that the side "adjacent" to the 60° angle is 7 units long. Notice that the cosine of the 60° angle is $\frac{7}{c}$, and we can use this fact to find c:

$$\begin{aligned} & \cos(60^\circ) = \frac{7}{c} \\ \Rightarrow\ & c \cdot \cos(60^\circ) = \frac{7}{c} \cdot c \\ \Rightarrow\ & c \cdot \frac{1}{2} = 7 \qquad \text{(since } \cos(60^\circ) = \tfrac{1}{2}\text{)} \\ \Rightarrow\ & c = \frac{7}{1/2} \\ \Rightarrow\ & c = 14 \end{aligned}$$

Now that we know the length of two sides of the triangle, we could use the Pythagorean Theorem to find the length of the third side, b. Instead, we'll use the fact that, on the triangle, the sine of 60° is $\frac{b}{c}$:

$$\begin{aligned} & \sin(60^\circ) = \frac{b}{c} \\ \Rightarrow\ & \sin(60^\circ) = \frac{b}{14} \qquad \text{(since } c = 14\text{)} \\ \Rightarrow\ & 14 \cdot \frac{\sqrt{3}}{2} = b \qquad \text{(since } \sin(60^\circ) = \tfrac{\sqrt{3}}{2}\text{)} \\ \Rightarrow\ & b = 14 \cdot \frac{\sqrt{3}}{2} \\ \Rightarrow\ & b = 7\sqrt{3} \end{aligned}$$

Let's summarize our findings: $A = 30^\circ$, $b = 7\sqrt{3}$, and $c = 14$.

Section II: Trigonometric Identities

Chapter 1: Review of Identities

Recall the following definition that we first studied in Section I: Chapter 3:

DEFINITION: An **identity** is an equation that is true for all values in the domains of the involved expressions.

In this chapter, we'll list the identities that we studied in Section I. If you haven't yet learned these identities, you should learn (i.e., understand and memorize) all of them.

In Section I, Chapter 3, we noticed the following identities:

Some Identities

$$\sin(\theta) = \sin(\theta + 2\pi)$$

$$\cos(\theta) = \cos(\theta + 2\pi)$$

$$\cos(\theta) = \sin\left(\theta + \tfrac{\pi}{2}\right)$$

$$\sin(\theta) = \cos\left(\theta - \tfrac{\pi}{2}\right)$$

$$\cos(-\theta) = \cos(\theta)$$

$$\sin(-\theta) = -\sin(\theta)$$

$$\sin(\theta) = \sin(\pi - \theta)$$

Also in Section I, Chapter 3, we defined the "other" trigonometric functions and obtained some identities in the process:

Other Trig Functions

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \qquad \cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$$

$$\sec(\theta) = \frac{1}{\cos(\theta)} \qquad \csc(\theta) = \frac{1}{\sin(\theta)}$$

Also in Section I, Chapter 3, we discovered the Pythagorean identities:

Pythagorean Identities

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

$$\tan^2(\theta) + 1 = \sec^2(\theta)$$

$$1 + \cot^2(\theta) = \csc^2(\theta)$$

Sometimes we need to use different forms of the Pythagorean Identities. By subtracting expressions from both sides of the different Pythagorean Identities, we can obtain the following identities. Be sure to convince yourself that these are all valid by adjusting the appropriate Pythagorean Identities.

More Pythagorean Identities

$$\sin^2(\theta) = 1 - \cos^2(\theta)$$

$$\cos^2(\theta) = 1 - \sin^2(\theta)$$

$$\tan^2(\theta) = \sec^2(\theta) - 1$$

$$\sec^2(\theta) - \tan^2(\theta) = 1$$

$$\cot^2(\theta) = \csc^2(\theta) - 1$$

$$\csc^2(\theta) - \cot^2(\theta) = 1$$

Section II: Trigonometric Identities

Chapter 2: The Laws of Sines and Cosines

In Section I, Chapter 9, we studied right triangle trigonometry and learned how we can use the sine and cosine functions to obtain information about right triangles. In this section we'll study how we can use sine and cosine to obtain information about non-right triangles. The triangle in Figure 1 is a non-right triangle since none of its angles measure 90°. We'll start by deriving the **Laws of Sines and Cosines** so that we can study non-right triangles.

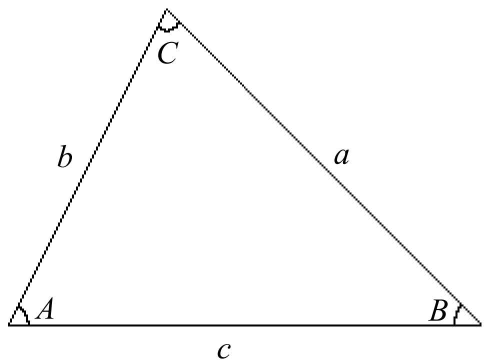
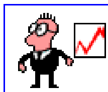


Figure 1

The Law of Sines

We'll work through the derivation of the Law of Sines here in the Lecture Notes but you can also watch a video of the derivation:



CLICK HERE to see a video showing the derivation of the Law of Sines.

To derive the Law of Sines, let's construct a segment h in the triangle given in Figure 1 that connects the vertex of angle C to the side c; this segment should be perpendicular to side c and is called a *height* of the triangle; see Figure 2.

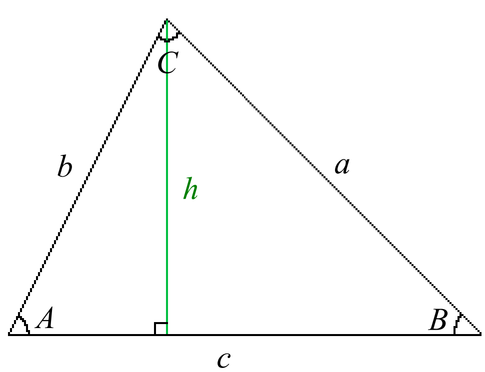


Figure 2

The segment h splits the triangle into two right triangles on which we can apply what we know about right triangle trigonometry; see Figure 3.

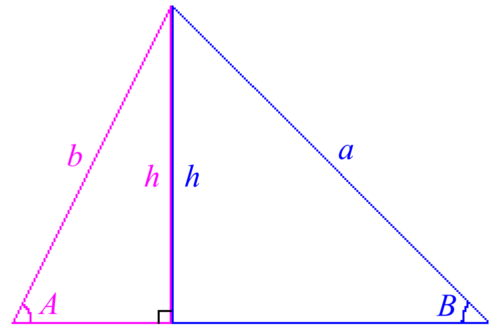


Figure 3

We can use the two right triangles in Figure 3 to obtain expressions for both $\sin(A)$ and $\sin(B)$:

$$\sin(A) = \frac{h}{b} \quad \text{and} \quad \sin(B) = \frac{h}{a}$$

We can now solve both of these equations for h :

$$\sin(A) = \frac{h}{b} \quad \text{and} \quad \sin(B) = \frac{h}{a}$$

$$\Rightarrow \; h = b\sin(A) \quad \text{and} \quad h = a\sin(B)$$

Now, since both of the h's represent the length of the same segment, they are equal. By setting the h's equal to each other we obtain the following:

$$b\sin(A) = a\sin(B)$$

This equation provides us with what is known as the Law of Sines. Typically, the law is written in terms of ratios. If we divide both sides by $a \cdot b$ we obtain the following.

$$b\sin(A) = a\sin(B)$$

$$\Rightarrow \frac{b\sin(A)}{a \cdot b} = \frac{a\sin(B)}{a \cdot b}$$

$$\Rightarrow \frac{\sin(A)}{a} = \frac{\sin(B)}{b}$$

(Note that the law also holds with angle C and side c since the analysis we've shown above also holds if we focus on this angle and side.)

THE LAW OF SINES

If a triangle's sides and angles are labeled like the triangle in Figure 4 then

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b}$$

This is an **identity** since it is true for *all* triangles.

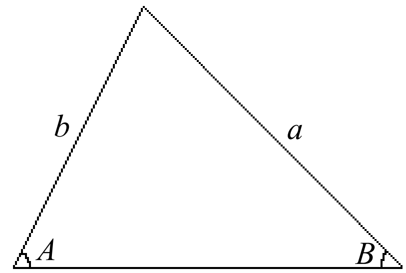


Figure 4

EXAMPLE 1: Find all of the missing angles and side-lengths of the triangle given in Fig, 5. (The triangle is not necessarily drawn to scale.)

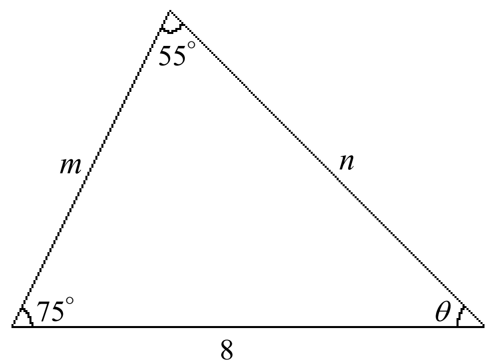


Figure 5

SOLUTION:

We can easily find θ since we know that the sum of the angle-measures in a triangle is always 180°. So

$$\theta + 55^\circ + 75^\circ = 180^\circ$$
$$\Rightarrow \qquad \theta = 50^\circ.$$

Now we can use the Law of Sines to find m and n. (Be sure your calculator is in degree mode to approximate these values.)

$$\frac{\sin(\theta)}{m} = \frac{\sin(55^\circ)}{8} \quad \Rightarrow \quad \frac{m}{\sin(\theta)} = \frac{8}{\sin(55^\circ)}$$
$$\Rightarrow \frac{m}{\sin(50^\circ)} = \frac{8}{\sin(55^\circ)}$$
$$\Rightarrow \qquad m = \frac{8 \cdot \sin(50^\circ)}{\sin(55^\circ)} \approx 7.48$$

and

$$\frac{\sin(75^\circ)}{n} = \frac{\sin(55^\circ)}{8} \quad \Rightarrow \quad \frac{n}{\sin(75^\circ)} = \frac{8}{\sin(55^\circ)}$$

$$\Rightarrow \quad n = \frac{8 \cdot \sin(75^\circ)}{\sin(55^\circ)} \approx 9.43$$

In Figure 6 we've drawn the given triangle and included all of its angle and side-length measures.

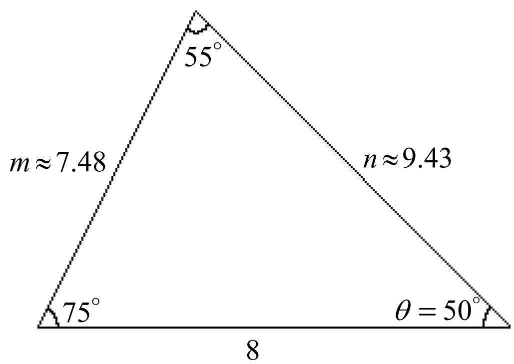


Figure 6

Notice that the longest side is opposite the largest angle and the shortest side is opposite the smallest angle. This is *always* true for a triangle and a *very* helpful fact to keep in mind so that you can make sure that your answer is even possibly correct.

Notice that the Law of Sines involves two angles and the two sides opposite those angles. In order to use the Law of Sines to find a missing part of missing part of a triangle, we need to know three of these for things. So the Law of Sines only is helpful if we know the length of two of the sides and the measure of the angle opposite one of these sides or if we know the measure of two angles and the length of the side opposite one of these angles. In the example above, we were able to use the Law of Sines since we were given the measure of two angles and the length of the side opposite one of these angles. Consider the triangle in Fig. 7:

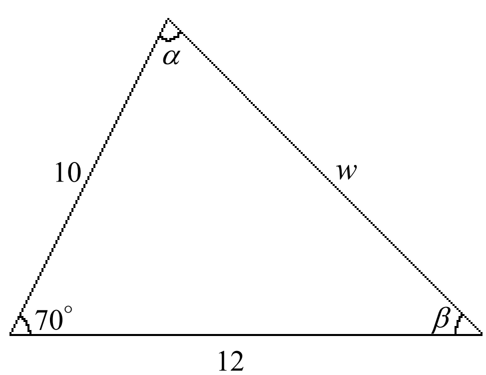
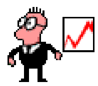


Figure 7

In this triangle, we are not given enough information to use the Law of Sines since we aren't given any "angle and side opposite" combination. (In other words, if we know a side's length, we don't know the opposite angle's measure, and if we know the angle's measure, we don't know the opposite side's length.) In order to find the missing measurements of this triangle, we need another law: the **Law of Cosines**. Let's derive the Law of Cosines just as we derived the Law of Sines earlier in this chapter.

The Law of Cosines

We'll work through the derivation of the Law of Cosines here in the Lecture Notes but you can also watch a video of the derivation:



CLICK HERE to see a video showing the derivation of the Law of Cosines.

To derive the Law of Cosines, let's start with a generic triangle and draw the height, h, just as we did when we derived the Law of Sines; see Figure 8.

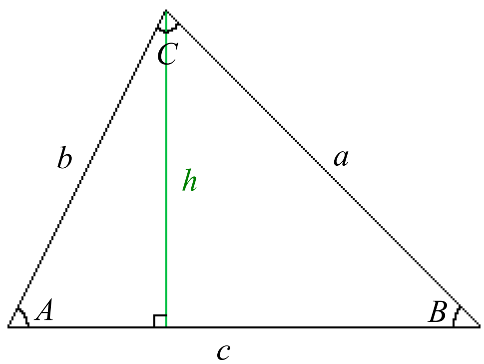


Figure 8

Again we want to consider the two right triangles induced by constructing the line h on the triangle given in Figure 8. This time we want to use the two pieces that the side c is split into. Let's call the segment on the left (the one closest to angle A) x and then the segments on the right must be $c-x$ units long. We've emphasized the two right triangles and labeled the two pieces of side c in Figure 9.

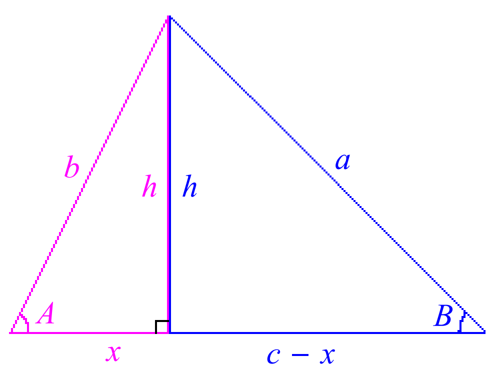


Figure 9

First, notice that the following is true:

$$\cos(A) = \tfrac{x}{b}$$
$$\Rightarrow \quad x = b\cos(A).$$

We'll use this fact later. Now let's apply the Pythagorean Theorem to each of the two triangles in Figure 9. The pink triangle on the left gives us

$$x^2 + h^2 = b^2,$$

and we can solve this for h^2 and obtain

$$h^2 = b^2 - x^2.$$

The blue triangle on the right gives us

$$(c - x)^2 + h^2 = a^2$$

and we can use the fact that $h^2 = b^2 - x^2$ to eliminate h from this equation:

$$(c - x)^2 + h^2 = a^2$$
$$\Rightarrow (c - x)^2 + (b^2 - x^2) = a^2.$$

Finally, we can simplify the left side of this equation and use the fact that $x = b\cos(A)$ to eliminate x:

$$(c - x)^2 + (b^2 - x^2) = a^2$$
$$\Rightarrow c^2 - 2cx + x^2 + b^2 - x^2 = a^2$$
$$\Rightarrow c^2 - 2cx + b^2 = a^2$$
$$\Rightarrow c^2 - 2c \cdot b\cos(A) + b^2 = a^2 \quad \text{(since } x = b\cos(A)\text{)}$$

This last equation is known as the Law of Cosines. Below we've re-written the law in its standard form.

THE LAW OF COSINES

If a triangle's sides and angles are labeled like the triangle in Figure 10 then

$$a^2 = b^2 + c^2 - 2bc \cdot \cos(A)$$

This is an **identity** since it is true for *all* triangles.

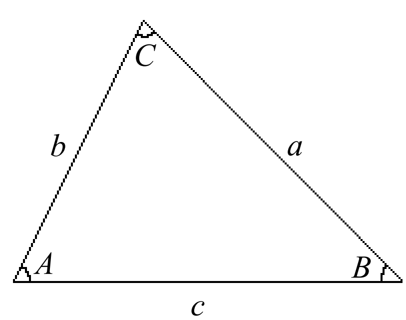


Figure 10

Notice that when $A = 90^\circ$, the Law of Cosines is equivalent to the Pythagorean theorem. (Verify this by substituting 90° for A.) For this reason, the **Law of Cosines is considered the generalization of the Pythagorean Theorem**.

EXAMPLE 2: Find all of the missing angles and side-lengths of the triangle given in Fig. 11. (The triangle is not necessarily drawn to scale.)

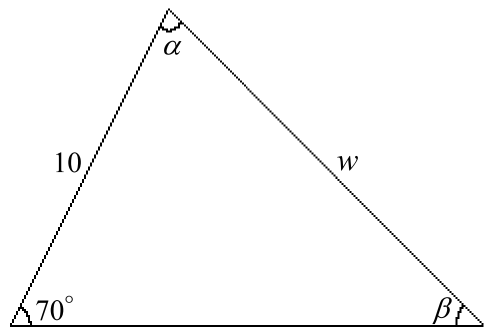


Figure 11

SOLUTION:

We can use the Law of Cosines to find w. (Be sure your calculator is in degree mode to approximate this value.)

$$w^2 = 12^2 + 10^2 - 2(12)(10)\cdot\cos(70^\circ)$$
$$\Rightarrow\ w^2 = 144 + 100 - 240\cdot\cos(70^\circ)$$
$$\Rightarrow\ w = \sqrt{144 + 100 - 240\cdot\cos(70^\circ)}$$
$$\Rightarrow\ w \approx 12.725$$

Now that we know w, we now know an angle and the side opposite that angle, so we can use the Law of Sines to find either of the other angles. Let's find β.

$$\frac{\sin(\beta)}{10} = \frac{\sin(70^\circ)}{w}$$
$$\Rightarrow\ \sin(\beta) = \frac{10\cdot\sin(70^\circ)}{w}$$
$$\Rightarrow\ \sin^{-1}\left(\sin(\beta)\right) = \sin^{-1}\left(\frac{10\cdot\sin(70^\circ)}{w}\right)$$
$$\Rightarrow\ \beta = \sin^{-1}\left(\frac{10\cdot\sin(70^\circ)}{w}\right)$$
$$\Rightarrow\ \beta \approx 47.6^\circ$$ using the fact that $w \approx 12.725$

Finally, we can find α by using the fact that the sum of the angle-measures in a triangle is always 180°:

$$\begin{aligned} &\alpha + \beta + 70^\circ = 180^\circ \\ \Rightarrow\quad &\alpha + 47.6^\circ + 70^\circ \approx 180^\circ \\ \Rightarrow\quad &\alpha \approx 62.4^\circ \end{aligned}$$

In Figure 12 we've drawn the given triangle and included all of its angle and side-length measures.

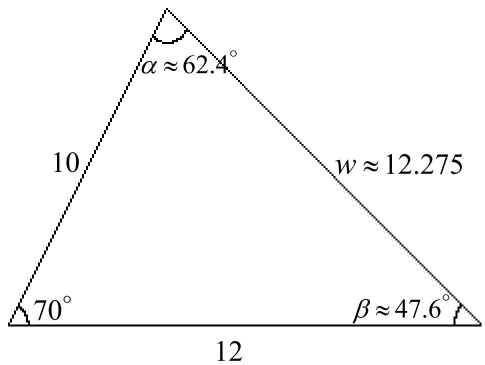


Figure 12

Notice that as with the triangle given in Figure 6 (and with *all* triangles) the longest side is opposite the largest angle and the shortest side is opposite the smallest angle. If this weren't the case, we would know that our answer was incorrect and know that we've made a mistake.

EXAMPLE 3: If a triangle has a side of length 6 units, a side of length 8 units, and the angle opposite the side of length 6 units measures 35°, find the missing side-length and the missing angle-measures.

SOLUTION:

First, we need to translate the information on a drawn triangle; see Figure 13.

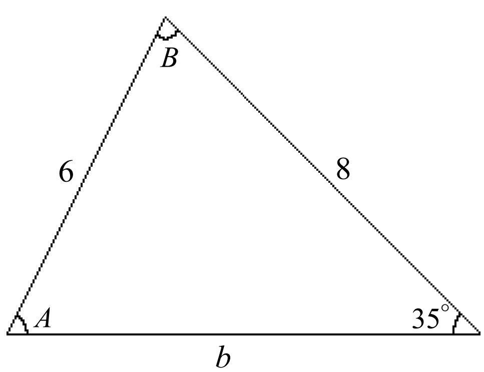


Figure 13

The way we've drawn the triangle could reasonable, but the information we've been given is ambiguous enough that there is an entirely different way we could draw a triangle with the given angle measure and side lengths. Imagine pivoting the 6 unit segment at the vertex of angle B. Since this segment is shorter than the 8 unit segment, it's possible that it intersects segment b at another location closer to the 35° angle. This will create a second possible angle A and a corresponding second possible angle B and second possible side b. In Figure 14, we've shown the two possible triangles with the same given information.

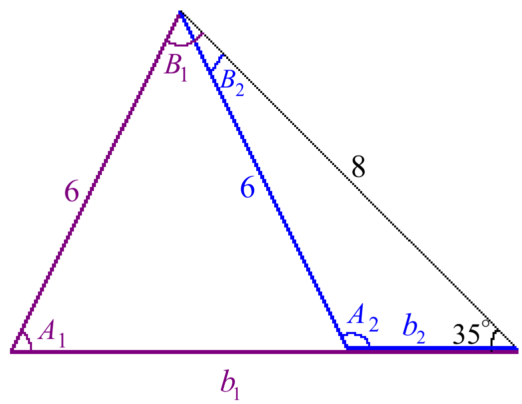


Figure 14

We'll start with the triangle containing A_1, B_1, and b_1. (Part of this triangle is purple in Figure 14.) First, let's use the Law of Sines to find A_1:

$$\frac{\sin(A_1)}{8} = \frac{\sin(35^\circ)}{6}$$

$$\Rightarrow \quad \sin(A_1) = \frac{8\sin(35^\circ)}{6}$$

$$\Rightarrow \sin^{-1}\left(\sin(A_1)\right) = \sin^{-1}\left(\frac{8\sin(35^\circ)}{6}\right)$$

$$\Rightarrow \quad A_1 = \sin^{-1}\left(\frac{8\sin(35^\circ)}{6}\right)$$

$$\Rightarrow \quad A_1 \approx 49.9^\circ.$$

We can now find B_1 by using the fact that the sum of the angle-measures in a triangle is always 180°:

$$A_1 + B_1 + 35^\circ = 180^\circ$$

$$\Rightarrow 49.9^\circ + B_1 + 35^\circ \approx 180^\circ$$

$$\Rightarrow \quad B_1 \approx 95.1^\circ.$$

Now that we know the measure of the angle opposite b_1, we can use the Law of Sines to find it:

$$\frac{\sin(B_1)}{b_1} = \frac{\sin(35^\circ)}{6} \quad \Rightarrow \quad \frac{\sin(95.1^\circ)}{b_1} \approx \frac{\sin(35^\circ)}{6}$$

$$\Rightarrow \quad \frac{b_1}{\sin(95.1^\circ)} \approx \frac{6}{\sin(35^\circ)}$$

$$\Rightarrow \quad b_1 \approx \frac{6 \cdot \sin(95.1^\circ)}{\sin(35^\circ)} \approx 10.42.$$

These values for A_1, B_1, and b_1 give us possible triangle with the given measurements. We've drawn this triangle (not to scale) in Figure 15.

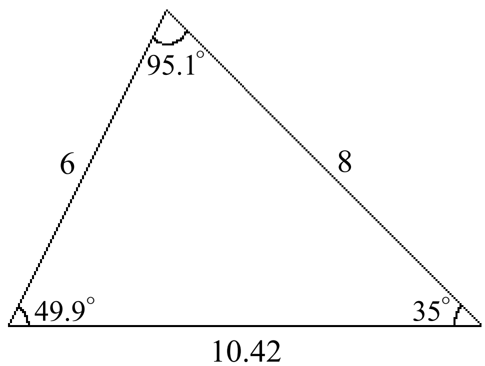


Figure 15

We've done everything correctly and found **one** triangle using all of the given information…so how will we find the values of A_2, B_2, and b_2 for the other triangle we drew in Figure 14? In order to find this other possible triangle, we have to remember the problem with using the inverse sine function, which we used to find A_1. Recall that the inverse sine function has range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$; thus, when we use the inverse sine function we will *only* obtain acute angles (i.e., angles that measure less than 90°). In order to find the obtuse (i.e., greater than 90°) possibility, we need to use the identity $\sin(\theta) = \sin(\pi - \theta)$ that we first noticed in Section I: Chapter 3. (Since we're currently working with degrees instead of radians, let's use $\sin(\theta) = \sin(180^\circ - \theta)$ instead.) Above we found that

$$\sin(A_1) = \frac{8\sin(35^\circ)}{6}$$

$$\Rightarrow \quad A_1 = \sin^{-1}\left(\frac{8\sin(35^\circ)}{6}\right) \approx 49.9^\circ.$$

The identity tells us that $\sin(A_1) = \sin(180^\circ - A_1)$, so we can let $A_2 = 180^\circ - A_1$ and we will have another angle, A_2, such that $\sin(A_2) = \dfrac{8\sin(35^\circ)}{6}$. Thus,

$$\begin{aligned} A_2 &= 180^\circ - A_1 \\ &\approx 180^\circ - 49.9^\circ \approx 130.1^\circ. \end{aligned}$$

We can now find B_2 by using the fact that the sum of the angle-measures in a triangle is always 180°:

$$\begin{aligned} A_2 + B_2 + 35^\circ = 180^\circ \quad &\Rightarrow \quad 130.1^\circ + B_2 + 35^\circ \approx 180^\circ \\ &\Rightarrow \quad B_2 \approx 14.9^\circ. \end{aligned}$$

Finally, we can use the Law of Sines to find b_2:

$$\begin{aligned} \frac{\sin(B_2)}{b_2} = \frac{\sin(35^\circ)}{6} \quad &\Rightarrow \quad \frac{\sin(14.9^\circ)}{b_2} \approx \frac{\sin(35^\circ)}{6} \\ &\Rightarrow \quad \frac{b_2}{\sin(14.9^\circ)} \approx \frac{6}{\sin(35^\circ)} \\ &\Rightarrow \quad b_2 \approx \frac{6 \cdot \sin(14.9^\circ)}{\sin(35^\circ)} \approx 2.69. \end{aligned}$$

These values for A_2, B_2, and b_2 give us another possible triangle with the given measurements. We've drawn this triangle (not to scale) in Figure 16.

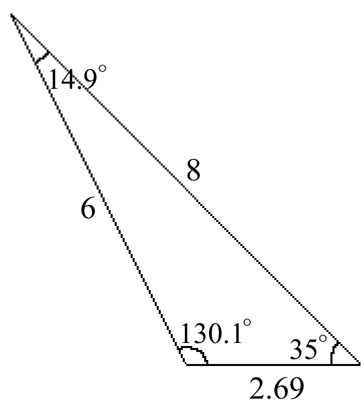


Figure 16

Whenever we're given two side-lengths and one angle measure and use the Law of Sines, it's possible that there will be two different triangles that satisfy the given information.

Section II: Trigonometric Identities

Chapter 3: Proving Trigonometric Identities

This quarter we've studied many important trigonometric identities. Because these identities are so useful, it is worthwhile to learn (or memorize) most of them. But there are many other identities that aren't particularly important (so they aren't worth memorizing) but they exist and they offer us an opportunity to learn another skill: proving mathematical statements. In this chapter we will *prove* that some equations are in fact identities.

Recall that an **identity** is an equation that is true for all values in the domains of the involved expressions. Thus, to prove an identity we need to show that the two sides of the equation are *always* equal. To accomplish this, we need to start with the expression on one side of the equation and use the rules of algebra as well as the identities that we've already studied to manipulate the expression until it's identical to the expression on the other side of the equation. Let's look at a few examples to help you make sense of this procedure.

EXAMPLE 1: Prove the identity $\sin(x) = \dfrac{\tan(x)}{\sec(x)}$.

As we mentioned above, we prove identities by manipulating the expression on one side of the equation (using the rules of algebra and the identities we already know) until it looks like the expression on the other side of the equation. We can start with either side of the equation, but it's usually most sensible to start with the "more complicated" side since it will be easier to manipulate it. In this example, $\dfrac{\tan(x)}{\sec(x)}$ is "more complicated" than $\sin(x)$, so let's start with $\dfrac{\tan(x)}{\sec(x)}$ and try to manipulate it until it looks like $\sin(x)$.

$$\frac{\tan(x)}{\sec(x)} = \frac{\frac{\sin(x)}{\cos(x)}}{\frac{1}{\cos(x)}} \qquad \leftarrow \text{since } \tan(x) = \frac{\sin(x)}{\cos(x)} \text{ and } \sec(x) = \frac{1}{\cos(x)}$$

$$= \frac{\sin(x)}{\cos(x)} \cdot \frac{\cos(x)}{1} \qquad \leftarrow \text{using the rules of algebra}$$

$$= \sin(x) \qquad \leftarrow \text{more algebra}$$

This is a *proof* of the identity $\sin(x) = \dfrac{\tan(x)}{\sec(x)}$ since we've shown that the right side is equivalent to the left side.

EXAMPLE 2: Prove the identity $\cot(x) + \tan(x) = \csc(x)\sec(x)$.

Here, both sides are equally "complicated" so it's not obvious which side we should start with. In such a case, just start with *either* side and see what happens. If you get stuck, start over using the other side. Let's start with the left side:

$$\cot(x) + \tan(x) = \frac{\cos(x)}{\sin(x)} + \frac{\sin(x)}{\cos(x)} \quad \leftarrow \text{since } \cot(x) = \frac{\cos(x)}{\sin(x)} \text{ and } \tan(x) = \frac{\sin(x)}{\cos(x)}$$

$$= \frac{\cos(x)}{\sin(x)} \cdot \frac{\cos(x)}{\cos(x)} + \frac{\sin(x)}{\cos(x)} \cdot \frac{\sin(x)}{\sin(x)} \quad \leftarrow \text{using the rules of algebra to obtain a common denominator}$$

$$= \frac{\cos^2(x) + \sin^2(x)}{\sin(x)\cos(x)}$$

$$= \frac{1}{\sin(x)\cos(x)} \quad \leftarrow \text{since } \cos^2(x) + \sin^2(x) = 1$$

$$= \frac{1}{\sin(x)} \cdot \frac{1}{\cos(x)} \quad \leftarrow \text{using more algebra}$$

$$= \csc(x)\sec(x) \quad \leftarrow \text{since } \csc(x) = \frac{1}{\sin(x)} \text{ and } \sec(x) = \frac{1}{\cos(x)}$$

This is a proof that $\cot(x) + \tan(x) = \csc(x)\sec(x)$ since we've shown that the left side is equivalent to the right side.

For the next examples, we'll need to use variations of the Pythagorean Identity discussed at the end of Section I: Chapter 3.

EXAMPLE 3: Prove the identity $\frac{\left(1 + \cos(t)\right)\left(1 - \cos(t)\right)}{\sin(t)} = \sin(t)$.

The left side is more "complicated", so we'll start with it:

$$\frac{\left(1 + \cos(t)\right)\left(1 - \cos(t)\right)}{\sin(t)} = \frac{1 - \cos(t) + \cos(t) - \cos^2(t)}{\sin(t)} \quad \leftarrow \text{using algebra}$$

$$= \frac{1 - \cos^2(t)}{\sin(t)} \quad \leftarrow \text{using algebra}$$

$$= \frac{\sin^2(t)}{\sin(t)} \quad \leftarrow \text{since } 1 - \cos^2(t) = \sin^2(t)$$

$$= \sin(t) \quad \leftarrow \text{using more algebra}$$

This is a proof that $\dfrac{(1+\cos(t))(1-\cos(t))}{\sin(t)} = \sin(t)$ since we've shown that the left side is equivalent to the right side.

EXAMPLE 4: Prove the identity $\dfrac{\cos(\theta)}{1-\sin(\theta)} = \dfrac{1+\sin(\theta)}{\cos(\theta)}$.

The proof of this identity requires a relatively commonly employed "trick" that is important to become familiar with. Sometimes it is useful to use the **conjugate** of some part (often the denominator) of the expression. The **conjugate** of the expression $a+b$ is the expression $a-b$, and vice versa. As you can see in the proof below, by multiplying the denominator by its conjugate, allows us to manipulate the left side so that it is equivalent to the right side.

$$\begin{aligned}
\frac{\cos(\theta)}{1-\sin(\theta)} &= \frac{\cos(\theta)}{1-\sin(\theta)} \cdot \frac{1+\sin(\theta)}{1+\sin(\theta)} && \leftarrow \text{using the conjugate of } 1-\sin(\theta) \\
&= \frac{\cos(\theta)(1+\sin(\theta))}{(1-\sin(\theta))(1+\sin(\theta))} \\
&= \frac{\cos(\theta)(1+\sin(\theta))}{1-\sin(\theta)+\sin(\theta)-\sin^2(\theta)} \\
&= \frac{\cos(\theta)(1+\sin(\theta))}{1-\sin^2(\theta)} \\
&= \frac{\cos(\theta)(1+\sin(\theta))}{\cos^2(\theta)} && \leftarrow \text{since } 1-\sin^2(\theta) = \cos^2(\theta) \\
&= \frac{\cancel{\cos(\theta)}(1+\sin(\theta))}{\cos^{\cancel{2}}(\theta)} \\
&= \frac{1+\sin(\theta)}{\cos(\theta)}
\end{aligned}$$

This is a proof that $\dfrac{\cos(\theta)}{1-\sin(\theta)} = \dfrac{1+\sin(\theta)}{\cos(\theta)}$ since we've shown that the left side is equivalent to the right side.

For the Example 5 we'll need to use a variation of the identity $\tan^2(\theta) + 1 = \sec^2(\theta)$ discussed at the end of Section I, Chapter 3.

EXAMPLE 5: Prove the identity $\dfrac{1}{\sec(u) - \tan(u)} - \dfrac{1}{\sec(u) + \tan(u)} = 2\tan(u)$.

To prove this identity, we will again use the **conjugate** of the denominators of the expressions on the left side of the equation:

$$\begin{aligned}
\frac{1}{\sec(u)-\tan(u)} - \frac{1}{\sec(u)+\tan(u)} &= \frac{1}{\sec(u)-\tan(u)}\cdot\frac{\sec(u)+\tan(u)}{\sec(u)+\tan(u)} - \frac{1}{\sec(u)+\tan(u)}\cdot\frac{\sec(u)-\tan(u)}{\sec(u)-\tan(u)} \\
&= \frac{\sec(u) + \tan(u) - \big(\sec(u) - \tan(u)\big)}{\big(\sec(u) - \tan(u)\big)\big(\sec(u) + \tan(u)\big)} \\
&= \frac{\sec(u) + \tan(u) - \sec(u) + \tan(u)}{\sec^2(u) - \sec(u)\tan(u) + \sec(u)\tan(u) - \tan^2(u)} \\
&= \frac{2\tan(u)}{\sec^2(u) - \tan^2(u)} \\
&= \frac{2\tan(u)}{1} \qquad \leftarrow \text{since } \sec^2(u) - \tan^2(u) = 1 \\
&= 2\tan(u)
\end{aligned}$$

Section II: Trigonometric Identities

Chapter 4: Sum and Difference Identities

In this chapter we'll study identities that allow us to change the form of expressions involving the sine or cosine of the sum or difference of two angles. Although we have the tools necessary to prove these identities, we won't bother proving them here. (You can see proofs in §6.5 in our textbook.) We'll just state the identities and then see some examples of how these identities can be used.

THE SUM- AND DIFFERENCE-OF-ANGLES IDENTITIES

sine :

$$\sin(A+B) = \sin(A)\cos(B) + \sin(B)\cos(A)$$
$$\sin(A-B) = \sin(A)\cos(B) - \sin(B)\cos(A)$$

cosine :

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$
$$\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B)$$

EXAMPLE 1: Use the sum-of-angles or the difference-of-angles identities to calculate the following.

a. $\cos(75^\circ)$ **b.** $\sin(-15^\circ)$ **c.** $\sin\left(\frac{11\pi}{12}\right)$

SOLUTIONS:

a. Since $75^\circ = 45^\circ + 30^\circ$, we can use the cosine of a sum-of-angles identity to calculate $\cos(75^\circ)$:

$$\begin{aligned}\cos(75^\circ) &= \cos(45^\circ + 30^\circ)\\ &= \cos(45^\circ)\cos(30^\circ) - \sin(45^\circ)\sin(30^\circ)\\ &= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2}\\ &= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

b. Since $-15^\circ = 30^\circ - 45^\circ$, we can use the sine of a difference-of-angles identity to calculate $\sin(-15^\circ)$:

$$\begin{aligned}\sin(-15^\circ) &= \sin(30^\circ - 45^\circ)\\ &= \sin(30^\circ)\cos(45^\circ) - \sin(45^\circ)\cos(30^\circ)\\ &= \frac{1}{2}\cdot\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}\\ &= \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}\\ &= \frac{\sqrt{2}-\sqrt{6}}{4}\end{aligned}$$

c. Since

$$\begin{aligned}\frac{11\pi}{12} &= \frac{3\pi}{12} + \frac{8\pi}{12}\\ &= \frac{\pi}{4} + \frac{2\pi}{3},\end{aligned}$$

we can use the sine of a sum-of-angles identity to calculate $\sin\left(\frac{11\pi}{12}\right)$:

$$\begin{aligned}\sin\left(\frac{11\pi}{12}\right) &= \sin\left(\frac{\pi}{4} + \frac{2\pi}{3}\right)\\ &= \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{2\pi}{3}\right) + \sin\left(\frac{2\pi}{3}\right)\cos\left(\frac{\pi}{4}\right)\\ &= \frac{\sqrt{2}}{2}\cdot\left(-\frac{1}{2}\right) + \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}\\ &= -\frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}\\ &= \frac{-\sqrt{2}+\sqrt{6}}{4}.\end{aligned}$$

EXAMPLE 2: Use the sum-of-angles or the difference-of-angles identities to calculate the following $\tan\left(\frac{7\pi}{12}\right)$.

SOLUTION:

To calculate $\tan\left(\frac{7\pi}{12}\right)$, we need to use the fact that $\tan\left(\frac{7\pi}{12}\right) = \frac{\sin\left(\frac{7\pi}{12}\right)}{\cos\left(\frac{7\pi}{12}\right)}$ along with the sum-of-angles identities since

$$\begin{aligned}\frac{7\pi}{12} &= \frac{3\pi}{12} + \frac{4\pi}{12}\\ &= \frac{\pi}{4} + \frac{\pi}{3}.\end{aligned}$$

Therefore,

$$\begin{aligned}\tan\left(\frac{7\pi}{12}\right) &= \frac{\sin\left(\frac{7\pi}{12}\right)}{\cos\left(\frac{7\pi}{12}\right)}\\ &= \frac{\sin\left(\frac{\pi}{4}+\frac{\pi}{3}\right)}{\cos\left(\frac{\pi}{4}+\frac{\pi}{3}\right)}\\ &= \frac{\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}\right)+\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right)}{\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}\right)-\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{3}\right)}\\ &= \frac{\frac{\sqrt{2}}{2}\cdot\frac{1}{2}+\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}\cdot\frac{1}{2}-\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}}\\ &= \frac{\frac{\sqrt{2}+\sqrt{6}}{4}}{\frac{\sqrt{2}-\sqrt{6}}{4}}\\ &= \frac{\sqrt{2}+\sqrt{6}}{\sqrt{2}-\sqrt{6}}\cdot\frac{\sqrt{2}+\sqrt{6}}{\sqrt{2}+\sqrt{6}} \qquad \text{(use the conjugate of the denominator to rationalize the denominator)}\\ &= \frac{2+2\sqrt{12}+6}{2-6}\\ &= \frac{8+4\sqrt{3}}{-4}\\ &= -2-\sqrt{3}\end{aligned}$$

In §6.5 in our textbook, sum and difference identities are given for tangent. We could have derived those identities here and then employed one of them in Example 2 but instead we've chosen to focus on the identities for sine and cosine, and then utilize these identities when working with tangent (i.e., there's no reason to waste our energy learning identities involving tangent if we can instead use what we already know about sine and cosine).

Section II: Trigonometric Identities

Chapter 5: Double-Angle and Half-Angle Identities

In this chapter we will find identities that will allow us to calculate $\sin(2\theta)$ and $\cos(2\theta)$ if we know the values of $\cos(\theta)$ and $\sin(\theta)$ (we call these "double-angle identities") and we will find identities that will allow us to calculate $\sin\left(\frac{\theta}{2}\right)$ and $\cos\left(\frac{\theta}{2}\right)$ if we know the values of $\cos(\theta)$ and $\sin(\theta)$ (we call these "half-angle identities").

Let's start by finding the **double-angle identities**. [Notice how we will derive these identities differently than in our textbook: our textbook uses the sum and difference identities but we'll use the laws of sine and cosine.] First we'll find an identity for $\sin(2\theta)$:

Recall that the definition of the cosine and sine functions tell us that $\cos(\theta)$ and $\sin(\theta)$ represent the horizontal and vertical coordinates, respectively, of the point specified by the angle θ on the unit circle; see Figure 1.

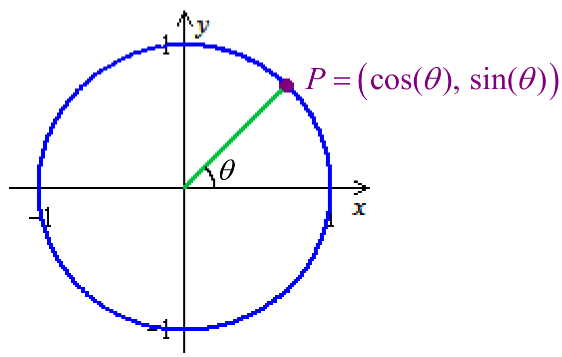


Figure 1: The unit circle with a point P specified by the angle θ.

We can construct a right triangle using the terminal side of angle θ. This triangle has hypotenuse of length 1 unit and sides of length $\cos(\theta)$ and $\sin(\theta)$; see Figure 2.

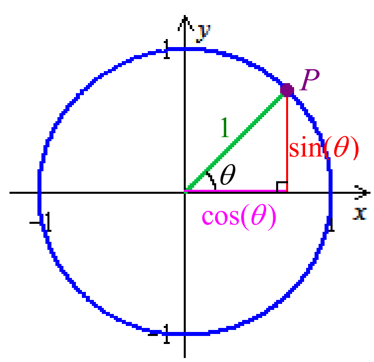


Figure 2: Right triangle with angle θ.

Although we will continue to focus on angle θ, let's label the angle whose vertex is point P; we'll call it α; see Figure 3.

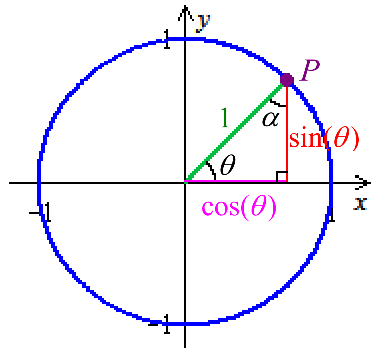


Figure 3

Notice that

$$\begin{aligned}\sin(\alpha) &= \frac{\cos(\theta)}{1} \\ &= \cos(\theta).\end{aligned}$$

We'll use this fact later.

Now let's construct the mirror-image of this triangle below the x-axis in Quadrant IV; see Fig. 4.

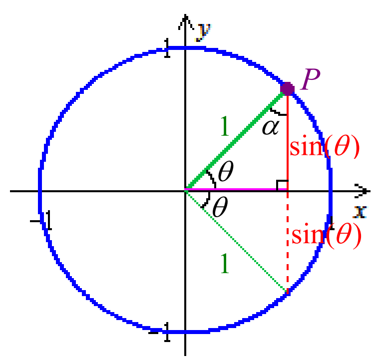


Figure 4

These two right triangles together form a larger non-right triangle that has an angle of measure 2θ; we've emphasized this triangle in Figure 5 by hiding the unit circle and the coordinate plane. (Note that the side opposite 2θ is of length $2\sin(\theta)$ since it consists of two segments each of length $\sin(\theta)$.)

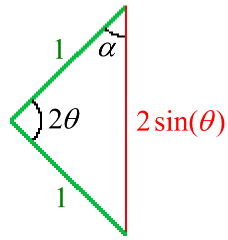


Figure 5

We can use this triangle to find the double-angle identities for cosine and sine. First, let's apply the **Law of Sines** to the triangle in Figure 5 to obtain the double-angle identity for sine.

The Law of Sines tells us that $\dfrac{\sin(2\theta)}{2\sin(\theta)} = \dfrac{\sin(\alpha)}{1}$; since $\sin(\alpha) = \cos(\theta)$ (from above), we can substitute $\cos(\theta)$ for $\sin(\alpha)$:

$$\begin{aligned} &\frac{\sin(2\theta)}{2\sin(\theta)} = \frac{\sin(\alpha)}{1} \\ \Rightarrow\ &\frac{\sin(2\theta)}{2\sin(\theta)} = \frac{\cos(\theta)}{1} \\ \Rightarrow\ &\sin(2\theta) = 2\sin(\theta)\cos(\theta). \end{aligned}$$

This last equation is the **double-angle identity for sine**. Notice that we can use this identity to obtain the value of $\sin(2\theta)$ if we know the values of $\cos(\theta)$ and $\sin(\theta)$.

Now let's find the double-angle identity for cosine. We can use the same triangle we constructed above (we've copied this triangle below in Figure 6), but apply the **Law of Cosines** instead of the Law of Sines.

$$\begin{aligned} &\left(2\sin(\theta)\right)^2 = 1^2 + 1^2 - 2\cdot 1\cdot 1\cdot\cos(2\theta) \\ \Rightarrow\ &4\sin^2(\theta) = 1 + 1 - 2\cos(2\theta) \\ \Rightarrow\ &4\sin^2(\theta) = 2 - 2\cos(2\theta) \\ \Rightarrow\ &2\cos(2\theta) = 2 - 4\sin^2(\theta) \\ \Rightarrow\ &\cos(2\theta) = 1 - 2\sin^2(\theta) \end{aligned}$$

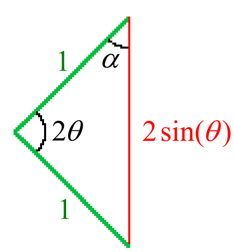


Figure 6

The last equation (above) is the **double-angle identity for cosine**. Notice that we can use this identity to obtain the value of $\cos(2\theta)$ if we know the value of $\sin(\theta)$. We can use the Pythagorean identity to obtain two other forms of the double-angle identity for cosine. Recall that the Pythagorean identity tells us that

$$\sin^2(\theta) + \cos^2(\theta) = 1$$
$$\Rightarrow \quad \sin^2(\theta) = 1 - \cos^2(\theta),$$

and we can now substitute $1 - \cos^2(\theta)$ for $\sin^2(\theta)$ in the double-angle identity to obtain another form of the identity:

$$\cos(2\theta) = 1 - 2\sin^2(\theta)$$
$$\Rightarrow \quad \cos(2\theta) = 1 - 2\left(1 - \cos^2(\theta)\right)$$
$$\Rightarrow \quad \cos(2\theta) = 1 - 2 + 2\cos^2(\theta)$$
$$\Rightarrow \quad \cos(2\theta) = 2\cos^2(\theta) - 1.$$

This last equation is another double-angle identity for cosine. We can obtain a third double-angle identity for cosine by substituting $\sin^2(\theta) + \cos^2(\theta)$ for 1:

$$\cos(2\theta) = 2\cos^2(\theta) - 1$$
$$\Rightarrow \quad \cos(2\theta) = 2\cos^2(\theta) - \left(\sin^2(\theta) + \cos^2(\theta)\right)$$
$$\Rightarrow \quad \cos(2\theta) = 2\cos^2(\theta) - \sin^2(\theta) - \cos^2(\theta)$$
$$\Rightarrow \quad \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta).$$

Below, we've summarized the double-angle identities for sine and cosine:

DOUBLE-ANGLE IDENTITIES

sine : $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$

cosine : $\begin{cases} \cos(2\theta) = 1 - 2\sin^2(\theta) \\ \cos(2\theta) = 2\cos^2(\theta) - 1 \\ \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) \end{cases}$

We could easily find double-angle identities for tangent by using the fact that $\tan(2\theta)=\dfrac{\sin(2\theta)}{\cos(2\theta)}$. But the resulting identities aren't easy to remember so it makes more sense to learn the identities for sine and cosine and use the fact that $\tan(2\theta)=\dfrac{\sin(2\theta)}{\cos(2\theta)}$ if you ever need to calculate $\tan(2\theta)$.

EXAMPLE 1: Suppose that $\sin(\alpha)=\frac{1}{3}$ and that α is in Quadrant II. Find $\sin(2\alpha)$, $\cos(2\alpha)$, and $\tan(2\alpha)$.

SOLUTION:

First, let's find $\cos(\alpha)$ since we need this value to use the double-angle identity for sine. To find $\cos(\alpha)$, let's use the Pythagorean identity:

$$\begin{aligned}
&\sin^2(\alpha)+\cos^2(\alpha)=1\\
\Rightarrow\quad &\left(\tfrac{1}{3}\right)^2+\cos^2(\alpha)=1\\
\Rightarrow\quad &\tfrac{1}{9}+\cos^2(\alpha)=1\\
\Rightarrow\quad &\cos^2(\alpha)=1-\tfrac{1}{9}\\
\Rightarrow\quad &\cos^2(\alpha)=\tfrac{8}{9}\\
\Rightarrow\quad &\cos(\alpha)=-\tfrac{2\sqrt{2}}{3}.
\end{aligned}$$

Note that we take the negative square root of $\frac{8}{9}$ since α is in Quadrant II so that $\cos(\alpha)$ must be negative.

Now we can use the double-angle identities to find $\sin(2\alpha)$ and $\cos(2\alpha)$. Let's start with $\sin(2\alpha)$:

$$\begin{aligned}
\sin(2\alpha)&=2\sin(\alpha)\cos(\alpha)\\
&=2\left(\tfrac{1}{3}\right)\left(-\tfrac{2\sqrt{2}}{3}\right)\\
&=-\tfrac{4\sqrt{2}}{9}.
\end{aligned}$$

To find $\cos(2\alpha)$, we can use any one of the three double-angle identities for cosine. Let's use $\cos(2\alpha) = 1 - 2\sin^2(\alpha)$:

$$\begin{aligned}\cos(2\alpha) &= 1 - 2\sin^2(\alpha) \\ &= 1 - 2\left(\tfrac{1}{3}\right)^2 \\ &= 1 - 2\cdot\tfrac{1}{9} \\ &= 1 - \tfrac{2}{9} \\ &= \tfrac{7}{9}.\end{aligned}$$

You should verify that the other double-angle identities for cosine give the same value for $\cos(2\alpha)$.

To find $\tan(2\alpha)$, we can use the fact that $\tan(2\alpha) = \dfrac{\sin(2\alpha)}{\cos(2\alpha)}$:

$$\begin{aligned}\tan(2\alpha) &= \frac{\sin(2\alpha)}{\cos(2\alpha)} \\ &= \frac{-\frac{4\sqrt{2}}{9}}{\frac{7}{9}} \\ &= -\frac{4\sqrt{2}}{9}\cdot\frac{9}{7} \\ &= -\frac{4\sqrt{2}}{7}.\end{aligned}$$

Half-Angle Identities

We can use the double-angle identities for cosine to derive **half-angle identities**.

Recall that $\cos(2\theta) = 1 - 2\sin^2(\theta)$ we can use this identity to find a half-angle identity for sine. Let $\alpha = 2\theta$. Then $\theta = \frac{\alpha}{2}$ and

$$\begin{aligned}
&\cos(2\theta) = 1 - 2\sin^2(\theta) \\
\Rightarrow\quad &\cos(\alpha) = 1 - 2\sin^2\left(\tfrac{\alpha}{2}\right) \\
\Rightarrow\quad &2\sin^2\left(\tfrac{\alpha}{2}\right) = 1 - \cos(\alpha) \\
\Rightarrow\quad &\sin^2\left(\tfrac{\alpha}{2}\right) = \frac{1 - \cos(\alpha)}{2} \\
\Rightarrow\quad &\sin\left(\tfrac{\alpha}{2}\right) = \pm\sqrt{\frac{1 - \cos(\alpha)}{2}}
\end{aligned}$$

(we choose the sign based on the quadrant of $\frac{\alpha}{2}$)

We can use $\cos(2\theta) = 2\cos^2(\theta) - 1$ to find a half-angle identity for cosine. Again, suppose that $\alpha = 2\theta$. Then $\theta = \frac{\alpha}{2}$ and

$$\begin{aligned}
&\cos(2\theta) = 2\cos^2(\theta) - 1 \\
\Rightarrow\quad &\cos(\alpha) = 2\cos^2\left(\tfrac{\alpha}{2}\right) - 1 \\
\Rightarrow\quad &1 + \cos(\alpha) = 2\cos^2\left(\tfrac{\alpha}{2}\right) \\
\Rightarrow\quad &\frac{1 + \cos(\alpha)}{2} = \cos^2\left(\tfrac{\alpha}{2}\right) \\
\Rightarrow\quad &\cos\left(\tfrac{\alpha}{2}\right) = \pm\sqrt{\frac{1 + \cos(\alpha)}{2}}
\end{aligned}$$

(we choose the sign based on the quadrant of $\frac{\alpha}{2}$)

Since these identities are valid for *any* value of α, we can express the identities in terms of θ since that's how we've been expressing our identities.

HALF-ANGLE IDENTITIES

sine : $\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1 - \cos(\theta)}{2}}$ (we choose the sign based on the quadrant of $\frac{\theta}{2}$)

cosine : $\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1 + \cos(\theta)}{2}}$ (we choose the sign based on the quadrant of $\frac{\theta}{2}$)

EXAMPLE 2: Recall from Example 1 that $\sin(\alpha) = \frac{1}{3}$ and that α is in Quadrant II. Find $\cos\left(\frac{\alpha}{2}\right)$, $\sin\left(\frac{\alpha}{2}\right)$, and $\tan\left(\frac{\alpha}{2}\right)$.

SOLUTION:

First, note that since α is in Quadrant II, $\frac{\pi}{2} < \alpha < \pi$. Thus,

$$\frac{\pi/2}{2} < \frac{\alpha}{2} < \frac{\pi}{2}$$
$$\Rightarrow \frac{\pi}{4} < \frac{\alpha}{2} < \frac{\pi}{2}$$

Thus, $\frac{\alpha}{2}$ is in Quadrant I so it's sine and cosine values are positive.

In Example 1 we used the Pythagorean identity to determine that $\cos(\alpha) = -\frac{2\sqrt{2}}{3}$; we can use this value and the half-angle identities to find $\cos\left(\frac{\alpha}{2}\right)$ and $\sin\left(\frac{\alpha}{2}\right)$:

$$\sin\left(\frac{\alpha}{2}\right) = +\sqrt{\frac{1-\cos(\alpha)}{2}} \quad \text{(we choose + since } \tfrac{\alpha}{2} \text{ is in Quadrant I)}$$
$$= \sqrt{\frac{\left(\frac{3}{3} - \left(-\frac{2\sqrt{2}}{3}\right)\right)}{2}}$$
$$= \sqrt{\frac{1}{2}\left(\frac{3+2\sqrt{2}}{3}\right)}$$
$$= \sqrt{\frac{3+2\sqrt{2}}{6}}$$

Similarly,

$$\cos\left(\frac{\alpha}{2}\right) = +\sqrt{\frac{1+\cos(\alpha)}{2}} \quad \text{(we choose + since } \tfrac{\alpha}{2} \text{ is in Quadrant I)}$$
$$= \sqrt{\frac{\left(\frac{3}{3} + \left(-\frac{2\sqrt{2}}{3}\right)\right)}{2}}$$
$$= \sqrt{\frac{1}{2}\left(\frac{3-2\sqrt{2}}{3}\right)}$$
$$= \sqrt{\frac{3-2\sqrt{2}}{6}}$$

Now we can find $\tan\left(\frac{\alpha}{2}\right)$ using the fact that $\tan\left(\frac{\alpha}{2}\right) = \frac{\sin\left(\frac{\alpha}{2}\right)}{\cos\left(\frac{\alpha}{2}\right)}$:

$$\begin{aligned}\tan\left(\frac{\alpha}{2}\right) &= \frac{\sin\left(\frac{\alpha}{2}\right)}{\cos\left(\frac{\alpha}{2}\right)}\\ &= \frac{\sqrt{\frac{3+2\sqrt{2}}{6}}}{\sqrt{\frac{3-2\sqrt{2}}{6}}}\\ &= \frac{\sqrt{3+2\sqrt{2}}}{\sqrt{3-2\sqrt{2}}}\end{aligned}$$

Note that it's easy to derive a half-angle identity for tangent but, as we discussed when we studied the double-angle identities, we can always use sine and cosine values to find tangent values so there's no reason to waste energy on additional identities for tangent.

Section III: Polar Coordinates and Complex Numbers

Chapter 1: Introduction to Polar Coordinates

We are all comfortable using rectangular (i.e., Cartesian) coordinates to describe points on the plane. For example, we've plotted the point $P = \left(\sqrt{3}, 1\right)$ on the coordinate plane in Figure 1.

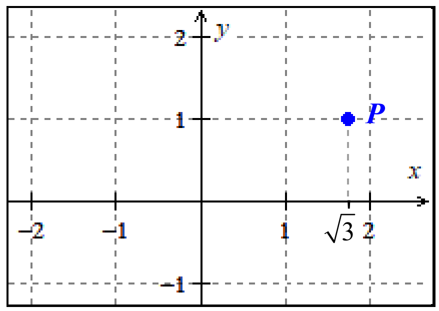


Figure 1: The point $P = \left(\sqrt{3}, 1\right)$ on the rectangular coordinate plane.

Instead of using these *rectangular* coordinates, we can use a *circular* coordinate system to describe points on the plane: **Polar Coordinates**. Ordered pairs in polar coordinates have form (r, θ) where r represents the point's distance from the origin and θ represents the angular displacement of the point with respect to the positive x-axis. Let's find the polar coordinates that describe P in Figure 1:

First let's find r, the distance from point P to the origin; in other words, we need to find the length of the segment labeled r in Figure 2. We can use the Pythagorean Theorem:

$$
\begin{aligned}
& r^2 = \left(\sqrt{3}\right)^2 + (1)^2 \\
\Rightarrow\ & r^2 = 3 + 1 \\
\Rightarrow\ & r^2 = 4 \\
\Rightarrow\ & r = 2
\end{aligned}
$$

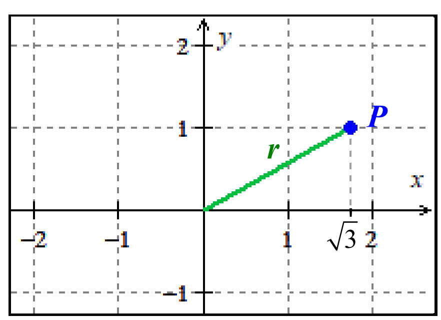


Figure 2

Now we need to find the angle between the positive x-axis and r; this angle is labeled θ in Figure 3.

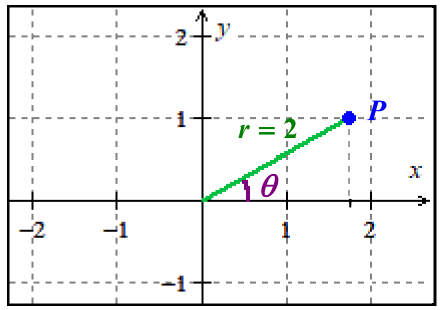


Figure 3

We can use the right triangle induced by the angle θ and the side r along with either sine or cosine to find the value of θ:

$$\cos(\theta) = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad \theta = \frac{\pi}{6}$$

Thus, in polar coordinates, $P = \left(2, \frac{\pi}{6}\right)$. We've plotted the point P on the polar coordinate plane in Figure 4.

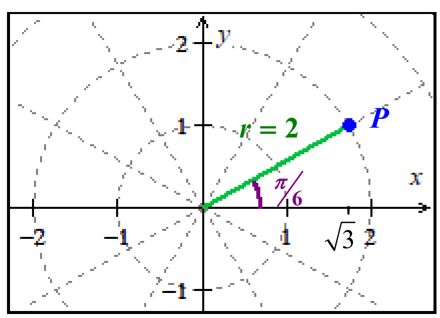


Figure 4: The point $P = \left(2, \frac{\pi}{6}\right)$ on the polar coordinate plane.

EXAMPLE 1: Plot the point $A=\left(10,\ \frac{5\pi}{4}\right)$ on the polar coordinate plane and determine the rectangular coordinates of point A.

SOLUTION:

To plot the point $A=\left(10,\ \frac{5\pi}{4}\right)$ we need to recognize that polar ordered pairs have form $(r,\ \theta)$, so $A=\left(10,\ \frac{5\pi}{4}\right)$ implies that

$$r=10 \quad \text{and} \quad \theta=\frac{5\pi}{4}.$$

We've plotted the point $A=\left(10,\ \frac{5\pi}{4}\right)$ on the polar coordinate plane in Figure 5.

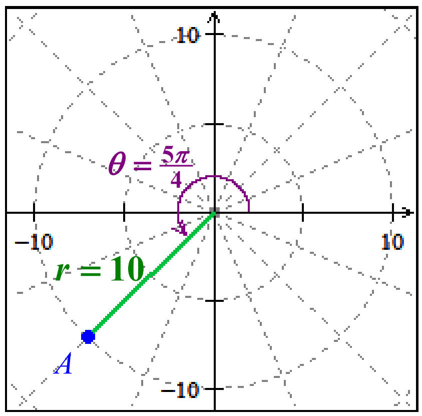


Figure 5

To find the rectangular coordinates of point A we can use the reference angle for θ, which is $\frac{\pi}{4}$, and the induced right triangle; see Figure 6.

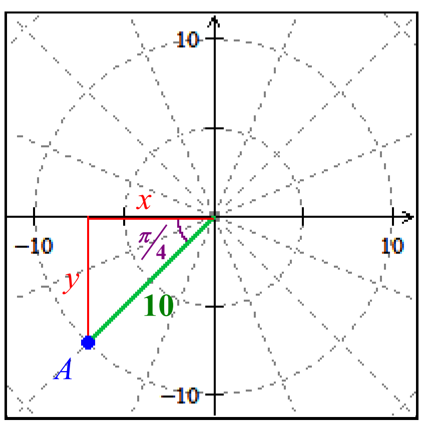


Figure 6

Using the triangle in Figure 6, we can see that

$$\begin{aligned} \cos\left(\tfrac{\pi}{4}\right) &= \tfrac{|x|}{10} \\ \Rightarrow \quad |x| &= 10\cos\left(\tfrac{\pi}{4}\right) \\ \Rightarrow \quad |x| &= 10\cdot\tfrac{\sqrt{2}}{2} \\ &= 5\sqrt{2} \end{aligned}$$

and

$$\begin{aligned} \sin\left(\tfrac{\pi}{4}\right) &= \tfrac{|y|}{10} \\ \Rightarrow \quad |y| &= 10\sin\left(\tfrac{\pi}{4}\right) \\ \Rightarrow \quad |y| &= 10\cdot\tfrac{\sqrt{2}}{2} \\ &= 5\sqrt{2}. \end{aligned}$$

Since point A is in Quadrant III, we know that both x and y are negative. Therefore, the rectangular coordinates of point A are $\left(-5\sqrt{2},\, -5\sqrt{2}\right)$.

EXAMPLE 2: Find the rectangular coordinates of a generic point $P = (r,\, \theta)$ on the polar coordinate plane.

SOLUTION:

In Figure 7, we've pointed P plotted in the polar plane.

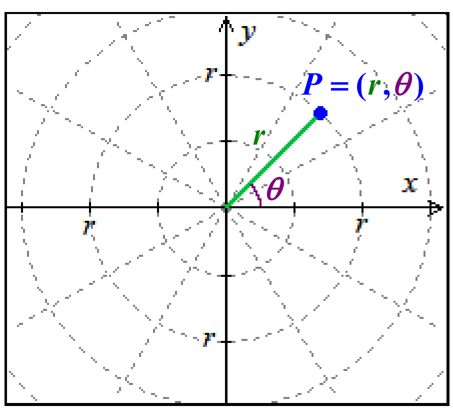


Figure 7

We can construct a right triangle and use trigonometry to obtain expressions for the horizontal and vertical coordinates of point P; see Figure 8 below.

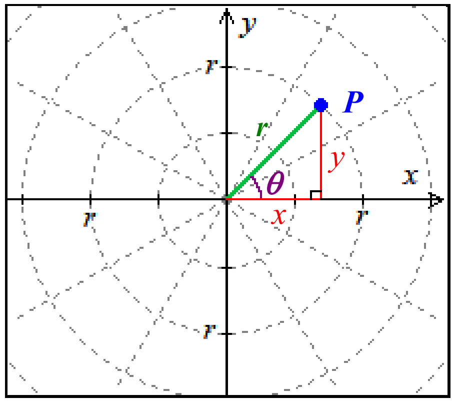


Figure 8

Based on the triangle in Figure 8, we can see that

$$\cos(\theta) = \frac{x}{r} \quad \text{and} \quad \sin(\theta) = \frac{y}{r}$$

$$\Rightarrow \quad x = r\cos(\theta) \qquad\qquad \Rightarrow \quad y = r\sin(\theta)$$

Thus, if $P = (r, \theta)$ is represents a point on the polar coordinate plane, then the rectangular coordinates of P are $(x, y) = (r\cos(\theta), r\sin(\theta))$. (Notice that we observed essentially the same fact in Section I: Chapter 3.) We can use what we've discovered to translate polar coordinates into rectangular coordinates.

The polar coordinates (r, θ) are equivalent to the rectangular coordinates

$$(x, y) = (r\cos(\theta), r\sin(\theta)).$$

Key Point:

Polar and rectangular ordered pairs cannot be set equal to each other. When ordered pairs are described as being equal, it means that they have the same coordinates so we can write something like $(0.75, 0.5) = \left(\frac{3}{4}, \frac{1}{2}\right)$ since $0.75 = \frac{3}{4}$ and $0.5 = \frac{1}{2}$ but we can't write $\left(10, \frac{5\pi}{4}\right) = \left(-5\sqrt{2}, -5\sqrt{2}\right)$ (from Example 1) since $10 \neq -5\sqrt{2}$ and $\frac{5\pi}{4} \neq -5\sqrt{2}$. In order to communicate that rectangular ordered pairs and polar ordered pairs describe the same location, we need to compose sentences like, "The rectangular ordered pair $\left(-5\sqrt{2}, -5\sqrt{2}\right)$ is equivalent to the polar ordered pair $\left(10, \frac{5\pi}{4}\right)$."

EXAMPLE 3: Plot the point $B = \left(-4, \frac{2\pi}{3}\right)$ on the polar coordinate plane and find the rectangular coordinates of the point.

SOLUTION:

To plot the point $B = \left(-4, \frac{2\pi}{3}\right)$ we need to recognize that polar ordered pairs have form (r, θ), so $B = \left(-4, \frac{2\pi}{3}\right)$ implies that

$$r = -4 \text{ and } \theta = \tfrac{2\pi}{3}.$$

Here, r is negative. This means that when we get to the terminal side of $\theta = \frac{2\pi}{3}$, instead of going "forward" 4 units into Quadrant II, we need to go "backwards" 4 units into Quadrant IV; see Figure 9.

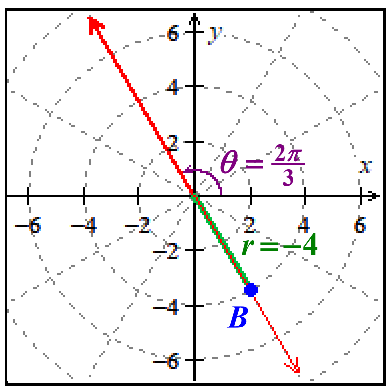


Figure 9

To find the rectangular coordinates of point B, we can use the conversion equations we derived in the previous example.

$$\begin{aligned} x &= r\cos(\theta) \\ &= -4\cos\left(\tfrac{2\pi}{3}\right) \\ &= -4\cdot\left(-\tfrac{1}{2}\right) \\ &= 2 \end{aligned} \quad \text{and} \quad \begin{aligned} y &= r\sin(\theta) \\ &= -4\sin\left(\tfrac{2\pi}{3}\right) \\ &= -4\cdot\left(\tfrac{\sqrt{3}}{2}\right) \\ &= -2\sqrt{3} \end{aligned}$$

Thus, the rectangular coordinates of B are $(x, y) = \left(2, -2\sqrt{3}\right)$.

Section III: Polar Coordinates and Complex Numbers

Chapter 2: Polar Equations and Functions

Just as we can create equations in rectangular coordinates, we can create equations in polar coordinates. For example, $r = 3\sin(\theta)$ is an equation in polar coordinates since it's an equation and it involves the polar coordinates r and θ.

When we have an equation in one coordinate system, we can often translate it into an equation in another coordinate system whose graph is identical. In Examples 1 and 2, we'll translate a polar equation into a rectangular equation, and vice versa.

EXAMPLE 1: Translate the polar equation $r = 3\sin(\theta)$ into an equation in rectangular coordinates whose graph is the same.

SOLUTION:

As we know, the following identities can be used to translate from polar coordinates (r, θ) to rectangular coordinates (x, y):

$$\begin{cases} r = \sqrt{x^2 + y^2} \\ x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases}$$

Since the equation $r = 3\sin(\theta)$ involves r, we'll certainly employ the identity $r = \sqrt{x^2 + y^2}$. Further, since the equation involves $\sin(\theta)$, we'll need to get an expression for $\sin(\theta)$ that contains only x and y (i.e., only rectangular coordinates):

$$y = r\sin(\theta)$$

$$\Rightarrow \quad \sin(\theta) = \frac{y}{r}$$

$$\Rightarrow \quad \sin(\theta) = \frac{y}{\sqrt{x^2 + y^2}} \qquad \text{since } r = \sqrt{x^2 + y^2}$$

So

$$r = 3\sin(\theta)$$

$$\Rightarrow \sqrt{x^2 + y^2} = 3 \cdot \frac{y}{\sqrt{x^2 + y^2}}$$

$$\Rightarrow \quad x^2 + y^2 = 3y$$

Therefore, the rectangular equation $x^2 + y^2 = 3y$ has the same graph as the polar equation $r = 3\sin(\theta)$.

EXAMPLE 2: Translate the rectangular equation $y = 4x - 3$ into an equation in polar coordinates whose graph is the same.

SOLUTION:

As we know, the following identities can be used to translate from rectangular coordinates (x, y) to polar coordinates (r, θ):

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases}$$

So

$$\begin{aligned} & y = 4x - 3 \\ \Rightarrow \quad & r\sin(\theta) = 4 \cdot r\cos(\theta) - 3 \\ \Rightarrow \quad & r\sin(\theta) - 4r\cos(\theta) = -3 \\ \Rightarrow \quad & r\big(\sin(\theta) - 4\cos(\theta)\big) = -3 \\ \Rightarrow \quad & r = -\frac{3}{\sin(\theta) - 4\cos(\theta)} \end{aligned}$$

Therefore, the polar equation $r = -\dfrac{3}{\sin(\theta) - 4\cos(\theta)}$ has the same graph as the rectangular equation $y = 4x - 3$.

Now let's discuss graphing functions in polar coordinates. Just as when we graph functions in rectangular coordinates, we can graph functions in polar coordinates by finding ordered pairs that satisfy the function. Ordered pairs in polar coordinates have the form (r, θ). **Notice that this convention for the polar ordered pair puts the *output* variable, *r*, first and the *input* variable, θ, second**. This is **different** from rectangular ordered pairs of the form (x, y) that have the *input* variable first and the *output* variable second.

EXAMPLE 3: Sketch the graph of the function $r = \theta$ on the polar coordinate plane.

SOLUTION:

To sketch the graph of $r = \theta$, we should find some ordered pairs (r, θ) that satisfy the function. The table below gives some values for $r = \theta$ that are plotted and connected in Figure 1 to form a graph of $r = \theta$.

θ	r	(approximately) (r, θ)
0	0	$(0, 0)$
$\frac{\pi}{4}$	$\frac{\pi}{4} \approx 0.79$	$\left(0.79, \frac{\pi}{4}\right)$
$\frac{\pi}{2}$	$\frac{\pi}{2} \approx 1.57$	$\left(1.57, \frac{\pi}{2}\right)$
$\frac{3\pi}{4}$	$\frac{3\pi}{4} \approx 2.36$	$\left(2.36, \frac{3\pi}{4}\right)$
π	$\pi \approx 3.14$	$(3.14, \pi)$
$\frac{5\pi}{4}$	$\frac{5\pi}{4} \approx 3.93$	$\left(3.93, \frac{5\pi}{4}\right)$
$\frac{3\pi}{2}$	$\frac{3\pi}{2} \approx 4.71$	$\left(4.71, \frac{3\pi}{2}\right)$
$\frac{7\pi}{4}$	$\frac{7\pi}{4} \approx 5.5$	$\left(5.5, \frac{7\pi}{4}\right)$
2π	$2\pi \approx 6.28$	$(6.28, 2\pi)$
$\frac{9\pi}{4}$	$\frac{9\pi}{4} \approx 7.07$	$\left(7.07, \frac{9\pi}{4}\right)$

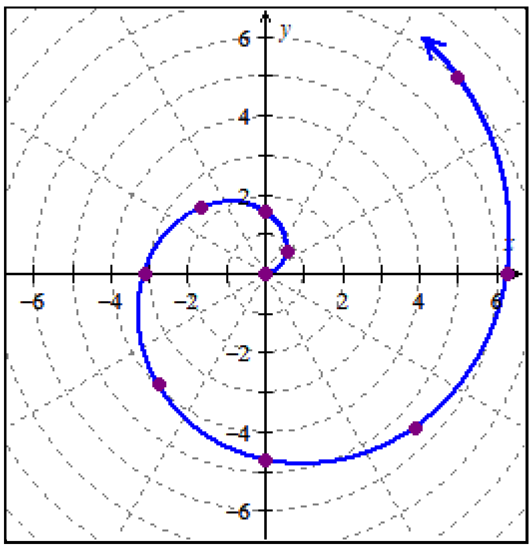


Figure 1: A graph of $r = \theta$.

The graph of this polar equation is called the **Archimedean Spiral**.

(Note that to graph polar equations on your calculator, you may need to change the graphing mode to ***POLAR***.)

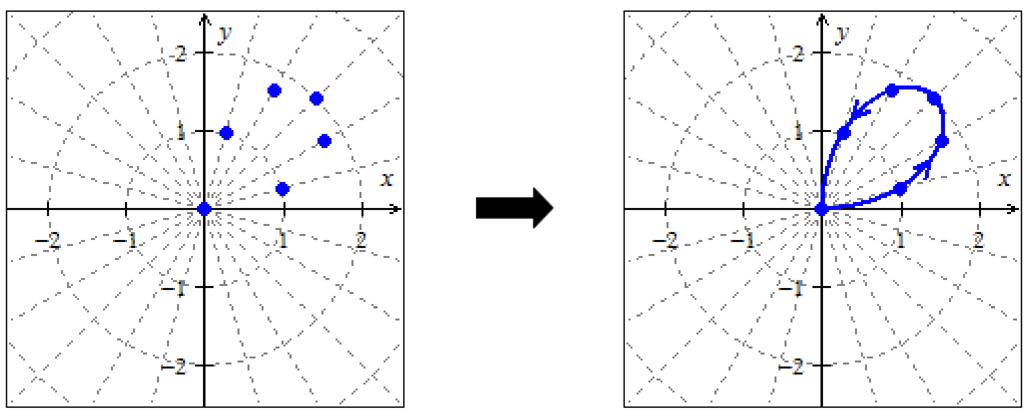
EXAMPLE 4: Sketch the graph of the $r = 2\sin(2\theta)$ on the polar coordinate plane.

SOLUTION:

To sketch the graph of $r = 2\sin(2\theta)$, we should find some ordered pairs $(r,\ \theta)$ that satisfy the function. The table below gives some values for $r = 2\sin(2\theta)$.

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
$r = 2\sin(2\theta)$	0	1	$\sqrt{3} \approx 1.7$	2	$\sqrt{3} \approx 1.7$	1	0
$(r,\ \theta)$ (approximately)	$(0,\ 0)$	$\left(1,\ \frac{\pi}{12}\right)$	$\left(1.7,\ \frac{\pi}{6}\right)$	$\left(2,\ \frac{\pi}{4}\right)$	$\left(1.7,\ \frac{\pi}{3}\right)$	$\left(1,\ \frac{5\pi}{12}\right)$	$\left(0,\ \frac{\pi}{2}\right)$

Let's plot these points and connect them in Figure 2:



Points on the graph of $r = 2\sin(2\theta)$.

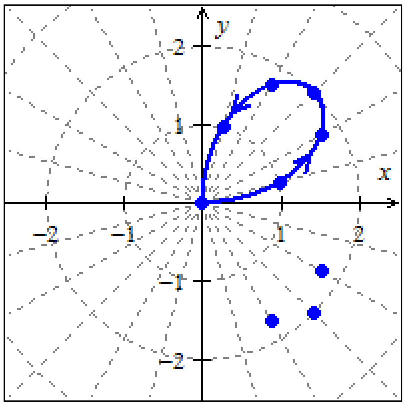
A portion of the graph of $r = 2\sin(2\theta)$.

Figure 2

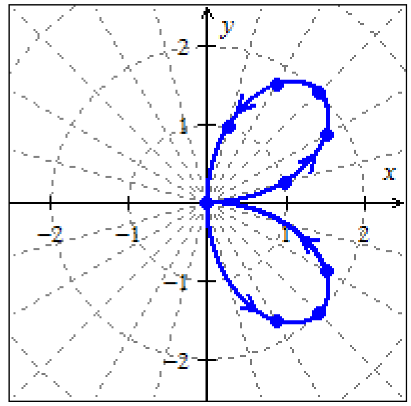
Now let's find some more points on the graph of $r = 2\sin(2\theta)$.

θ	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$r = 2\sin(2\theta)$	$-\sqrt{3} \approx -1.7$	-2	$-\sqrt{3} \approx -1.7$	0
$(r,\ \theta)$ (approximately)	$\left(-1.7,\ \frac{2\pi}{3}\right)$	$\left(-2,\ \frac{3\pi}{4}\right)$	$\left(-1.7,\ \frac{5\pi}{6}\right)$	$(0,\ \pi)$

Let's plot these points in Figure 3 (below). Keep in mind that negative r-values are plotted in the opposite quadrant than the angle θ. Since all of these angles are in Quadrant II, we will end up plotting points in Quadrant IV.



More points on the graph of $r = 2\sin(2\theta)$.

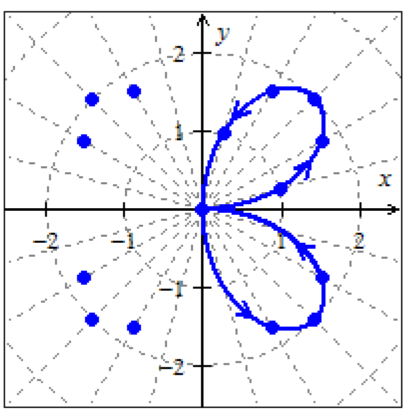


A larger portion of the graph of $r = 2\sin(2\theta)$.

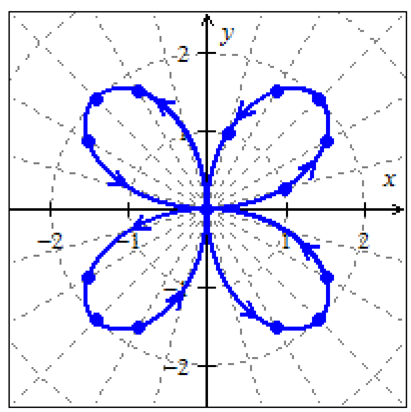
Figure 3

Now we can find a few more points and finish our graph of $r = 2\sin(2\theta)$; see Figure 4.

θ	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$r = 2\sin(2\theta)$	$\sqrt{3} \approx 1.7$	2	$\sqrt{3} \approx 1.7$	0	$-\sqrt{3} \approx -1.7$	-2	$-\sqrt{3} \approx -1.7$	0
(r, θ) (approximately)	$\left(1.7, \frac{7\pi}{6}\right)$	$\left(2, \frac{5\pi}{4}\right)$	$\left(1.7, \frac{4\pi}{3}\right)$	$\left(0, \frac{3\pi}{2}\right)$	$\left(-1.7, \frac{5\pi}{3}\right)$	$\left(-2, \frac{7\pi}{4}\right)$	$\left(-1.7, \frac{11\pi}{6}\right)$	$(0, 2\pi)$



More points on the graph of $r = 2\sin(2\theta)$.



The compete graph of $r = 2\sin(2\theta)$.

Figure 4

You should make sure you can get this graph of $r = 2\sin(2\theta)$ on your graphing calculator. Don't forget to change the graphing mode of your calculator to ***polar***.

EXAMPLE 5: Sketch the graph of the $r = 3$ on the polar coordinate plane.

SOLUTION:

Here, since $r = 3$, the distance from the origin is *always* 3 units. So no matter what θ is, $r = 3$. This gives us a **circle** of radius 3; see Figure 5.

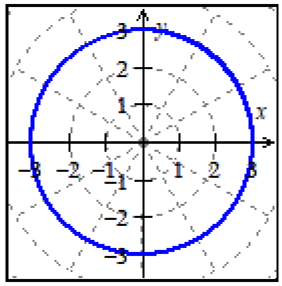
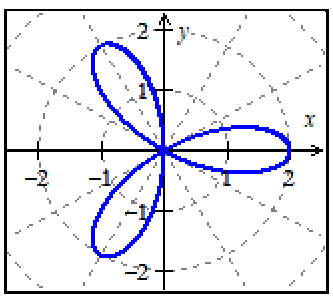
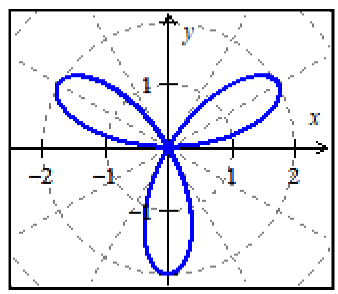


Figure 5: A graph of $r = 3$.

Below are the graphs of a few other functions defined via polar coordinates. You should graph them on your graphing calculator.

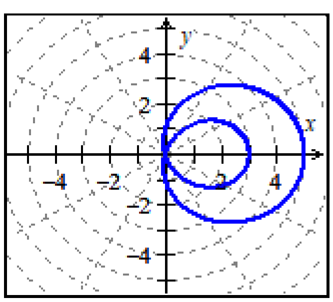


$r = 2\cos(3\theta)$

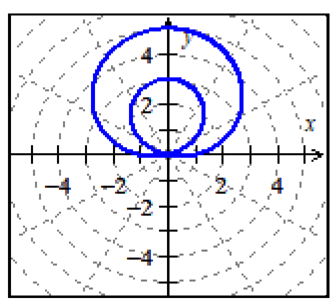


$r = 2\sin(3\theta)$

Figure 6: These graphs and the graph given in Figure 3 are called "**roses**".



$r = 1 + 4\cos(\theta)$



$r = 1 + 4\sin(\theta)$

Figure 7: These graphs are called "**limaçons**".

Section III: Polar Coordinates and Complex Numbers

Chapter 3: Complex Numbers

In this chapter we'll study how we can employ what we know about polar coordinates and trigonometry to represent complex numbers. Let's start by reviewing complex numbers.

Recall from Section I: Chapter 0 the definition of the set of complex numbers:

$$\mathbb{C} = \left\{ x \,\middle|\, x = a + bi \text{ and } a, b \in \mathbb{R} \text{ and } i = \sqrt{-1} \right\}.$$

If a complex number has the form $a + bi$, we say that its *real part* is a and its *imaginary part* is b.

EXAMPLE 1: Which of the following is a complex number?

a. $s = 2 + 5i$ **b.** $t = \frac{3}{2} - 3i$ **c.** $u = 3i$ **d.** $v = -4$

SOLUTION:

a. $s = 2 + 5i$ is a complex number since it has the form $a + bi$ where $a = 2$ and $b = 5$. The real part of s is 2 and the imaginary part of s is 5.

b. $t = \frac{3}{2} - 3i$ is a complex number since it has the form $a + bi$ where $a = \frac{3}{2}$ and $b = -3$. The real part of t is $\frac{3}{2}$ and the imaginary part of t is -3.

c. $u = 3i$ is a complex number since it has the form $a + bi$ where $a = 0$ and $b = 3$. The real part of u is 0 and the imaginary part of u is 3.

d. $v = -4$ is a complex number since it has the form $a + bi$ where $a = -4$ and $b = 0$. The real part of v is -4 and the imaginary part of v is 0.

Because a complex number has *two* parts, we can use the *two dimensional* rectangular coordinate plane to plot complex numbers. We use the horizontal axis to represent the real part and the vertical axis to represent the complex part. Thus, the complex number $a + bi$ can be represented by the point (a, b) on the rectangular coordinate plane; see Figure 1.

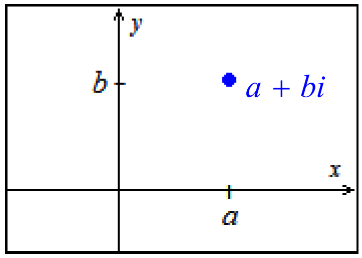


Figure 1


EXAMPLE 2: Plot the following complex numbers on the coordinate plane in Figure 2.

a. $s = 2 + 5i$ **b.** $t = \frac{3}{2} - 3i$ **c.** $u = 3i$ **d.** $v = -4$

SOLUTION:

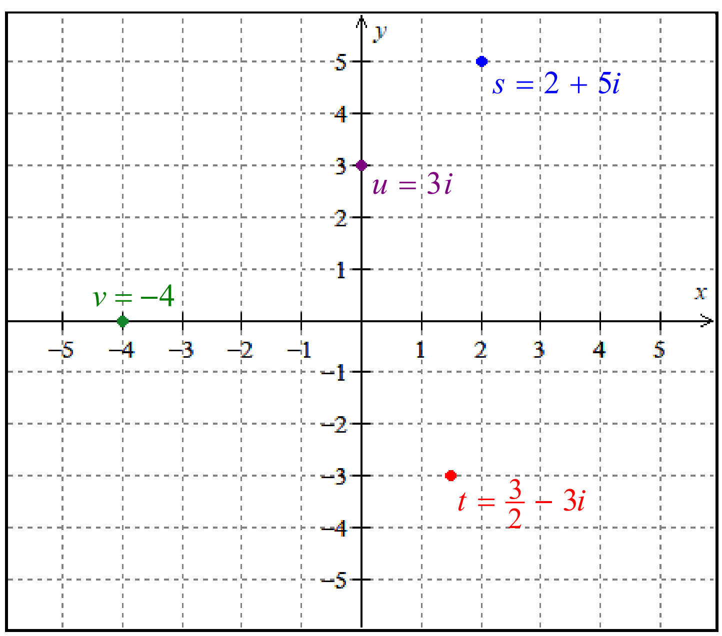


Figure 2

As we studied in Section III: Chapter 1, the rectangular ordered pair (a, b) can be represented in polar coordinates (r, θ) where r represents the distance the point is from the origin and θ represents the angle between the positive x-axis and the segment connecting the origin and the point; see Figure 3.

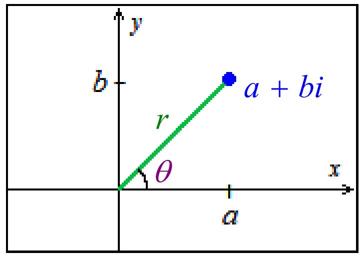


Figure 3

We know that if the rectangular pair (a, b) represents the same point as the polar pair (r, θ), then $a = r\cos(\theta)$ and $b = r\sin(\theta)$. Thus,

$$\begin{aligned} a + bi &= r\cos(\theta) + r\sin(\theta)\cdot i \\ &= r\left(\cos(\theta) + \sin(\theta)\cdot i\right) \end{aligned}$$

Using this fact, we can establish a surprising connection between exponential function e^x and complex numbers. If you continue studying mathematics and take a calculus sequence, you have an opportunity to see why this equation is true but, for now, you need to just accept it and learn to work with it.

EULER'S FORMULA

$$e^{i\theta} = \cos(\theta) + \sin(\theta)\cdot i$$

By multiplying both sides of Euler's formula by r, we obtain the formula

$$re^{i\theta} = r\cos(\theta) + r\sin(\theta)\cdot i$$

that allows us to write any complex number in **polar form**:

The **polar form** of the complex number $z = a + bi$ is

$$z = re^{i\theta}$$

where $r = \sqrt{a^2 + b^2}$ and $\tan(\theta) = \frac{b}{a}$.

EXAMPLE: Express the complex number $z = 6e^{i\cdot\frac{5\pi}{6}}$ in the form $z = a + bi$.

SOLUTION:

$$\begin{aligned} z &= 6e^{i\cdot\frac{5\pi}{6}} \\ &= 6\cos\left(\tfrac{5\pi}{6}\right) + 6\sin\left(\tfrac{5\pi}{6}\right)\cdot i \\ &= 6\cdot\left(-\tfrac{\sqrt{3}}{2}\right) + 6\cdot\left(\tfrac{1}{2}\right)\cdot i \\ &= -3\sqrt{3} + 3i \end{aligned}$$

Therefore, the complex number $z = 6e^{i\cdot\frac{5\pi}{6}}$ can be expressed as $z = -3\sqrt{3} + 3i$.

EXAMPLE: Find the polar form $z = re^{i\theta}$ of the complex number $z = 3 - 3i$.

SOLUTION:

We can associate the complex number $z = 3 - 3i$ with the rectangular ordered pair $(3, -3)$, and then translate this ordered pair into polar coordinates (r, θ), and finally use this polar ordered pair to obtain the polar form $z = re^{i\theta}$. First, let's find r:

$$\begin{aligned} r &= \sqrt{(3)^2 + (-3)^2} \\ &= \sqrt{9 + 9} \\ &= 3\sqrt{2}. \end{aligned}$$

Now, let's find θ:

$$\begin{aligned} &\tan(\theta) = \tfrac{-3}{3} \\ \Rightarrow\quad &\theta = \tan^{-1}(-1) \\ \Rightarrow\quad &\theta = -\tfrac{\pi}{4} \end{aligned}$$

Thus, the complex number $z = 3 - 3i$ can be expressed in polar form $z = 3\sqrt{2}\,e^{i\cdot\left(-\frac{\pi}{4}\right)}$.

EXAMPLE: Find $\sqrt[4]{-1}$ using the polar form of -1.

SOLUTION:

Since $-1 = -1 + 0 \cdot i$, we can associate the number -1 with the point $(-1, 0)$ on the coordinate plane. The point $(-1, 0)$ is 1 unit from the origin at an angle π with respect to the positive x-axis so we can represent it in polar coordinates as $(1, \pi)$. Thus, in polar form, $-1 = 1 \cdot e^{\pi i} = e^{\pi i}$. Therefore,

$$\begin{aligned}\sqrt[4]{-1} &= (-1)^{1/4} \\ &= \left(e^{\pi i}\right)^{1/4} \\ &= e^{\pi \cdot \frac{1}{4} i} \\ &= \cos\left(\tfrac{\pi}{4}\right) + \sin\left(\tfrac{\pi}{4}\right) \cdot i \\ &= \tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{2} i\end{aligned}$$

EXAMPLE: Find $\left(-1 + \sqrt{3} \cdot i\right)^{1/2}$ using the polar form of $-1 + \sqrt{3} \cdot i$.

SOLUTION:

We can associate the number $-1 + \sqrt{3} \cdot i$ with the point $\left(-1, \sqrt{3}\right)$ on the coordinate plane and then translate this point into polar coordinates (r, θ):

$$\begin{aligned}r &= \sqrt{(-1)^2 + (\sqrt{3})^2} \\ &= \sqrt{1 + 3} \\ &= \sqrt{4} \\ &= 2\end{aligned}$$

Now, let's find θ:

$$\begin{aligned}\theta &= \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right) + \pi \\ &= -\frac{\pi}{3} + \pi \\ &= \frac{2\pi}{3}\end{aligned}$$

(note that we add π since we know that the point is in the 2^{nd} quadrant)

Therefore, we can represent the point $\left(-1, \sqrt{3}\right)$ in polar coordinates as $\left(2, \frac{2\pi}{3}\right)$ and, in polar form, $-1 + \sqrt{3} \cdot i = 2e^{\frac{2\pi}{3}i}$. So:

$$\begin{aligned}
\left(-1 + i\sqrt{3}\right)^{1/2} &= \left(2e^{\frac{2\pi}{3}i}\right)^{1/2} \\
&= 2^{\frac{1}{2}} e^{\frac{2\pi}{3}\cdot\frac{1}{2}i} \\
&= \sqrt{2}e^{\frac{2\pi}{3}\cdot\frac{1}{2}i} \\
&= \sqrt{2}e^{\frac{\pi}{3}i} \\
&= \sqrt{2}\cos\left(\frac{\pi}{3}\right) + \sqrt{2}\sin\left(\frac{\pi}{3}\right)\cdot i \\
&= \sqrt{2}\cdot\frac{1}{2} + \sqrt{2}\cdot\frac{\sqrt{3}}{2}i \\
&= \frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{2}i
\end{aligned}$$

Section IV: Vectors

Chapter 1: Introduction to Vectors

Vectors are mathematical objects used to represent physical quantities like velocity, force, and displacement. Unlike ordinary numbers (or **scalars**), vectors describe *both* magnitude and direction. So, for example, we can describe the velocity (i.e., the speed *and* direction) of an object with a vector.

DEFINITION: A **vector** is a mathematical object used to represent a physical quantity that has both a *magnitude* (i.e., size) and a *direction*.

In order to distinguish between vectors from scalars (i.e., numbers) we need to use a different notation to denote vectors. In this course, we will use a small arrow above the vector name to denote a vector, so that $\vec{v}$ and $\vec{s}$ represent vectors while v and s represent scalars. (Note that our textbook uses bold text to represent vectors but this isn't possible for handwritten work; instead the "arrow notation" is necessary for handwritten work so I prefer to also use arrows in typed work.)

In this class we will focus on **two-dimensional vectors**. A two-dimensional vectors can be represented by an **arrow** on the coordinate plane. The **length** of the arrow represents the **magnitude** of the vector and the **direction** that the arrow points represents the direction of the vector. (We traditionally use the **angle between the positive x-axis and the arrow** to describe the **direction** of the vector.)

EXAMPLE 1: The vector $\vec{v}$ is depicted as an arrow on the coordinate plane in Figure 1.

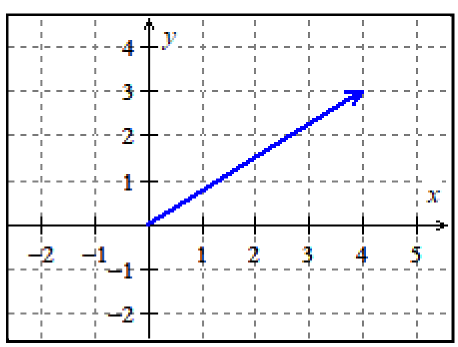


Figure 1: Arrow representing $\vec{v}$.

The **tip** of the vector is where the arrow ends and the **tail** of the vector is where the arrow begins. Thus, the tip of $\vec{v}$ is at the point $(4, 3)$ and the tail of the vector is at the origin, $(0, 0)$.

As mentioned above, the **length** of the arrow represents the **magnitude** of the vector. We denote the magnitude of vector $\vec{v}$ by $\|\vec{v}\|$. To find the magnitude of $\vec{v}$, we need to find the length of the arrow; we can do this by thinking of the arrow as being the hypotenuse of a right-triangle with side lengths 4 and 3 (see Figure 2) and then use the Pythagorean Theorem to find $\|\vec{v}\|$.

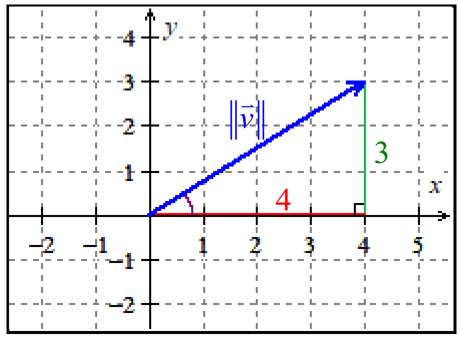


Figure 2

$$\|\vec{v}\|^2 = 3^2 + 4^2$$
$$\Rightarrow \|\vec{v}\|^2 = 9 + 16$$
$$\Rightarrow \|\vec{v}\|^2 = 25$$
$$\Rightarrow \quad \|\vec{v}\| = 5$$

So the magnitude of $\vec{v}$ is 5 units.

We can find the angle between the positive x-axis and the arrow to describe the **direction** of the vector. We've denoted this angle by θ in Figure 3.

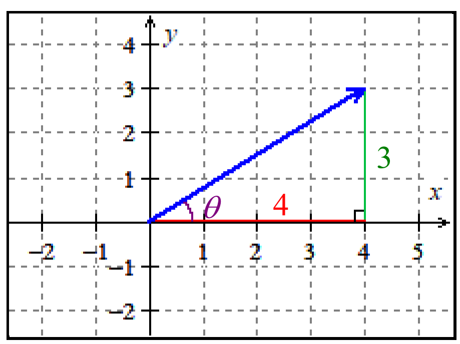


Figure 3: The components of $\vec{v}$.

We can use the trigonometry that we learned earlier in the course to find θ:

$$\tan(\theta) = \tfrac{3}{4}$$
$$\Rightarrow \qquad \theta = \tan^{-1}\left(\tfrac{3}{4}\right)$$
$$\Rightarrow \qquad \theta \approx 36.87^\circ$$

Although the magnitude and direction of the vector describe it completely, it is often useful to describe a vector by using its **horizontal and vertical components**. The *horizontal component* of $\vec{v}$ in Figure 3 (above) is 4 units and a *vertical component* of vector $\vec{v}$ is 3 units. Thus, we say that the **component form of vector $\vec{v}$** is $\langle 4, 3 \rangle$.

It is important to recognize that we could translate this vector anywhere in the coordinate plane and it would still be the same vector. For example, all of the arrows in Figure 4 represent $\vec{v}$ since all of these vectors have a horizontal component of 4 and a vertical component of 3.

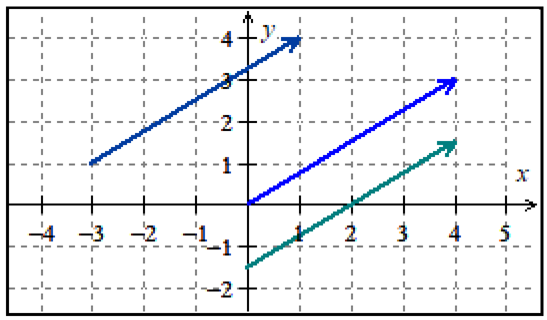


Figure 4: Three copies of $\vec{v}$.

EXAMPLE 2: Find the component form of the vector $\vec{s}$ given in Figure 5 below.

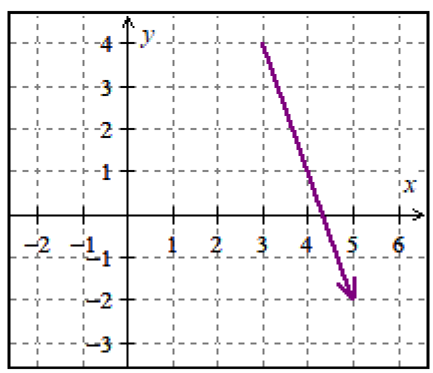


Figure 5: $\vec{s}$.

SOLUTION:

To find the *horizontal component* of $\vec{s}$ we need to determine the *horizontal distance* between the tip and the tail of the vector's arrow, and to find the *vertical component* of $\vec{s}$, we need to determine the *vertical distance* between the tip and the tail of the vector's arrow. As we can see in Figure 6 (below), the *horizontal component* of $\vec{s}$ is 2 units and the *vertical component* of $\vec{s}$ is -6 units. Note that the vertical component is negative since the arrow travels *down 6 units*, or *vertically −6 units*. The component form of $\vec{s}$ is $\langle 2, -6 \rangle$.

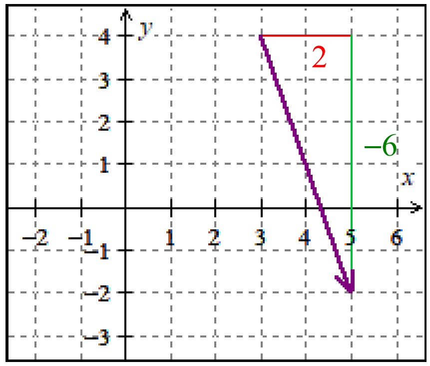


Figure 6: The components of $\vec{s}$.

If we translate vector $\vec{s}$ so that its tail is at the origin we see that its tip is at the point $(2,-6)$; see Figure 7.

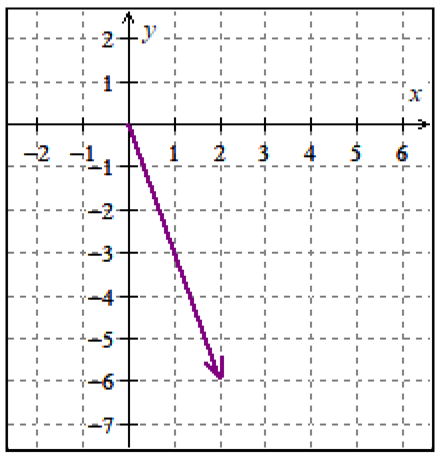


Figure 7: $\vec{s}$ translated so that its tail is at the origin.

Notice that in Example 1, the tail of $\vec{v}$ is at the origin and its tip is at the point $(4, 3)$, and the component form of $\vec{v}$ is $\langle 4, 3\rangle$.) In general, a vector with component form $\langle a, b\rangle$ can be represented by an arrow on the coordinate plane whose tail is at the origin and whose tip is at the point (a, b).

EXAMPLE 3: Find the component form of the vector $\vec{r}$ given in Figure 8.

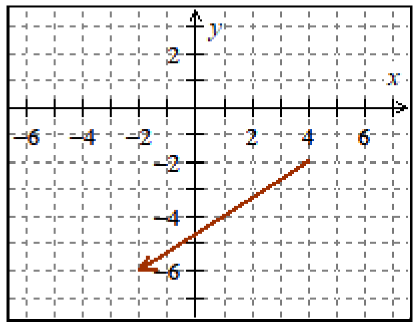


Figure 8: $\vec{r}$.

SOLUTION:

As we can see in Figure 9, the *horizontal component* of $\vec{r}$ is -6 units and the *vertical component* of $\vec{r}$ is -4 units so the component form of $\vec{r}$ is $\langle -6, -4 \rangle$.

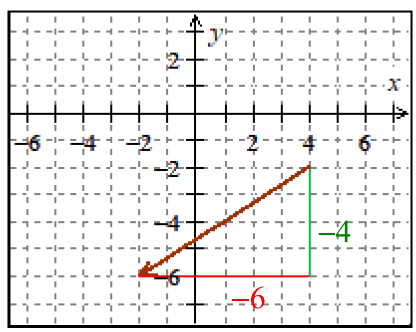


Figure 9: The components of $\vec{r}$.

We can translate $\vec{r} = \langle -6, -4 \rangle$ so that its tail is at the origin (see Figure 10); its tip is at the point $(-6, -4)$ which agrees with what we noticed above.

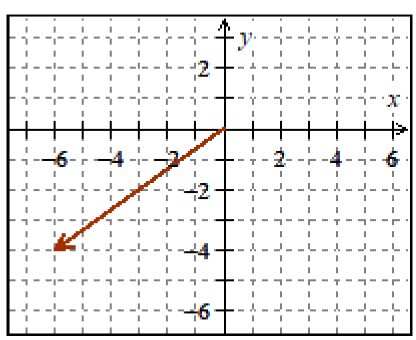


Figure 10: $\vec{r}$ translated so that its tail is at the origin.

Vectors Operations

We can multiply any vector by a scalar (i.e., a number) and we can add or subtract any two vectors.

When we **multiply a vector by a scalar**, we simply multiply the respective components of the vector by the scalar. Thus, if $\vec{a} = \langle a_1, a_2 \rangle$ and $k \in \mathbb{R}$, then $k\vec{a} = \langle ka_1, ka_2 \rangle$.

EXAMPLE 4: Let $\vec{v} = \langle 4, 3 \rangle$ (from Example 1).

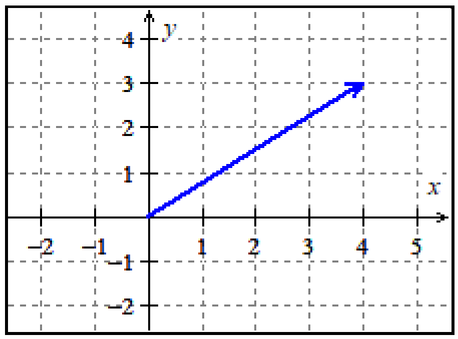


Figure 11: $\vec{v}$.

a. Find $2\vec{v}$. **b.** Find $\frac{1}{2}\vec{v}$. **c.** Find $-\vec{v}$.

SOLUTION:

a.
$$\begin{aligned} 2\vec{v} &= 2\cdot\langle 4, 3 \rangle \\ &= \langle 2\cdot 4, 2\cdot 3 \rangle. \\ &= \langle 8, 6 \rangle \end{aligned}$$

In Figure 12 we've drawn an arrow representing $2\vec{v}$. Notice that $2\vec{v}$ is **twice** as long as $\vec{v}$ yet it points **in the same direction**.

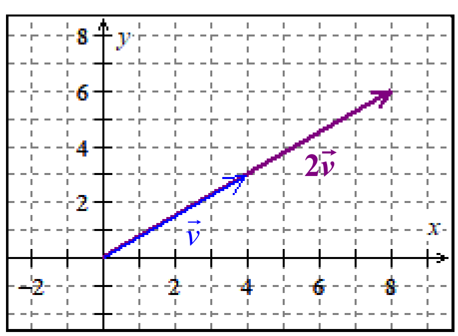


Figure 12: $\vec{v}$ and $2\vec{v}$.

b. $\frac{1}{2}\vec{v} = \frac{1}{2}\cdot\langle 4, 3\rangle$

$= \left\langle \frac{1}{2}\cdot 4, \frac{1}{2}\cdot 3\right\rangle.$

$= \left\langle 2, \frac{3}{2}\right\rangle$

In Figure 13 we've drawn an arrow representing $\frac{1}{2}\vec{v}$. Notice that $\frac{1}{2}\vec{v}$ is **half** as long as $\vec{v}$ yet it **points in the same direction**.

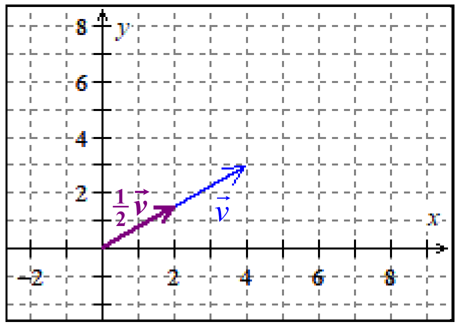


Figure 13: $\vec{v}$ and $\frac{1}{2}\vec{v}$.

c. $-\vec{v} = -1\cdot\vec{v}$

$= -1\cdot\langle 4, 3\rangle$

$= \langle -1\cdot 4, -1\cdot 3\rangle.$

$= \langle -4, -3\rangle$

In Figure 14 we've drawn an arrow representing $-\vec{v}$. Notice that $-\vec{v}$ is the **same** length as $\vec{v}$ yet it points **in the opposite direction**.

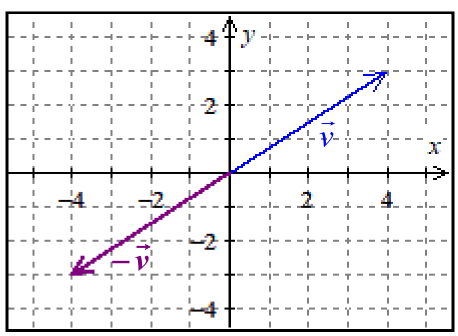


Figure 14: $\vec{v}$ and $-\vec{v}$.

If $\vec{a} = \langle a_1, a_2\rangle$ is a vector and $k \in \mathbb{R}$ then $k\vec{a} = \langle ka_1, ka_2\rangle$ has magnitude $|k|\cdot\|\vec{a}\|$. If $k > 0$ then $k\vec{a}$ points in the same direction as $\vec{a}$; if $k < 0$ then $k\vec{a}$ points in the opposite direction as $\vec{a}$.

When we **add or subtract vectors**, we simply add the respective components of the vectors. Thus, if $\vec{a} = \langle a_1, a_2 \rangle$ and $\vec{b} = \langle b_1, b_2 \rangle$, then $\vec{a} + \vec{b} = \langle a_1 + b_1, a_2 + b_2 \rangle$ and $\vec{a} - \vec{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$

EXAMPLE 5: Let $\vec{v} = \langle 4, 3 \rangle$ (from Example 1) and $\vec{s} = \langle 2, -6 \rangle$ (from Example 2). Find $\vec{v} + \vec{s}$ and $\vec{v} - \vec{s}$.

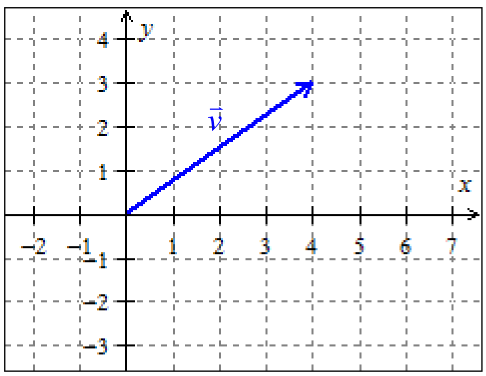


Figure 15a: $\vec{v}$.

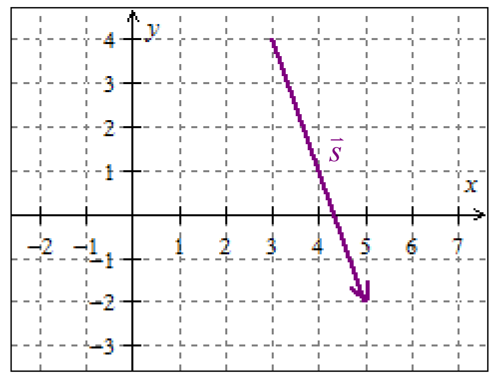


Figure 15b: $\vec{s}$.

SOLUTION:

Let's start by finding $\vec{v} + \vec{s}$:

$$\begin{aligned}\vec{v} + \vec{s} &= \langle 4, 3 \rangle + \langle 2, -6 \rangle \\ &= \langle 4 + 2, 3 + (-6) \rangle \\ &= \langle 6, -3 \rangle.\end{aligned}$$

We can also add vectors by using arrows on the coordinate plane by connecting the tip of the first arrow to the tail of the second arrow. In Figure 16 we show $\vec{v} + \vec{s}$:

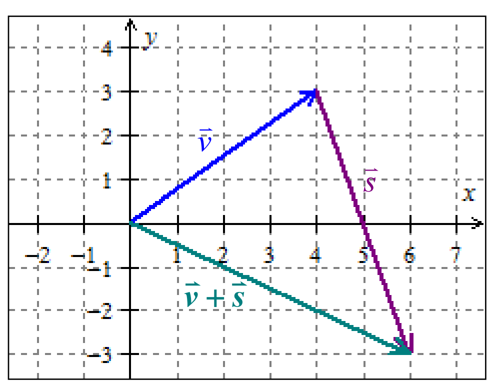


Figure 16: $\vec{v} + \vec{s}$.

Notice that the arrow for $\vec{v} + \vec{s} = \langle 6, -3 \rangle$ starts at the origin and ends at the point $(6, -3)$ which should give us confidence that these two ways of adding vectors (using components and using arrows) are equivalent.

Now, let's find $\vec{v} - \vec{s}$:

$$\begin{aligned}\vec{v} - \vec{s} &= \langle 4, 3\rangle - \langle 2, -6\rangle \\ &= \langle 4-2, 3-(-6)\rangle \\ &= \langle 2, 9\rangle\end{aligned}$$

We can also subtract vectors by using arrows on the coordinate plane. Notice that

$$\vec{v} - \vec{s} = \vec{v} + (-\vec{s})$$

Thus, we can obtain $\vec{v} - \vec{s}$ by adding $\vec{v}$ and $-\vec{s}$, i.e., by connecting the tip $\vec{v}$ with the tail of $-\vec{s}$; see Figure 17.

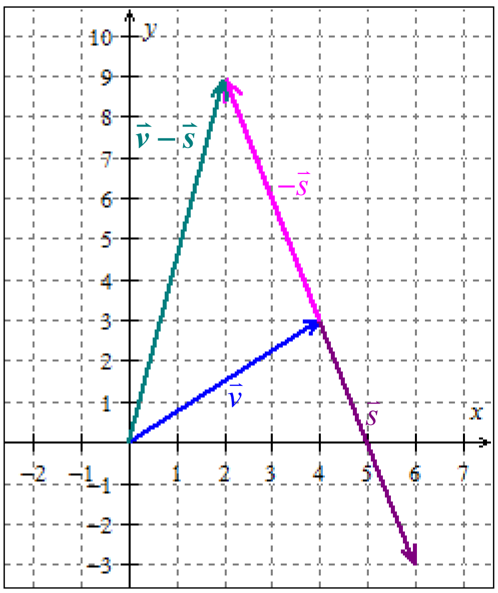


Figure 17

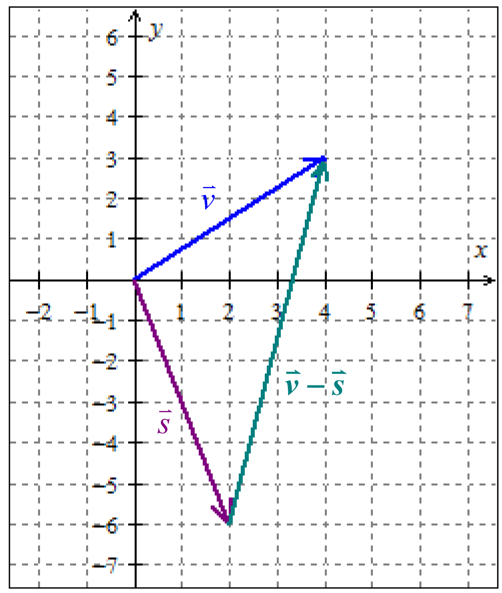


Figure 18

Notice that the arrow for $\vec{v} - \vec{s} = \langle 2, 9\rangle$ starts at the origin and ends at the point $(2, 9)$ which should give us confidence that these two ways of subtracting vectors (using components and using arrows) are equivalent. Notice that if we start both $\vec{v}$ and $\vec{s}$ at the origin, then $\vec{v} - \vec{s}$ is equivalent to the vector that starts at the tip of $\vec{s}$ and ends at the tip of $\vec{v}$; see Figure 18.

In order to facilitate the communication and manipulation of vectors, it is useful to consider **unit vectors**.

DEFINITION: A **unit vector** is a vector whose magnitude is 1 unit. So if $\vec{a}$ is a unit vector then $\|\vec{a}\| = 1$.

The **standard unit vectors** are the unit vectors that point in the horizontal and vertical directions.

DEFINITION:

The vector $\vec{i}$ is the unit vector that points in the **positive horizontal direction**. Since its horizontal component is 1 and its vertical component is 0, we see that $\vec{i} = \langle 1, 0 \rangle$.

The vector $\vec{j}$ is the unit vector that points in the **positive vertical direction**. Since its horizontal component is 0 and its vertical component is 1, we see that $\vec{j} = \langle 0, 1 \rangle$.

Since $\vec{i}$ and $\vec{j}$ are unit vectors, $\|\vec{i}\| = 1$ and $\|\vec{j}\| = 1$.

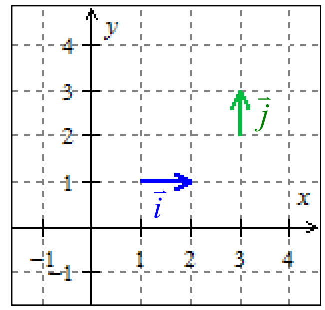


Figure 19: Unit vectors $\vec{i}$ and $\vec{j}$.

We can use vectors $\vec{i}$ and $\vec{j}$ to describe all other two-dimensional vectors. For example, in order to describe $\vec{v} = \langle 4, 3 \rangle$ (from Example 1) we can use vectors $\vec{i}$ and $\vec{j}$:

$$\begin{aligned} \vec{v} &= \langle 4, 3 \rangle \\ &= \langle 4, 0 \rangle + \langle 0, 3 \rangle \\ &= 4 \cdot \langle 1, 0 \rangle + 3 \cdot \langle 0, 1 \rangle \\ &= 4\vec{i} + 3\vec{j} \end{aligned}$$

Similarly, we can represent $\vec{s} = \langle 2, -6 \rangle$ and $\vec{r} = \langle -6, -4 \rangle$ (from the examples above) using vectors $\vec{i}$ and $\vec{j}$:

$$\begin{aligned} \vec{s} &= \langle 2, -6 \rangle \\ &= \langle 2, 0 \rangle + \langle 0, -6 \rangle \\ &= 2 \cdot \langle 1, 0 \rangle + (-6) \cdot \langle 0, 1 \rangle \\ &= 2\vec{i} + \left(-6\vec{j}\right) \\ &= 2\vec{i} - 6\vec{j} \end{aligned}$$

and

$$\begin{aligned} \vec{r} &= \langle -6, -4 \rangle \\ &= \langle -6, 0 \rangle + \langle 0, -4 \rangle \\ &= -6 \cdot \langle 1, 0 \rangle + (-4) \cdot \langle 0, 1 \rangle \\ &= -6\vec{i} + \left(-4\vec{j}\right) \\ &= -6\vec{i} - 4\vec{j} \end{aligned}$$

In general, if $\vec{a} = \langle a_1, a_2 \rangle$ is a vector, then $\vec{a} = a_1\vec{i} + a_2\vec{j}$.

EXAMPLE 6: Find the magnitude and direction of the vector $\vec{p} = -3\vec{i} + 7\vec{j}$.

SOLUTION:

First, notice that we can write $\vec{p}$ in component form as $\langle -3, 7 \rangle$. We've drawn an arrow representing $\vec{p}$ in Figure 20.

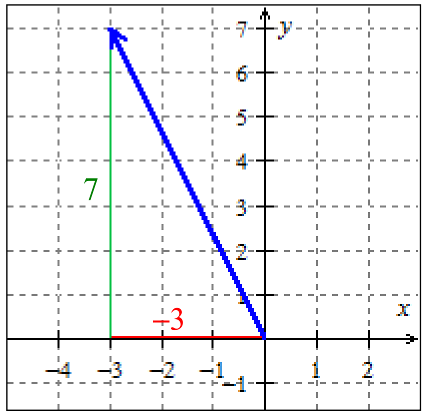


Figure 20: $\vec{p} = -3\vec{i} + 7\vec{j}$

We can use the Pythagorean Theorem to find $\|\vec{p}\|$, the magnitude of $\vec{p}$:

$$\|\vec{p}\|^2 = (-3)^2 + (7)^2$$
$$\Rightarrow \|\vec{p}\|^2 = 9 + 49$$
$$\Rightarrow \|\vec{p}\| = \sqrt{58}$$

So the magnitude of $\vec{p}$ is $\sqrt{58}$ units.

To describe the direction $\vec{p}$, we can find the angle that the vector makes with the positive x-axis; we've labeled this angle θ in Figure 21. To do this, we can first find the "reference angle" (labeled α in Figure 21) and then subtract this angle from 180° to find θ.

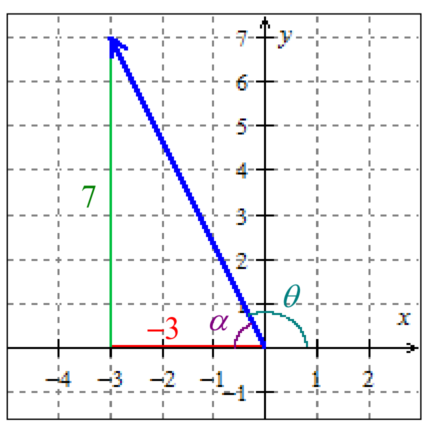


Figure 21

$$\tan(\alpha) = \tfrac{7}{3}$$ we use positive 3 since it represents a length in the "reference triangle".

$$\Rightarrow \quad \alpha = \tan^{-1}\left(\tfrac{7}{3}\right)$$

$$\Rightarrow \quad \alpha \approx 66.8^\circ$$

Thus,

$$\begin{aligned} \theta &\approx 180^\circ - 66.8^\circ \\ &= 113.2^\circ \end{aligned}$$

so $\vec{p}$ makes an angle of about 113.2° with the positive x-axis.

EXAMPLE 7: Suppose that the vector $\vec{u}$ is represented by an arrow on the coordinate plane whose tail is at the point $(-2, 1)$ and tip is at the point $(5, 6)$. Find the components of $\vec{u}$.

SOLUTION:

First, lets draw the arrow that represents $\vec{u}$.

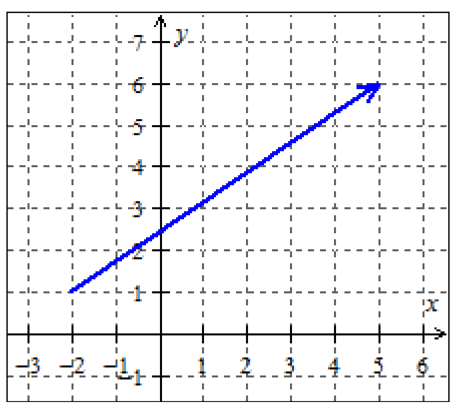


Figure 22: $\vec{u}$

To find the horizontal component of $\vec{u}$, we need to determine the difference in x-values between the tip and the tail:

$$5 - (-2) = 7$$

So the horizontal component is 7.

To find the vertical component of $\vec{u}$, we need to determine the difference in y-values between the tip and the tail:

$$6 - 1 = 5$$

So the vertical component is 5.

Thus, $\vec{u} = \langle 7, 5 \rangle = 7\vec{i} + 5\vec{j}$; see Figure 23.

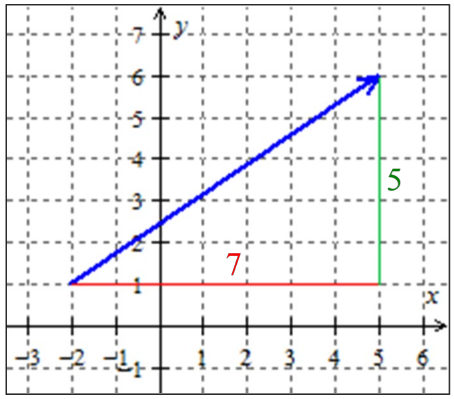


Figure 23: $\vec{u} = \langle 7, 5 \rangle$

> In general, if an arrow representing vector $\vec{v}$ has its tail at the point (x_1, y_1) and its tip at the point (x_2, y_2), then
>
> $$\vec{v} = \langle (x_2 - x_1), (y_2 - y_1) \rangle$$
> $$= (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j}.$$

EXAMPLE 8: Suppose that the vector $\vec{m}$ makes an angle of 37° with respect to the positive x-axis and that $\|\vec{m}\| = 20$. Find the horizontal and vertical components of $\vec{m}$.

SOLUTION:

First, lets draw the arrow that represents $\vec{m}$.

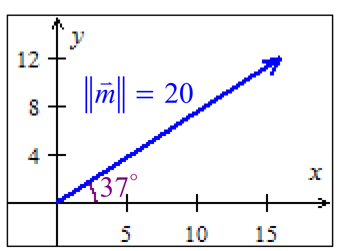


Figure 24: $\vec{m}$.

If we think of the arrow as being the hypotenuse of a right-triangle, we can use we can use right-triangle trigonometry to find the components of $\vec{m}$. (In Figure 25 we've labeled the horizontal component m_1 and the vertical component m_2.)

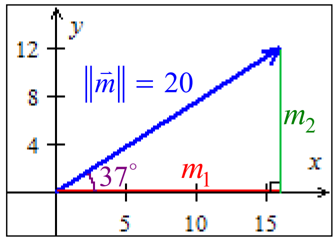


Figure 25

$$\cos(37^\circ) = \frac{m_1}{20} \quad \text{and} \quad \sin(37^\circ) = \frac{m_2}{20}$$

$$\Rightarrow \quad m_1 = 20\cos(37^\circ) \qquad \Rightarrow \quad m_2 = 20\sin(37^\circ)$$

$$\Rightarrow \quad m_1 \approx 15.97 \qquad \Rightarrow \quad m_2 \approx 12.04$$

Thus,

$$\begin{aligned} \vec{m} &= \left\langle 20\cos(37^\circ),\ 20\sin(37^\circ) \right\rangle \\ &\approx \left\langle 15.97,\ 12.04 \right\rangle \\ &\approx 15.97\vec{i} + 12.04\vec{j}. \end{aligned}$$

In general, if vector $\vec{v}$ makes an angle θ with the positive x-axis then, in component form,

$$\begin{aligned} \vec{v} &= \left\langle \|\vec{v}\|\cos(\theta),\ \|\vec{v}\|\sin(\theta) \right\rangle \\ &= \|\vec{v}\|\cos(\theta)\,\vec{i} + \|\vec{v}\|\sin(\theta)\,\vec{j}. \end{aligned}$$

On the next page, we'll list some properties of vector addition and scalar multiplication.

Properties of Vector Addition and Scalar Multiplication

If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors and a and b are scalars (i.e., $a, b \in \mathbb{R}$) then the following properties hold true:

1. **Commutativity of Vector Addition:** $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

2. **Associativity of Vector Addition:** $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

3. **Associativity of Scalar Multiplication:** $a(b\vec{v}) = (ab)\vec{v}$

4. **Distributivity:** $(a + b)\vec{v} = a\vec{v} + b\vec{v}$

   and

   $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$

5. **Identities:** $\vec{v} + \vec{0} = \vec{v}$ and $1 \cdot \vec{v} = \vec{v}$

You can check these properties by choosing some particular vectors and scalars and calculating the left and right side of each equation to see that they are equal.

Section IV: Vectors

Chapter 2: The Dot Product

In the previous chapter we studied how to add and subtract vectors and how to multiply vectors by scalars. In this chapter we will study how to multiply one vector by another using an operation called the **dot product**. Since we are focusing on two-dimensional vectors in this course, we will define the dot product in terms of two-dimensional vectors:

DEFINITION: If $\vec{u} = \langle u_1, u_2 \rangle$ and $\vec{v} = \langle v_1, v_2 \rangle$, then **the dot product of $\vec{u}$ and $\vec{v}$**, denoted $\vec{u} \cdot \vec{v}$, is defined as follows:

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 .$$

Thus, to compute the dot product of two vectors, we simply multiply the horizontal components of the two vectors and the vertical components of the two vectors and then add the results. It is important to note that the dot product produces a **scalar**.

EXAMPLE 1: If $\vec{a} = \langle 3, -9 \rangle$ and $\vec{b} = \langle 6, -1 \rangle$, find $\vec{a} \cdot \vec{b}$.

SOLUTION:

$$\begin{aligned} \vec{a} \cdot \vec{b} &= \langle 3, -9 \rangle \cdot \langle 6, -1 \rangle \\ &= 3 \cdot 6 + (-9) \cdot (-1) \\ &= 18 + 9 \\ &= 27 \end{aligned}$$

Notice that the result is a scalar, not a vector.

Now let's state some properties of the dot product. Afterwards, we will justify each of the properties.

Properties of the Dot Product

If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors then the following properties hold true:

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$ (commutative property)

2. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$ (distributive property)

3. $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$

4. $\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cos(\theta)$ where θ is the angle between $\vec{u}$ and $\vec{v}$.

The fourth property in our list is probably the most interesting since it suggests that the dot product can be used to measure the alignment of two vectors. Before discussing this property, let's justify the other properties in our list.

---

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$ (commutative property)

To justify this statement, let's compute $\vec{u} \cdot \vec{v}$ for some generic vectors $\vec{u}$ and $\vec{v}$ and show that the result is equal to $\vec{v} \cdot \vec{u}$.

Let $\vec{u} = \langle u_1, u_2 \rangle$ and $\vec{v} = \langle v_1, v_2 \rangle$. Then

$$\begin{aligned} \vec{u} \cdot \vec{v} &= \langle u_1, u_2 \rangle \cdot \langle v_1, v_2 \rangle \\ &= u_1 v_1 + u_2 v_2 \\ &= v_1 u_1 + v_2 u_2 \quad \text{(since scalar multiplication is commutative)} \\ &= \vec{v} \cdot \vec{u} \end{aligned}$$

---

2. $\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}$ (distributive property)

To justify this statement, let's compute $\vec{u}\cdot(\vec{v}+\vec{w})$ for some generic vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ and show that the result is equal to $\vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}$.

Let $\vec{u} = \langle u_1, u_2\rangle$, $\vec{v} = \langle v_1, v_2\rangle$, and $\vec{w} = \langle w_1, w_2\rangle$. Then

$$\begin{aligned}
\vec{u}\cdot(\vec{v}+\vec{w}) &= \langle u_1, u_2\rangle\cdot\left(\langle v_1, v_2\rangle + \langle w_1, w_2\rangle\right)\\
&= \langle u_1, u_2\rangle\cdot\left(\langle v_1 + w_1, v_2 + w_2\rangle\right)\\
&= u_1(v_1 + w_1) + u_2(v_2 + w_2)\\
&= u_1v_1 + u_1w_1 + u_2v_2 + u_2w_2 \quad \text{(since scalar multiplication is distributive)}\\
&= u_1v_1 + u_2v_2 + u_1w_1 + u_2w_2 \quad \text{(since scalar addition is commutative)}\\
&= \langle u_1, u_2\rangle\cdot\langle v_1, v_2\rangle + \langle u_1, u_2\rangle\cdot\langle w_1, w_2\rangle\\
&= \vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}
\end{aligned}$$

---

3. $\vec{v}\cdot\vec{v} = \|\vec{v}\|^2$

To justify this statement, let's compute $\vec{v}\cdot\vec{v}$ for some generic vector $\vec{v}$ and show that the result is equal to $\|\vec{v}\|^2$.

Let $\vec{v} = \langle v_1, v_2\rangle$. Then

$$\begin{aligned}
\vec{v}\cdot\vec{v} &= \langle v_1, v_2\rangle\cdot\langle v_1, v_2\rangle\\
&= v_1v_1 + v_2v_2\\
&= v_1^{\,2} + v_2^{\,2}\\
&= \|\vec{v}\|^2 \quad \left(\text{since } \|\vec{v}\| = \sqrt{v_1^{\,2} + v_2^{\,2}}\right)
\end{aligned}$$

4. $\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cos(\theta)$ where θ is the angle between $\vec{u}$ and $\vec{v}$.

To justify this statement, let's first draw two vectors $\vec{u}$ and $\vec{v}$ so that θ is the angle between the vectors; see Figure 1.

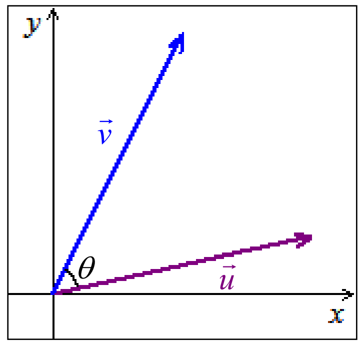


Figure 1

Now let's construct the vector $\vec{w} = \vec{v} - \vec{u}$. Recall from the previous chapter that we can obtain the vector $\vec{v} - \vec{u}$ by drawing an arrow that starts at the tip of $\vec{u}$ and ends at the tip of $\vec{v}$. We've drawn $\vec{w} = \vec{v} - \vec{u}$ in Figure 2.

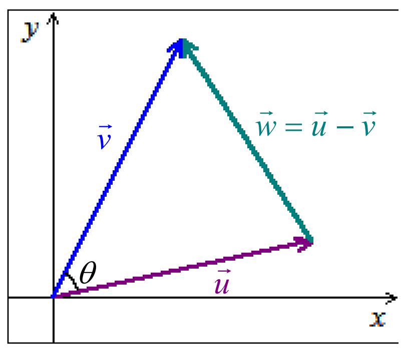


Figure 2

If we think of the arrows as being line segments instead of arrows, we have a triangle with side lengths $\|\vec{u}\|$, $\|\vec{v}\|$, and $\|\vec{w}\|$ and angle θ opposite the side $\|\vec{w}\|$; see Figure 3.

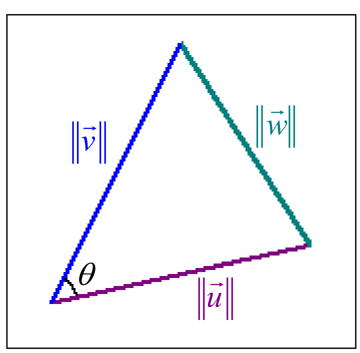


Figure 3

Now we can use the Law of Cosines to obtain an equation that relates the magnitudes of the vectors and angle θ.

$$\|\vec{w}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\cdot\|\vec{v}\|\cdot\cos(\theta) \qquad \langle 1 \rangle$$

We can use this equation to obtain the statement given in fourth property in the table above. First, let's find $\|\vec{w}\|^2$. Recall that $\vec{w} = \vec{v} - \vec{u}$. So

$$\|\vec{w}\|^2 = \vec{w}\cdot\vec{w} \qquad \text{(using the property 3 from the table above)}$$
$$= (\vec{v} - \vec{u})\cdot(\vec{v} - \vec{u})$$
$$= \vec{v}\cdot\vec{v} - \vec{u}\cdot\vec{v} - \vec{v}\cdot\vec{u} + \vec{u}\cdot\vec{u} \qquad \text{(using the property 2 from the table above)}$$
$$= \vec{v}\cdot\vec{v} - \vec{u}\cdot\vec{v} - \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{u} \qquad \text{(using the property 1 from the table above)}$$
$$= \|\vec{v}\|^2 - 2\vec{u}\cdot\vec{v} + \|\vec{u}\|^2 \qquad \text{(using the property 3 from the table above)}$$

We can now substitute $\|\vec{v}\|^2 - 2\vec{u}\cdot\vec{v} + \|\vec{u}\|^2$ for $\|\vec{w}\|^2$ in the equation labeled $\langle 1 \rangle$, above, and simplify to obtain equation given in the forth property in the table:

$$\cancel{\|\vec{v}\|^2} - 2\vec{u}\cdot\vec{v} + \cancel{\|\vec{u}\|^2} = \cancel{\|\vec{u}\|^2} + \cancel{\|\vec{v}\|^2} - 2\|\vec{u}\|\cdot\|\vec{v}\|\cdot\cos(\theta)$$
$$\Rightarrow \qquad \cancel{-2}\vec{u}\cdot\vec{v} = \cancel{-2}\|\vec{u}\|\cdot\|\vec{v}\|\cdot\cos(\theta)$$
$$\Rightarrow \qquad \vec{u}\cdot\vec{v} = \|\vec{u}\|\cdot\|\vec{v}\|\cdot\cos(\theta)$$

We can use the fact that $\vec{u}\cdot\vec{v} = \|\vec{u}\|\cdot\|\vec{v}\|\cos(\theta)$ to see that the dot product is intimately related to the alignment of the vectors $\vec{u}$ and $\vec{v}$.

EXAMPLE 2: If the angle between $\vec{u}$ and $\vec{v}$ is $\theta = 90^\circ$ (i.e., if $\vec{u}$ and $\vec{v}$ are perpendicular), find $\vec{u} \cdot \vec{v}$.

SOLUTION:

$$\begin{aligned} \vec{u} \cdot \vec{v} &= \|\vec{u}\| \cdot \|\vec{v}\| \cdot \cos(\theta) \\ &= \|\vec{u}\| \cdot \|\vec{v}\| \cdot \cos(90^\circ) \\ &= \|\vec{u}\| \cdot \|\vec{v}\| \cdot 0 \\ &= 0 \end{aligned}$$

So if $\vec{u}$ and $\vec{v}$ are perpendicular, $\vec{u} \cdot \vec{v} = 0$.

EXAMPLE 3: If $\vec{u}$ and $\vec{v}$ are non-zero vectors and $\vec{u} \cdot \vec{v} > 0$, what can you say about the angle θ between vectors $\vec{u}$ and $\vec{v}$. What if $\vec{u} \cdot \vec{v} < 0$?

SOLUTION:

Recall that $\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cos(\theta)$. Since $\vec{u} > 0$ and $\vec{v} > 0$, we see that the sign of $\vec{u} \cdot \vec{v}$ depends on the sign of $\cos(\theta)$.

- If $\cos(\theta) > 0$ then $\vec{u} \cdot \vec{v} > 0$; since $\cos(\theta) > 0$ when $0 \le \theta < 90^\circ$, we can conclude that $\vec{u} \cdot \vec{v} > 0$ when the angle between the vectors is acute (i.e., less than 90°).

- If $\cos(\theta) < 0$ then $\vec{u} \cdot \vec{v} < 0$; since $\cos(\theta) < 0$ when $90^\circ < \theta \le 180^\circ$, we can conclude that $\vec{u} \cdot \vec{v} < 0$ when the angle between the vectors is obtuse (i.e., greater than 90°).

In the two examples above we see that the dot product can be used to learn about the alignment of two vectors. In fact, the dot product can be used to find the angle between two vectors; see Example 4, below.

EXAMPLE 4: Find the angle between the vectors $\vec{a} = \langle 3, -9 \rangle$ and $\vec{b} = \langle 6, -1 \rangle$ from Ex. 1.

SOLUTION:

To find the angle θ between the two vectors we will use the fact that $\vec{a} \cdot \vec{b} = \|\vec{a}\| \cdot \|\vec{b}\| \cdot \cos(\theta)$.

Recall from Example 1 that $\vec{a} \cdot \vec{b} = 27$. Let's find $\|\vec{a}\|$ and $\|\vec{b}\|$:

$$\begin{aligned} \|\vec{a}\| &= \sqrt{(3)^2 + (-9)^2} \\ &= \sqrt{9 + 81} \\ &= 3\sqrt{10} \end{aligned}$$

and

$$\begin{aligned} \|\vec{b}\| &= \sqrt{(6)^2 + (-1)^2} \\ &= \sqrt{36 + 1} \\ &= \sqrt{37} \end{aligned}$$

Thus,

$$\begin{aligned} \vec{a} \cdot \vec{b} &= \|\vec{a}\| \cdot \|\vec{b}\| \cdot \cos(\theta) \\ \Rightarrow \quad 27 &= 3\sqrt{10} \cdot \sqrt{37} \cdot \cos(\theta) \\ \Rightarrow \quad \cos(\theta) &= \tfrac{27}{3\sqrt{10} \cdot \sqrt{37}} \\ \Rightarrow \quad \theta &= \cos^{-1}\left(\tfrac{9}{\sqrt{370}}\right) \approx 62.1^\circ \end{aligned}$$

So the angle between $\vec{a}$ and $\vec{b}$ is about 62.1°; see Figure 4.

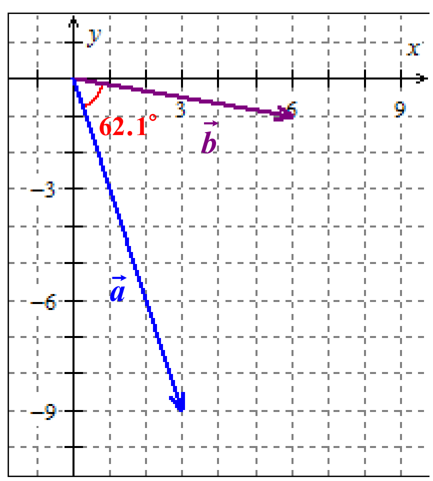


Figure 4