Vectors

Vectors are mathematical objects used to represent physical quantities like velocity, force, and displacement. Unlike ordinary numbers (or **scalars**), vectors have *both* magnitude and direction. So, for example, we can use a vector to describe the velocity of an object (i.e., the speed *and* direction).

> **DEFINITION:** A **vector** is a mathematical object that has both a *magnitude* (i.e., size) and a *direction*.

In order to distinguish between vectors from scalars (i.e., numbers) we need to use a different notation to denote vectors. In this class, we will use a small arrow above the vector name to denote a vector, so that $\vec{v}$ and $\vec{s}$ represent vectors while v and s represent scalars.

In this class we will focus on **two-dimensional vectors**. A two-dimensional vector can be represented by an **arrow** on the coordinate plane. The **length** of the arrow represents the **magnitude** of the vector and the **direction** of the arrow represents the direction of the vector. (We traditionally use the **angle between the positive x-axis and the arrow** to describe the **direction** of the vector.)

EXAMPLE 1: The vector $\vec{v}$ is depicted as an arrow on the coordinate plane in Figure 1.

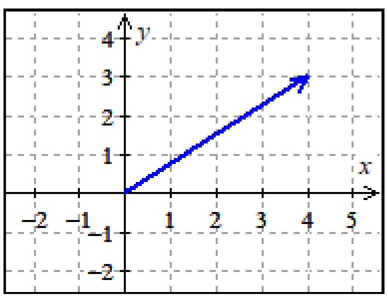


Figure 1: Arrow representing vector $\vec{v}$.

The **tip** of the vector is where the arrow ends and the **tail** of the vector is where the arrow begins. Thus, the tip of $\vec{v}$ is at the point $(4, 3)$ and the tail of the vector is at the origin, $(0, 0)$.

As mentioned above, the **length** of the arrow represents the **magnitude** of the vector. We denote the magnitude of vector $\vec{v}$ by $\|\vec{v}\|$. To find the magnitude of $\vec{v}$, we need to find the length of the arrow; we can do this by thinking of the arrow as being the hypotenuse of a right-triangle with side lengths 4 and 3 and then use the Pythagorean Theorem to find $\|\vec{v}\|$:

We can find the angle between the positive x-axis and the arrow to describe the **direction** of the vector. We've denoted this angle by θ in Figure 2.

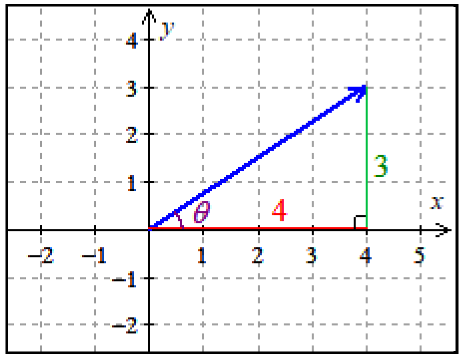


Figure 2

We can use the trigonometry that we studied earlier this quarter to find θ:

Although the magnitude and direction of the vector describe it completely, it is often useful to describe a vector by using its **horizontal and vertical components**. The *horizontal component* of $\vec{v}$ in Figure 2 is 4 units and a *vertical component* of vector $\vec{v}$ is 3 units. Thus, we say that the **component form of vector** $\vec{v}$ is $\langle 4, 3 \rangle$.

It's important to recognize that we could translate this vector anywhere in the coordinate plane and it would still be the same vector. For example, all of the arrows in Figure 3 represent $\vec{v}$ since all of these vectors have a horizontal component of 4 units and a vertical component of 3 units.

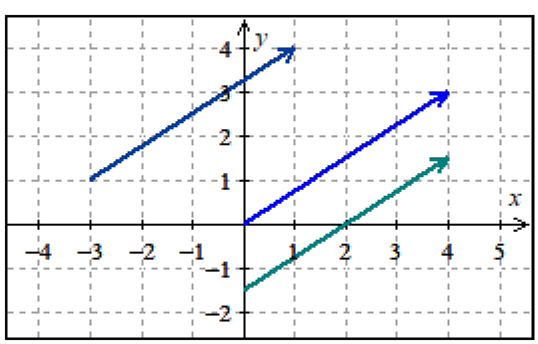


Figure 3: Three copies of $\vec{v}$.

Vectors Operations

We can **multiply any vector by a scalar** (i.e., a number) and we can **add or subtract any two vectors**.

When we **multiply a vector by a scalar**, we simply multiply the respective components of the vector by the scalar. Thus, if $\vec{a} = \langle a_1, a_2 \rangle$ and $k \in \mathbb{R}$, then $k\vec{a} = \langle ka_1, ka_2 \rangle$.

EXAMPLE 2: Let $\vec{v} = \langle 4, 3 \rangle$ (from Example 1). Find and draw vectors $\vec{m} = 2\vec{v}$ and $\vec{n} = -2\vec{v}$.

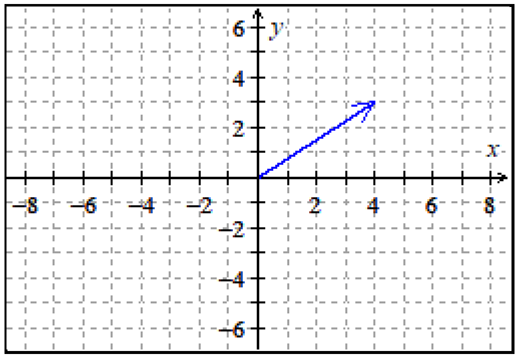


Figure 4: Vector $\vec{v}$.

If $\vec{a} = \langle a_1, a_2 \rangle$ is a vector and $k \in \mathbb{R}$ then $k\vec{a} = \langle ka_1, ka_2 \rangle$ has magnitude $|k| \cdot \|\vec{a}\|$. If $k > 0$ then $k \cdot \vec{a}$ points in the same direction as $\vec{a}$; if $k < 0$ then $k \cdot \vec{a}$ points in the opposite direction as $\vec{a}$.

When we **add**, we simply add the respective components of the vectors. Thus, if $\vec{a} = \langle a_1, a_2 \rangle$ and $\vec{b} = \langle b_1, b_2 \rangle$, then $\vec{a} + \vec{b} = \langle a_1 + b_1, a_2 + b_2 \rangle$ and $\vec{a} - \vec{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$

EXAMPLE 3: Let $\vec{v} = \langle 4, 3 \rangle$ (from Example 1) and $\vec{s} = \langle 2, -6 \rangle$. Find $\vec{v} + \vec{s}$.

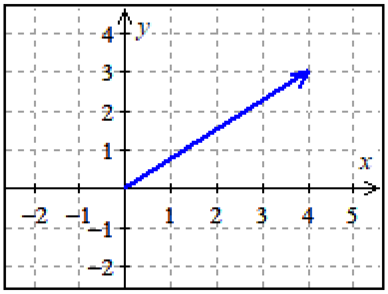


Figure 3: Vector $\vec{v}$.

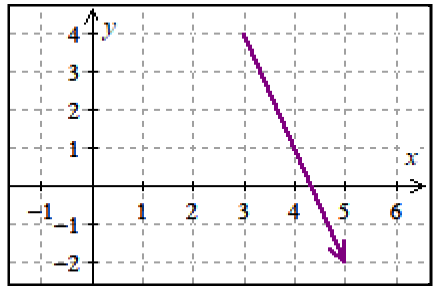
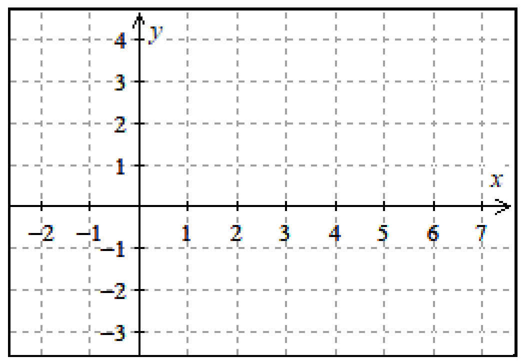


Figure 4: Vector $\vec{s}$.

Let's find $\vec{v} + \vec{s}$:



We can also add vectors by using arrows on a coordinate plane:

Figure 5: Adding vectors graphically.

Properties of Vector Addition and Scalar Multiplication

If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors and a and b are scalars (i.e., $a, b \in \mathbb{R}$) then the following properties hold true:

1. **Commutativity of Vector Addition:** $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
2. **Associativity of Vector Addition:** $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
3. **Associativity of Scalar Multiplication:** $a(b\vec{v}) = (ab)\vec{v}$
4. **Distributivity:** $(a + b)\vec{v} = a\vec{v} + b\vec{v}$ and $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$
5. **Identities:** $\vec{v} + \vec{0} = \vec{v}$ and $1 \cdot \vec{v} = \vec{v}$

In order to facilitate the communication and manipulation of vectors, it is useful to consider **unit vectors**.

DEFINITION: A **unit vector** is a vector whose magnitude is 1 unit. So if $\vec{a}$ is a unit vector then $\|\vec{a}\| = 1$.

The **standard unit vectors** are the unit vectors that point in the horizontal and vertical directions.

DEFINITION:

The vector $\vec{\boldsymbol{i}}$ is the unit vector that points in the **positive horizontal direction**. Since its horizontal component is 1 and its vertical component is 0, we see that $\vec{i} = \langle 1, 0 \rangle$.

The vector $\vec{\boldsymbol{j}}$ is the unit vector that points in the **positive vertical direction**. Since its horizontal component is 0 and its vertical component is 1, we see that $\vec{j} = \langle 0, 1 \rangle$.

Note that since $\vec{i}$ and $\vec{j}$ are *unit vectors*, $\|\vec{i}\| = 1$ and $\|\vec{j}\| = 1$.

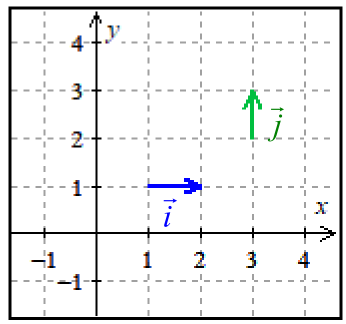


Figure 6: Unit vectors $\vec{i}$ and $\vec{j}$.

We can use vectors $\vec{i}$ and $\vec{j}$ to describe all other two-dimensional vectors. For example, we can describe $\vec{v} = \langle 4, 3 \rangle$ (from Example 1) using vectors $\vec{i}$ and $\vec{j}$: along with scalar multiplication and vector addition:

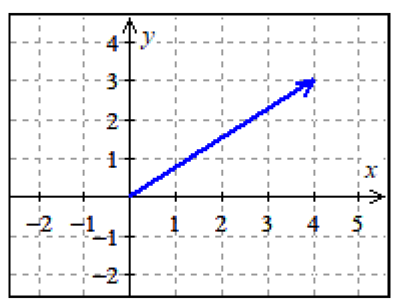


Figure 7: Vector $\vec{v}$.

In general, if $\vec{a} = \langle a_1, a_2 \rangle$ is a vector, then __________________.

EXAMPLE 4: Suppose that the vector $\vec{m}$ makes an angle of 37° with respect to the positive x-axis and $\|\vec{m}\| = 20$. Find the horizontal and vertical components of $\vec{m}$.

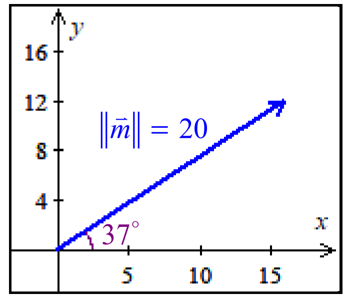


Figure 8: Vector $\vec{m}$

In general, if vector $\vec{v}$ makes an angle θ with the positive x-axis then, in component form,

The Dot Product

We've studied how to add and subtract vectors and how to multiply vectors by scalars. Now we'll study how to multiply one vector by another. This type of multiplication is called the **dot product**. Since we are focusing on two-dimensional vectors in this class, we will define the dot product in terms of two-dimensional vectors:

DEFINITION: If $\vec{u} = \langle u_1, u_2 \rangle$ and $\vec{v} = \langle v_1, v_2 \rangle$, then **the dot product of $\vec{u}$ and $\vec{v}$**, denoted $\vec{u} \cdot \vec{v}$, is defined as follows:

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2$$

Thus, to compute the dot product of two vectors, we simply multiply the horizontal components of the two vectors and the vertical components of the two vectors and then add the results. It is important to note that the dot product produces a **scalar**.

EXAMPLE 5: If $\vec{a} = \langle 3, -9 \rangle$ and $\vec{b} = \langle 6, -1 \rangle$, find $\vec{a} \cdot \vec{b}$.

Properties of the Dot Product

If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors then the following properties hold true:

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$ (commutative property)

2. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$ (distributive property)

3. $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$

4. $\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cos(\theta)$ where θ is the angle between $\vec{u}$ and $\vec{v}$.

We the dot product can be used to find the angle between two vectors.

EXAMPLE 6: Find the angle between vectors $\vec{a} = \langle 3, -9 \rangle$ and $\vec{b} = \langle 6, -1 \rangle$ from Example 5.

EXAMPLE 7: If the angle between $\vec{u}$ and $\vec{v}$ is $\theta = 90^{\circ}$ (i.e., if $\vec{u}$ and $\vec{v}$ are perpendicular), find $\vec{u} \cdot \vec{v}$.

EXAMPLE 8: If $\vec{u}$ and $\vec{v}$ are non-zero vectors and $\vec{u} \cdot \vec{v} > 0$, what can you say about the angle θ between vectors $\vec{u}$ and $\vec{v}$. What if $\vec{u} \cdot \vec{v} < 0$?