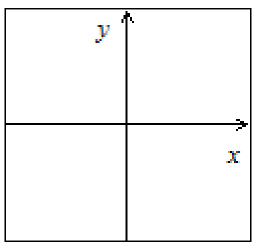
Power and Polynomial Functions

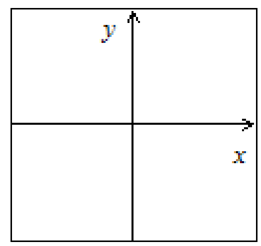
DEFINITION: A **power function** is a function of the form $f(x) = ax^n$ where $n \in \mathbb{Z}^{\text{nonneg}}$ (i.e., n is a nonnegative integer) and $a \in \mathbb{R}$ (A particular power function will have constants in place of a and n, leaving x as the only variable).

EXAMPLE: Compare the graphs of $s(x) = x^2$ and $t(x) = x^3$.



$s(x) = x^2$

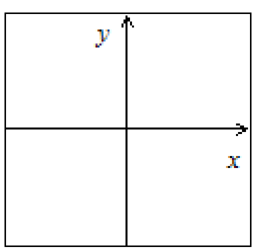
x	$s(x) = x^2$



$t(x) = x^3$

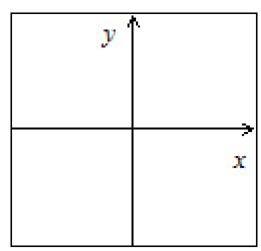
x	$t(x) = x^3$

EXAMPLE: Compare the graphs of $u(x) = -x^2$ and $w(x) = -x^3$.



$u(x) = -x^2$

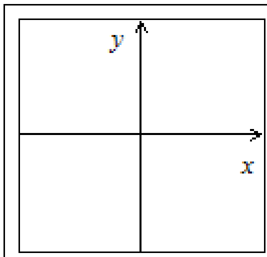
x	$u(x) = -x^2$



$w(x) = -x^3$

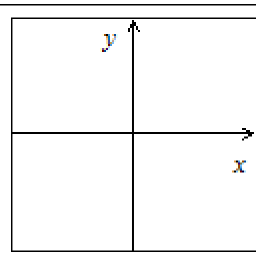
x	$w(x) = -x^3$

EXAMPLE: Compare the graphs of $f(x) = ax^p$ where p is an even positive integer and $g(x) = ax^q$. where q is an odd positive integer. (Note $a \in \mathbb{R}$.)



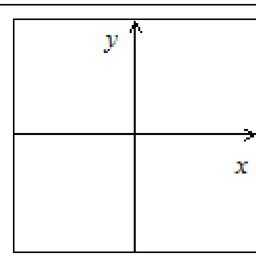
$a > 0$

or



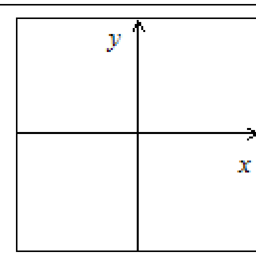
$a < 0$

$f(x) = ax^p$, p is even



$a > 0$

or



$a < 0$

$g(x) = ax^q$, q is odd

DEFINITION: A polynomial function is a function of the form:

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where $n \in \mathbb{Z}^{\text{nonneg}}$ (i.e., n is a nonnegative integer) and each $a_i \in \mathbb{R}$. (So a particular polynomial will have constants in place of all the n's and a_i's, leaving x as the only variable). Notice that we could also define a polynomial as a *sum of power functions*.

Here is some terminology we will use while studying polynomials:

- The number n is called the **degree** of the polynomial. It represents the largest power that appears in the rule for the function.
- The constants $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ are called **coefficients**.
- Each power function $a_k x^k$ in this sum is called a **term**.
- The highest-powered term, $a_n x^n$, is called the **leading term**.
- The number a_n is called the **leading coefficient**.
- The term a_0 is called the **constant term**.
- A polynomial is written in **standard form** if its terms are arranged from highest power to lowest power, reading from left to right. (The functions given in the example below are written in standard form.)

EXAMPLE: The functions given below are all polynomial functions.

a. $a(x) = 5x^3 - 8x^2 + 7x - 10$ **b.** $b(x) = 2x - 7$

c. $c(x) = x^8 + 7x^5 - 3x^4 - 10x^2 + 2x - 12$ **d.** $d(x) = -4x^6$

EXAMPLE: The function $a(x) = 5x^3 - 8x^2 + 7x - 10$ is a ___*degree* polynomial written in ___________*form*. The ________ ______ is $5x^3$, the *constant term* is _____, and the *coefficients* are ______________.

The domain of a polynomial function is ____________________.

Polynomial functions have extremely “nice” graphs since they are *smooth* and *continuous*. Their graphs have **no sharp corners**, **no gaps**, and **no jumps**. Let's draw a graph below that is definitely ***NOT*** a polynomial function:

Let's draw some graphs below that do represent polynomial functions. They need to be ***smooth*** **and** ***continuous*** since there aren't any sharp corners, gaps, or jumps in a polynomial's graph.

Based on your previous coursework, you should already know a great deal about 2^{nd} degree polynomial functions.

Let's consider a 4th degree polynomial function and then compare it to what we know about 2nd degree polynomials.

EXAMPLE: Consider the 4th degree polynomial function

$$g(x) = 3x^4 + 2x^3 + 5x^2 - 7x + 13.$$

Let's investigate the behavior of g when the input values get extremely large.

$$\begin{aligned} g(100) &= \mathbf{3\times 100^4} + 2\cdot 100^3 + 5\cdot 100^2 - 7\cdot 100 + 13 \\ &= \mathbf{300,000,000} + 2,000,000 + 50,000 - 700 + 13 \\ &= \mathbf{300,000,000} + 2,049,313 \\ &= \end{aligned}$$

$$\begin{aligned} g(1000) &= \mathbf{3\times 1000^4} + 2\cdot 1000^3 + 5\cdot 1000^2 - 7\cdot 1000 + 13 \\ &= \mathbf{3,000,000,000,000} + 2,000,000,000 + 5,000,000 - 7,000 + 13 \\ &= \mathbf{3,000,000,000,000} + 2,004,993,013 \\ &= \end{aligned}$$

$$\begin{aligned} g(10,000) &= \mathbf{3\times 10,000^4} + 2\cdot 10,000^3 + 5\cdot 10,000^2 - 7\cdot 10,000 + 13 \\ &= \mathbf{30,000,000,000,000,000} + 2,000,000,000,000 + 500,000,000 - 70,000 + 13 \\ &= \mathbf{30,000,000,000,000,000} + 2,000,499,930,013 \\ &= \end{aligned}$$

What we can see above is that the leading term is MUCH larger than all the other terms when the x-values are large. In fact, as the x-values get larger, the leading term represents a larger and larger fraction of the output value. Mathematicians often say that the leading term dominates the other terms of the polynomial when the x-values are sufficiently large. In fact, when viewed on a large enough scale, the graph of the polynomial function $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$ looks like the graph of its leading term, the power function $y = a_n x^n$. This behavior is called the **long-run behavior** of the polynomial.

Key Point: The **long-run behavior** (i.e., as $x \to \pm\infty$) of the graph of a polynomial function is determined by its leading term. Thus, the leading term of a polynomial function is the **dominant term** when the inputs are increase or decrease without bound (i.e., get large in absolute value).

EXAMPLE: Compare the graphs of $g(x) = 3x^4 + 2x^3 + 5x^2 - 7x + 13$ and $h(x) = 3x^4$.

Let's take a second look at 2nd degree polynomial functions. We know that the graphs of quadratic functions with positive leading coefficients are parabolas opening upwards. Compare that to the graph of $g(x) = 3x^4 + 2x^3 + 5x^2 - 7x + 13$. The graph of g has similar long-run behavior as quadratic functions with positive leading coefficients! In fact, **any** even degree polynomial with a positive leading coefficient has similar long-run behavior as quadratic functions with positive leading coefficients.

A similar argument can be made about odd degree polynomials: The long-run behaviors of their graphs are determined by their leading terms. Based on what we observed about power functions, this means that **any** odd degree polynomial has similar long-run behavior as a 3rd degree polynomial.

EXAMPLE: Which of the following could be the algebraic rule for the function graphed in Figure 1? What about Figure 2?

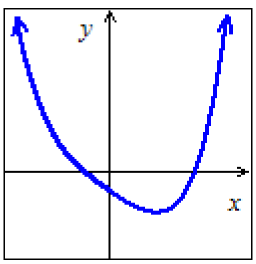


Figure 1

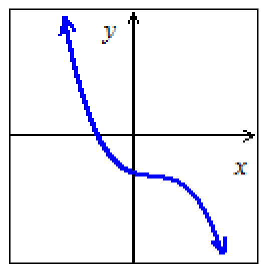


Figure 2

a. $a(x) = 3x^5 + x^2 - 2x - 2$

b. $b(x) = -3x^3 - x - 2$

c. $c(x) = -3x^6 + 3x^4 - 2x^2 - 2$

d. $d(x) = 3x^4 + x^2 - 5x - 2$

e. $e(x) = -3x^4 + 2x^3 + x^2 - 7$

f. $f(x) = x^3 + 2x^2 - 7$

g. $g(x) = x^6 - 5x^3 + 2x - 7$

h. $h(x) = -2x^3 + 3x^2 - 2x - 7$

Graphing Polynomial Functions

Based on what we've studied thus far, we should be able to recognize that polynomials like $f(x) = x^4 - 3x^3 - 63x^2 + 27x + 486$ and $g(x) = x^4 - 12x^3 - 27x^2 + 270x + 648$ have similar long-run behavior. Since they are both 4th degree polynomials with a positive leading coefficient, we know that their graphs must have arrows pointing up at the extreme left- and right-sides (i.e., the outputs of both functions increases without bound as the inputs increase without bound and as the inputs decrease without bound). See Figure 3.

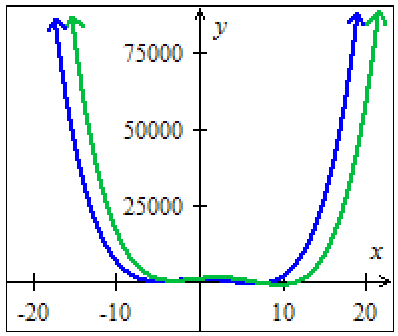


Figure 3: $y = f(x)$ and $y = g(x)$

Although $f(x) = x^4 - 3x^3 - 63x^2 + 27x + 486$ and $g(x) = x^4 - 12x^3 - 27x^2 + 270x + 648$ have similar long-run behavior, they are *not* identical functions! Let's study the *short-run* behavior of their graphs to see how these functions differ.

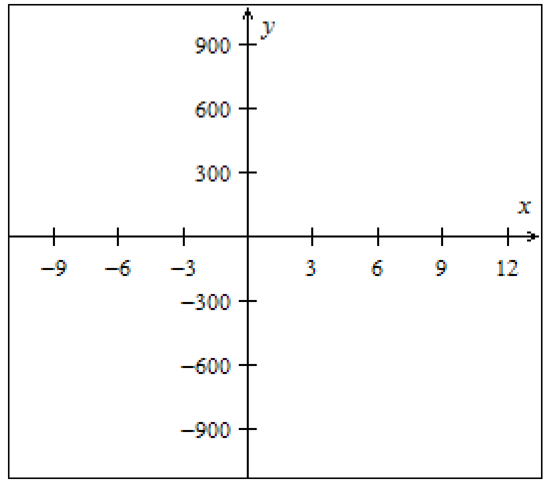
The **short-run behavior** of the graph of a function concerns graphical features that occur when the input values aren't very large. (It's hard to specify what "not large" means since it will be different for each function, but we'll look for particular graphical features rather than look within a particular interval, so we don't need to worry about being more specific.)

- Clearly, $x = 0$ isn't a large x-value, so the ***y*-intercept** will be part of the short-run behavior of a polynomial function's graph. (Notice that the y-coordinate of the y-intercept of a polynomial function is its constant term.)

- **Roots** (or **zeros**) are another important part of the **short-run** behavior of the graph of a polynomial function. To find the **roots** of a polynomial function, we can write it in **factored form**. (Recall what we know about 2nd degree polynomials.

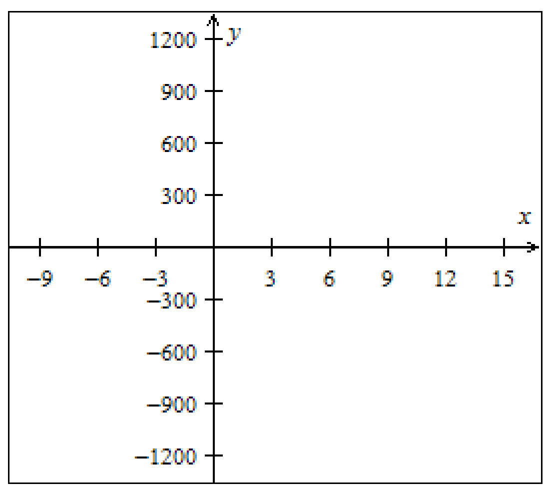
Let's use the short- and long-run behavior of $y = f(x)$ and $y = g(x)$ to sketch graphs of both functions on the next page.

a. $f(x) = x^4 - 3x^3 - 63x^2 + 27x + 486$



Draw the graph of $y = f(x)$.

b. $g(x) = x^4 - 12x^3 - 27x^2 + 270x + 648$



Draw the graph of $y = g(x)$.

EXAMPLE: Write an algebraic rule for the polynomial function p graphed Figure 4. Note that the graph passes through the point $(-3, 18)$.

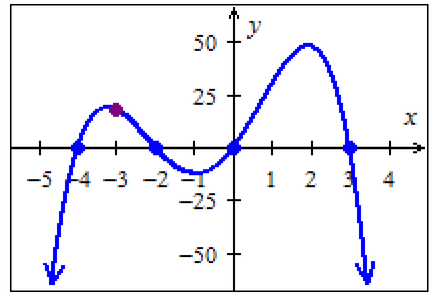


Figure 4: The graph of $y = p(x)$.

EXAMPLE: Write an algebraic rule for the polynomial function h graphed in Figure 5. Note that the y-intercept of h is $(0, 13)$.

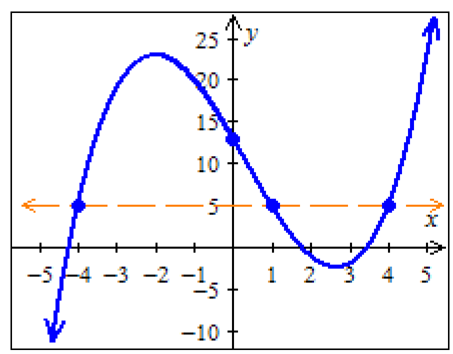


Figure 5: The graph of $y = h(x)$.

Properties of Polynomial Functions

- The graph of a polynomial is a smooth unbroken curve. (By "smooth" we mean that the graph does not have any sharp corners as turning points.)

- The graph of a polynomial always exhibits the characteristic that as $|x|$ gets very large, $|y|$ gets very large.

- If p is a polynomial of degree n, then the polynomial equation $p(x)=0$ has *at most* n distinct solutions; that is, p has at most n zeros. This is equivalent to saying that the graph of $y = p(x)$ crosses the x-axis at most n times. Thus a 5^{th} degree polynomial can have *at most* five x-intercepts.

- The graph of a polynomial function of degree n can have *at most* $n-1$ turning points. These points are the *relative maximums* or *relative minimums* of the function. For example, the graph of a polynomial of degree five can have *at most* four turning points. In particular, the graph of a quadratic (2^{nd} degree) polynomial function always has exactly one turning point – its vertex.

EXAMPLE: What is the minimum possible degree of the polynomial function in Figure 6?

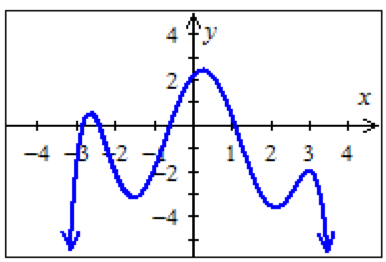


Figure 6

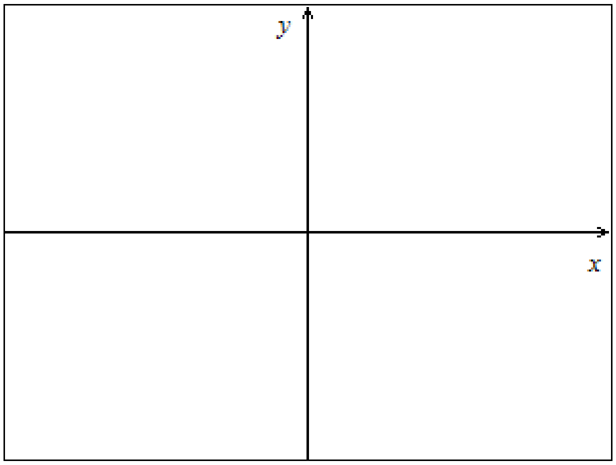
SOLUTION:

The polynomial function graphed in Figure 6 has four zeros and five turning points. The properties of polynomials tell us that a polynomial function with four zeros must have a degree of at least four. These properties also tell us that if a polynomial has degree *n* then it can have at most $n-1$ turning points. In other words the degree of a polynomial must be at least one more than the number of turning points. Since this graph has five turning points, the degree of the polynomial must be at least six.

Keep in mind that although a 6^{th} degree polynomial *may* have as many as six real zeros, it need **not** have that many. The graph in Figure 6 only has four real zeros.

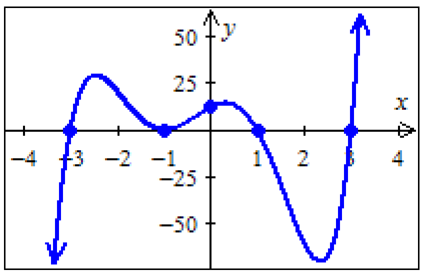
Practice for Power and Polynomial Functions

1. Sketch a graph of the function $m(x) = (x^2 + 3x - 10)(x^2 - 4)$.



Draw the graph of $y = m(x)$.

2. Write an algebraic rule for the polynomial function w graphed below. Note that the y-intercept of w is $(0, 12)$.



The graph of $y = w(x)$.