Section I: Functions and Their Graphs

Unit 8: Graph Transformations:

Combinations of Transformations

In this unit, we will investigate how to work with combinations of the different graph transformations that we studied in Units 6 and 7. Before we see what happens when we apply more than one transformation to a single function, let's use the following example to review the different transformations we've studied.

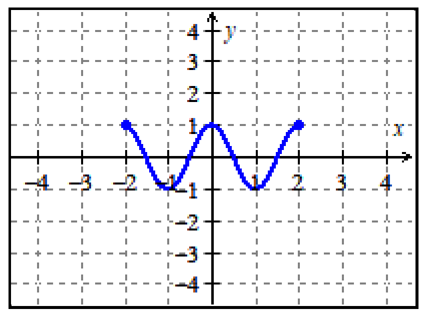
EXAMPLE: A function $y = m(x)$ is defined by the graph below. Match each transformation of m given in ***(a)*** – ***(d)*** with one of the graphs in ***(i)*** – ***(vi)*** below.

(a) $y = m(2x)$

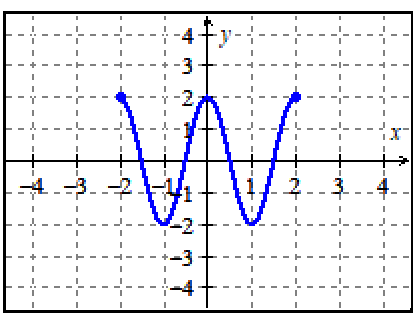
(b) $y = 2m(x)$

(c) $y = m(x) + 2$

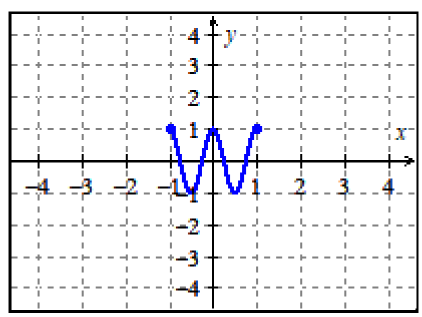
(d) $y = m(x + 2)$



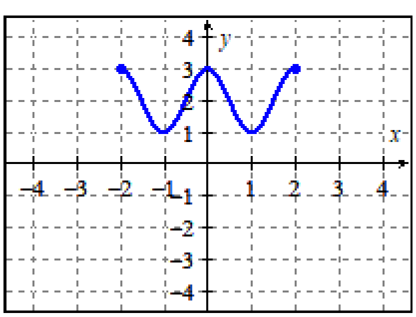
A graph of $y = m(x)$.



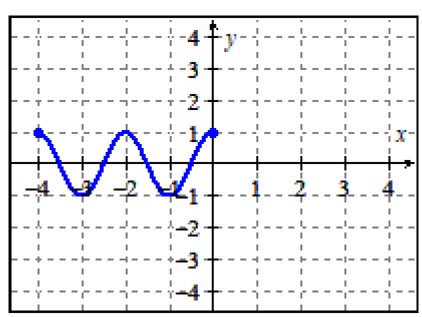
(i)



(ii)



(iii)



(iv)

SOLUTION:

Since $y = m(2x)$ is a horizontal compression of $y = m(x)$ by a factor of $\frac{1}{2}$, the graph **(*ii*)** must represent the function in **(*a*)**.

Since $y = 2m(x)$ is a vertical stretch of $y = m(x)$ by a factor of 2, the graph **(*i*)** must represent the function in **(*b*)**.

Since $y = m(x) + 2$ is a shift up 2 units of $y = m(x)$, the graph of **(*iii*)** must represent the function in **(*c*)**.

Since $y = m(x + 2)$ is a shift left 2 units of $y = m(x)$, the graph of **(*iv*)** must represent the function in **(*d*)**.

When transformations are combined, the order that you perform the transformations *matters*!! In practice, there are usually many different reasonable sequences of transformations that work. As long as you provide a correct sequence, you need not worry about the order you choose. But, in general, the order that you list the transformations **matters**, so if you mix up their order you won't obtain the proper graph. In the generalization below, we suggest an order to use each time you are confronted with a transformation problem. It isn't the only possible generalization, but we think it is the most useful.

SUMMARY OF GRAPH TRANSFORMATIONS

Suppose that f and g are functions such that $g(x) = a \cdot f\left(b(x - h)\right) + k$ and $a, b, h, k \in \mathbb{R}$. In order to transform the graph of the function f into the graph of g…

1st: horizontally stretch or compress the graph of f by a factor of $\frac{1}{|b|}$ (*stretch* if $0 < |b| < 1$ and *compress* if $|b| > 1$); if $b < 0$, *reflect* the graph about the y-axis.

2nd: shift the graph horizontally h units (shift to the *right* if h is positive and to *left* if h is negative).

3nd: vertically stretch or compress the graph by a factor of $|a|$ (*stretch* if $|a| > 1$ and *compress* if $0 < |a| < 1$) and, if $a < 0$, *reflect* the graph about the x-axis.

4th: shift the graph vertically k units (shift *up* if k is positive and *down* if k is negative).

The *order* in which these transformations are performed *matters*!

EXAMPLE: Describe a sequence of transformations that warp the graph of $r(x) = x^3$ into the graph of $t(x) = (-x-1)^3 + 2$.

SOLUTION:

First, let's rewrite $t(x)$ in terms r, and adjust its form:

$$\begin{aligned} t(x) &= (-x-1)^3 + 2 \\ &= r(-x-1) + 2 \\ &= r(-(x+1)) + 2 \end{aligned}$$

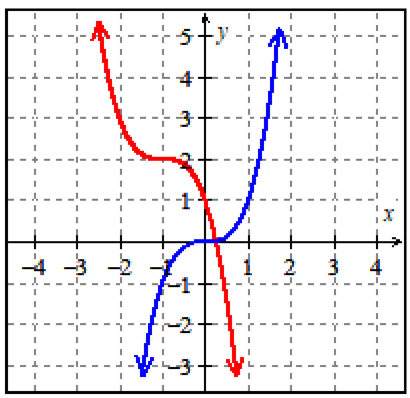
So to warp the graph of $y = r(x)$ into the graph of $y = t(x)$,

1st: Reflect the graph of $r(x) = x^3$ about the y-axis.

2nd: Shift left one unit.

3rd: Shift up two units to obtain the graph of $y = t(x)$.

Below, we've graphed $r(x) = x^3$ and $t(x) = (-x-1)^3 + 2$.



Graphs of $r(x) = x^3$ and $t(x) = (-x-1)^3 + 2$.

EXAMPLE: Describe a sequence of transformations that would warp the graph of $y = f(x)$ into the graph of $g(x) = 5f\left(\frac{1}{4}x - 2\right) + 7$.

SOLUTION:

First let's write $g(x)$ in terms of f and adjust its form:

$$\begin{aligned} g(x) &= 5f\left(\tfrac{1}{4}x - 2\right) + 7 \\ &= 5f\left(\tfrac{1}{4}(x - 8)\right) + 7 \end{aligned}$$

So to warp the graph of $y = f(x)$ into the graph of $y = g(x)$:

1st: horizontally stretch the graph by a factor of 4.

2nd: shift right 8 units.

3rd: vertically stretch by a factor of 5.

4th: shift up 7 units.

EXAMPLE: Describe a sequence of transformations that warps the graph of $s(x) = \sqrt{x}$ into the graph of $u(x) = -3\sqrt{2x - 7} + 4$.

SOLUTION:

First, let's write $u(x)$ in terms of s and adjust its form:

$$\begin{aligned} u(x) &= -3\sqrt{2x - 7} + 4 \\ &= -3 \cdot s\left(2x - 7\right) + 4 \\ &= -3 \cdot s\left(2\left(x - \tfrac{7}{2}\right)\right) + 4 \end{aligned}$$

We can now see that a sequence of transformations that warps the graph of $y = s(x)$ into the graph of $y = u(x)$ is the following:

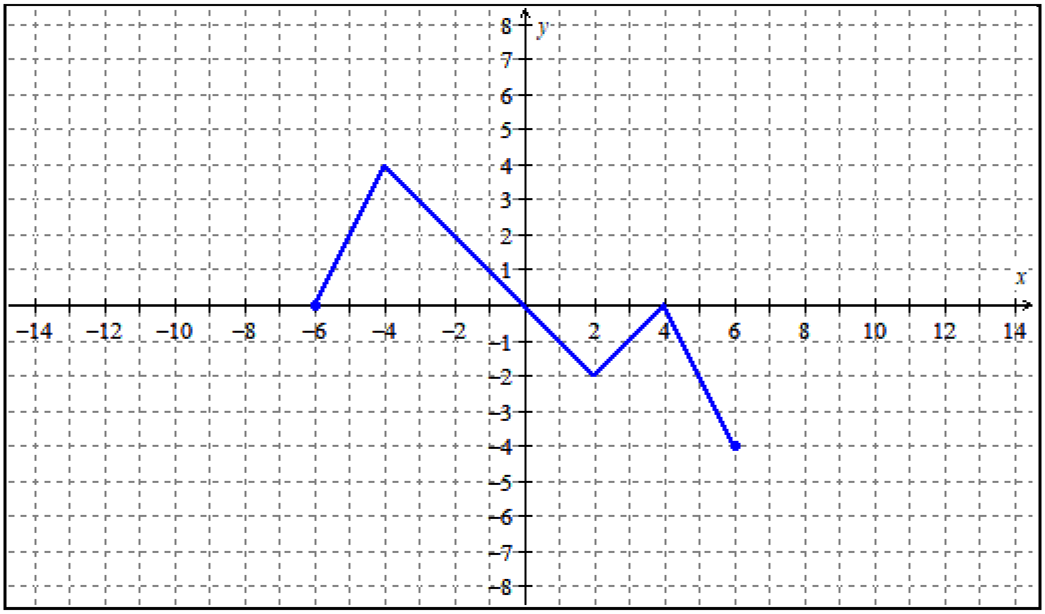
1st: Compress the graph horizontally by a factor of $\frac{1}{2}$.

2nd: Shift right $\frac{7}{2}$ units.

3rd: Stretch vertically by a factor of 3 and reflect about the x-axis.

4th: Shift up 4 units.

EXAMPLE: A graph of the function $y = f(x)$ is given below. Sketch a graph of $w(x) = -\frac{1}{2} f(x) + 3$.



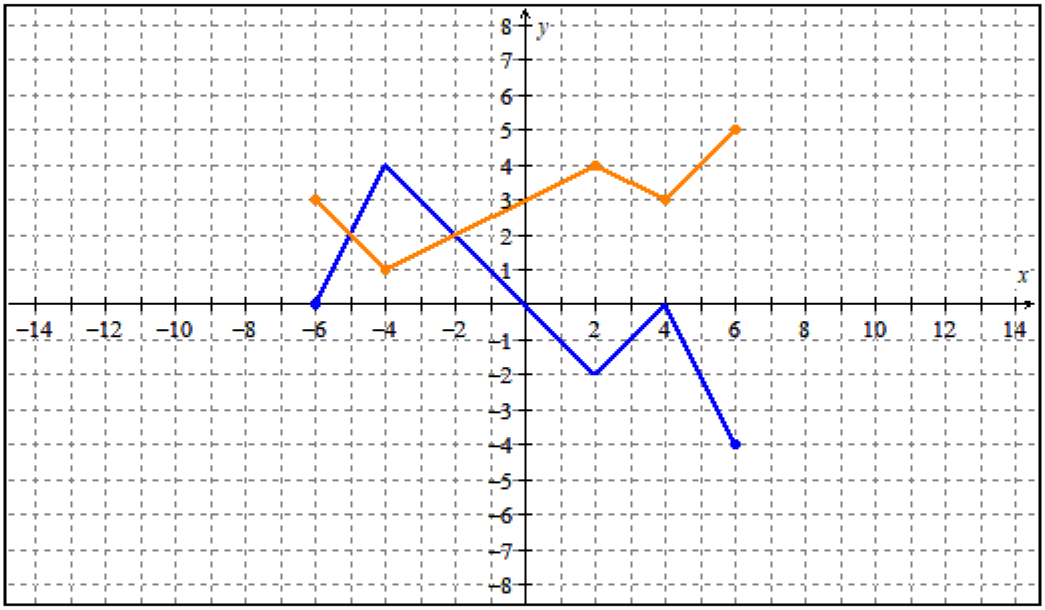
A graph of $y = f(x)$.

SOLUTION:

To sketch the graph of $w(x) = -\frac{1}{2} f(x) + 3$, we need to vertically compress the graph of $y = f(x)$ by a factor of $\frac{1}{2}$ and reflect the graph about the x-axis. Then we need to shift the graph up 3 units.

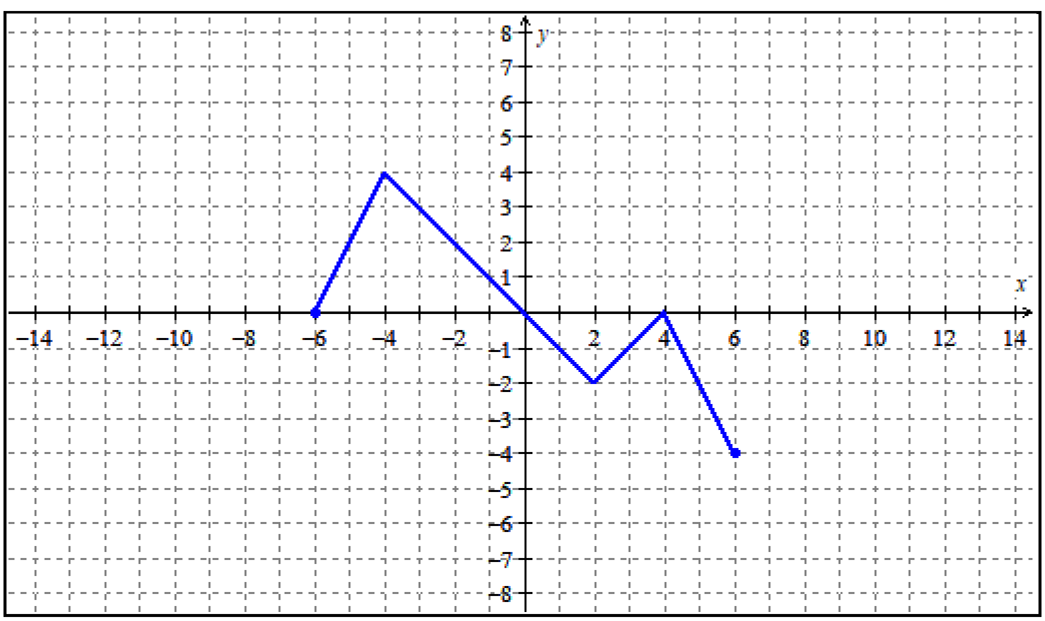
Let's just think about what happens to a few key points. Reflecting about the x-axis and stretching vertically by a factor of $\frac{1}{2}$ will not affect the points $(-6, 0)$ or $(4, 0)$. So after shifting up 3 units, these points will end up at $(-6, 3)$ and $(4, 3)$. The point $(-4, 4)$ becomes $(-4, 2)$ after we compress it by a factor of $\frac{1}{2}$, then it becomes $(-4, -2)$ after reflecting about the x-axis. Finally shifting up 3 units brings takes $(-4, -2)$ to $(-4, 1)$. The point $(2, -2)$ will end up at the point $(2, 4)$ and the point $(6, -4)$ will end up at the point $(6, 5)$. Verify these for yourself.

Now we can use these key points to sketch $w(x) = -\frac{1}{2} f(x) + 3$. (A graph of $w(x) = -\frac{1}{2} f(x) + 3$ is given below.)



Graphs of $y = f(x)$ and $w(x) = -\frac{1}{2} f(x) + 3$.

EXAMPLE: A graph of the function $y = f(x)$ is given below. Sketch a graph of $t(x) = -f\left(\frac{x}{2} + 1\right) - 3$.



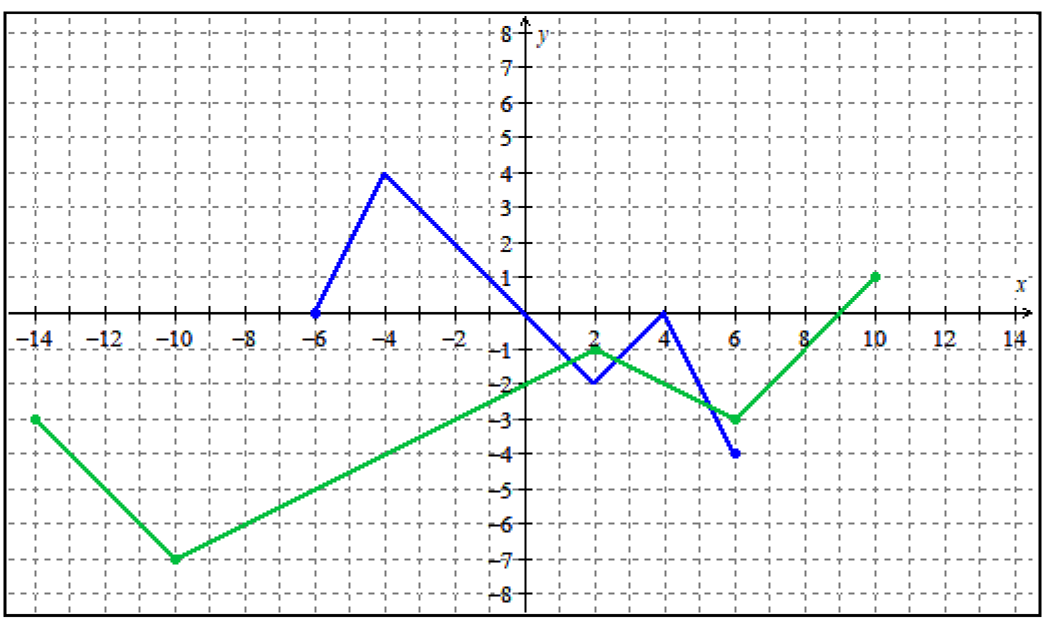
A graph of $y = f(x)$.

SOLUTION:

Let's start by rewriting $t(x)$ in terms of f (trying to mimic the form given in the Summary of Graph Transformations above).

$$\begin{aligned} t(x) &= -f\left(\frac{x}{2}+1\right)-3 \\ &= -f\left(\frac{1}{2}(x+2)\right)-3 \end{aligned}$$

So, to sketch the graph of $t(x)=-f\left(\frac{x}{2}+1\right)-3$, we need to reflect $y=f(x)$ about the x-axis, then stretch horizontally by a factor of 2. Next, shift left 2 units and then shift down 3 units. The points $(-6, 0)$ and $(4, 0)$ are not affected by the reflection about the x-axis. However, stretching horizontally by a factor of 2 takes the points $(-6, 0)$ and $(4, 0)$ to the points $(-12, 0)$ and $(8, 0)$, respectively. Shifting left 2 units and then shifting down 3 units takes the points $(-12, 0)$ and $(8, 0)$ to the points $(-14, -3)$ and $(6, -3)$, respectively. The point $(-4, 4)$ moves to $(-4, -4)$ upon reflection about the x-axis, then to $(-8, -4)$ upon stretching horizontally by a factor of 2. Finally shifting left 2 units and then shifting down 3 units takes the point $(-8, -4)$ to the point $(-10, -7)$. The point $(2, -2)$ will end up at the point $(2, -1)$ and the point $(6, -4)$ will end up at the point $(10, 1)$. You should verify these for yourself. Now we can use the points we just found to sketch $t(x)=-f\left(\frac{x}{2}+1\right)-3$; see the graph below.



Graphs of $y=f(x)$ and $t(x)=-f\left(\frac{x}{2}+1\right)-3$.

EXAMPLE: Describe a sequence of transformations that warp the graph of $f(x) = x^2$ into the graph of $g(x) = x^2 - 6x + 1$.

SOLUTION:

In order to determine how to transform the graph of $y = f(x)$ into the graph of $y = g(x)$, we need to write of $g(x)$ in terms of f. We can do this by **completing-the-square**. [To review the completing-the-square procedure, see Appendix A (pages A38-A39), §2.2 (page 140), and §2.4 (page 151) in our textbook.]

$$g(x) = x^2 - 6x + 1$$

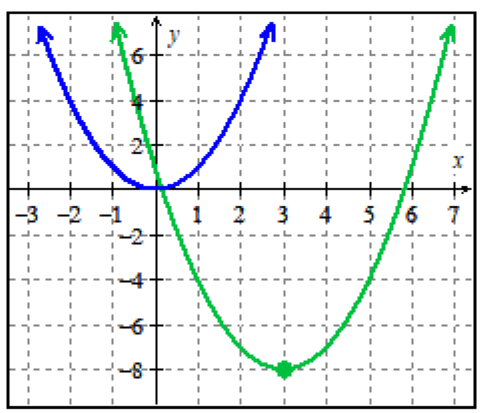
$$= x^2 - 6x + 9 - 9 + 1$$ (add and subtract $\left(\frac{1}{2}(-6)\right)^2 = 9$ to create a perfect square trinomial)

$$= (x^2 - 6x + 9) - 8$$

$$= (x - 3)^2 - 8$$

$$= f(x - 3) - 8$$

So, to transform the graph of $y = f(x)$ into the graph of $g(x) = f(x - 3) - 8$, first shift $y = f(x)$ right 3 units, then down 8 units; see the graph below. You should recognize $y = g(x)$ as a parabola opening upwards with vertex $(3, -8)$.



Graphs of $f(x) = x^2$ and $g(x) = f(x - 3) - 8$.

EXAMPLE: Describe a sequence of transformations that warp the graph $f(x) = x^2$ into the graph of $p(x) = 2x^2 - 3x + 5$.

SOLUTION:

First let's use the *completing-the-square* procedure in order to write of $p(x)$ in terms of f:

$$
\begin{aligned}
p(x) &= 2x^2 - 3x + 5 \\
&= 2\left(x^2 - \tfrac{3}{2}x\right) + 5 \qquad \text{(factor out the coefficient of the squared term from the first two terms)} \\
&= 2\left(x^2 - \tfrac{3}{2}x + \tfrac{9}{16} - \tfrac{9}{16}\right) + 5 \qquad \text{(add and subtract } \left(\tfrac{1}{2}\left(-\tfrac{3}{2}\right)\right)^2 = \tfrac{9}{16} \text{ to create a perfect square trinomial)} \\
&= 2\left(x^2 - \tfrac{3}{2}x + \tfrac{9}{16}\right) - 2\cdot\tfrac{9}{16} + 5 \\
&= 2\left(x - \tfrac{3}{4}\right)^2 - \tfrac{9}{8} + 5 \\
&= 2\left(x - \tfrac{3}{4}\right)^2 + \tfrac{31}{8} \\
&= 2f\left(x - \tfrac{3}{4}\right) + \tfrac{31}{8}
\end{aligned}
$$

So, to transform the graph of $y = f(x)$ into the graph of $y = p(x)$:

1st: Shift the graph right $\frac{3}{4}$ of a unit.

2nd: Vertically stretch by a factor of 2.

3rd: Shift up $\frac{31}{8}$ units.