Section II: Exponential and Logarithmic Functions

Unit 1: Introduction to Exponential Functions

Exponential functions are functions in which the variable appears in the exponent. For example, \( f(x) = 80 \cdot (0.35)^x \) is an exponential function since the independent variable, \( x \), appears in the exponent. One way to characterize exponential functions is to say that they represent quantities that change at a constant percentage rate.

**EXAMPLE:** When Rodney first got his job in 2018 he earned $41,000 per year. After every year, Rodney receives a 10% raise.

- **After one year,** Rodney gets a 10% raise; his salary becomes:

  \[
  41000 + 41000(0.10) = 41000(1 + 0.10) = 41000(1.10)
  \]

  So, to find his new salary, we must multiply his original salary by 1.10.

- **After one year of receiving** 41000(1.10) dollars (i.e., after his second year), Rodney gets another raise of 10%. His salary becomes:

  \[
  41000(0.10) + (41000(1.10))(0.10) = 41000(1.10)(1 + 0.10) = 41000(1.10)^2
  \]

  So, to find his new salary, we must multiply his original salary by \((1.10)^2\).

- **After his third year,** Rodney gets another raise of 10%. His salary becomes

  \[
  41000(1.10)^2 + \left(41000(1.10)^2\right)(0.10) = 41000(1.10)^2(1 + 0.10) = 41000(1.10)^3
  \]

  So, to find his new salary, we must multiply his original salary by \((1.10)^3\).
• We can now write a formula for Rodney’s salary \( s(t) \) (in dollars) after he has worked at the job \( t \) years:

\[
s(t) = 41000(1.10)^t
\]

This is obviously an exponential function (since the variable is in the exponent). Thus, we can see why exponential functions represent quantities that change at a constant percent rate. Note that this function works when \( t = 0 \) because

\[
s(0) = 41000(1.10)^0 = 41000(1) = 41000
\]

and $41,000 is Rodney's initial salary.

Below is another example that shows us that exponential functions represent quantities that change at a constant percentage rate.

**EXAMPLE:** Suppose that the population of the Expo Nation this year is 150,000. If the population decreases at a rate of 8% each year, find a function \( p \) that represents the population of the Expo Nation \( t \) years from now.

*population this year:*

150,000

*population after 1 year:*

\[
150000 - 150000(0.08) = 150000(1 - 0.08)
\]

\[
= 150000(0.92)
\]

*population after 2 years:*

\[
150000(0.92) - 150000(0.92)(0.08) = 150000(0.92)(1 - 0.08)
\]

\[
= 150000(0.92)(0.92)
\]

\[
= 150000(0.92)^2
\]

*population after 3 years:*

\[
150000(0.92)^2 - 150000(0.92)^2(0.08) = 150000(0.92)^2(1 - 0.08)
\]

\[
= 150000(0.92)^2(0.92)
\]

\[
= 150000(0.92)^3
\]
Observing the pattern above, we can deduce that the population of the Expo Nation after \( t \) years is given by the function \( p(t) = 150000(0.92)^t \).

Again, note that this function works when \( t = 0 \) because

\[
p(0) = 150000(0.92)^0 = 150000(1) = 150000
\]

and the initial population of Expo Nation was 150,000.

In order to generalize about exponential functions, we need to analyze the “structure” of both of the exponential functions \( s(t) = 41000(1.10)^t \) and \( p(t) = 150000(0.92)^t \) that we found in the examples above.

- In both functions the “initial value” (Rodney’s initial salary of $41,000 and Expo Nation’s initial population of 150,000) plays the same role:

\[ s(t) = 41000(1.10)^t \quad \text{and} \quad p(t) = 150000(0.92)^t \]

- Also, in both functions the number under the exponent (the “base” of the exponential function) is \( 1 + r \) where \( r \) is the decimal representation of the percent rate of change per units of \( t \):

\[
s(t) = 41000(1.10)^t \Rightarrow 1.10 = 1 + 0.10 \quad \text{⇒ Rodney’s raise: 10% per year.}
\]
\[
p(t) = 150000(0.92)^t \Rightarrow 0.92 = 1 + (−0.08) \quad \text{⇒ Population loss: 8% per year.}
\]

We can use the information above to obtain a definition of an exponential function:

**Definition:** An **exponential function** has the form \( f(x) = a \cdot b^x \) where \( a \) is the initial value (i.e., \( a = f(0) \)) and \( b \) is the growth factor (\( b = 1 + r \) where \( r \) is the decimal representation of the percent rate of change per unit of \( x \)).

**Note:** If \( r > 0 \), then \( b > 1 \), and the resulting function exhibits exponential growth.

If \(-1 < r < 0\), then \( b < 1 \), and the resulting function exhibits exponential decay.

**Also:** \( b \) is always positive. (Recall that \( b = 1 + r \), and it must be the case that \( r > −1 \): If the \( r < −1 \), the rate of change would be less than \(-100\%\) per unit of time which would represent losing more than everything in one unit of time; if \( r = −1 \), the rate of change would be \(-100\%\) per unit of time which would represent losing everything in the first unit of time, and then the function would be the constant function, \( y = 0 \), not an exponential function. Since \( r > −1 \), \( b = 1 + r > 0 \).)
Graphs of Exponential Functions

We already know what happens to the graphs of functions when we multiply their rules by positive and negative constants. Thus, all we need to determine is the shape of a generic exponential function and we will then be able to determine the shape of any exponential function.

There are basically two classes of exponential functions:

1. \( f(x) = a \cdot b^x \) with \( b > 1 \)
2. \( f(x) = a \cdot b^x \) with \( 0 < b < 1 \)

The next two examples will help us determine the shape of the graphs of these two classes of exponential functions.

**EXAMPLE:** Sketch a graph of \( h(x) = 2^x \). Note that this is an exponential function of the form \( h(x) = a \cdot b^x \) where \( a = 1 \) and \( b = 2 \).

**SOLUTION:**

In order to graph \( h \) we will create a table of values that we can use to form ordered pairs. Then we will plot the ordered pairs and connect our dots in an appropriate manner.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( h(x) )</th>
<th>( (x, h(x)) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>¼</td>
<td>(−2, ½)</td>
</tr>
<tr>
<td>-1</td>
<td>½</td>
<td>(−1, ½)</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>(0, 1)</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>(1, 2)</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>(2, 4)</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>(3, 8)</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>(4, 16)</td>
</tr>
</tbody>
</table>

**DEFINITION:** A horizontal asymptote is a horizontal line that the graph of a function gets arbitrarily close to as the input values get very large (or very small).

The function \( h(x) = 2^x \) graphed in the previous example has a horizontal asymptote at \( y = 0 \) (the \( x \)-axis).
EXAMPLE: Sketch a graph of \( g(x) = \left(\frac{1}{2}\right)^x \). Note that this is an exponential function of the form \( g(x) = a \cdot b^x \) where \( a = 1 \) and \( b = \frac{1}{2} \).

SOLUTION:

In order to graph \( g \) we will create a table of values that we can use to form ordered pairs. Then we will plot the ordered pairs and connect our dots in an appropriate manner.

\[
\begin{array}{|c|c|c|}
\hline
x & g(x) & (x, g(x)) \\
\hline
-4 & 16 & (-4, 16) \\
-3 & 8 & (-3, 8) \\
-2 & 4 & (-2, 4) \\
-1 & 2 & (-1, 2) \\
0 & 1 & (0, 1) \\
1 & \frac{1}{2} & (1, \frac{1}{2}) \\
2 & \frac{1}{4} & (2, \frac{1}{4}) \\
\hline
\end{array}
\]

Note that the horizontal asymptote for \( y = g(x) \) is the \( x \)-axis.

Based on the two examples above, we can conclude that the graph of an exponential function of the form \( f(x) = a \cdot b^x \) is increasing if \( b > 1 \) and decreasing if \( 0 < b < 1 \). (Technically we need to also state that \( a \) is positive for this increasing/decreasing behavior; the next example might clarify why this is so.)

EXAMPLE: Let’s use our understanding of graph transformations to predict how the graphs of \( m(x) = 4 \cdot 2^x \) and \( n(x) = -7 \cdot 2^x \) compare with the graph of \( h(x) = 2^x \). Then we’ll use a graphing calculator or other graphing utility to sketch graphs of \( h, m, \) and \( n \) so we can confirm our predictions. Finally, we’ll draw a conclusion about the role of \( a \) on the graph of an exponential function of the form \( f(x) = a \cdot b^x \).
**SOLUTION:**

We aren’t going to graph these functions here (since you can do that yourself on your graphing calculator) but we will discuss how we can use graph transformations to predict how the graph will look.

To compare \( m(x) = 4 \cdot 2^x \) with \( h(x) = 2^x \), we need to write \( m(x) \) in terms of \( h(x) \):

\[
m(x) = 4 \cdot 2^x = 4 \cdot h(x)
\]

Since \( m(x) \) is \( h(x) \) multiplied by 4 on the “outside,” we know that to graph \( y = m(x) \) we need to stretch the graph of \( y = h(x) \) vertically by a factor of 4. So if we perform this transformation to the \( y \)-intercept of \( y = h(x) \), which is \((0, 1)\), by a factor of 4, we see that the \( y \)-intercept of \( y = m(x) \) is \((0, 4)\). In Figure 3 we’ve graphed \( y = h(x) \) and \( y = m(x) \).

![Figure 3: \( h(x) = 2^x \) and \( m(x) = 4 \cdot 2^x \)](image)

To compare \( n(x) = -7 \cdot 2^x \) with \( h(x) = 2^x \), we need to write \( n(x) \) in terms of \( h(x) \).

\[
n(x) = -7 \cdot 2^x = -7 \cdot h(x)
\]

Since \( n(x) \) is \( h(x) \) multiplied by -7 on the “outside,” we know that to graph \( y = n(x) \) we need to reflect the graph of \( y = h(x) \) about the \( x \)-axis and stretch the graph of \( y = h(x) \) vertically by a factor of 7. So if we perform these transformations to the \( y \)-intercept of \( y = h(x) \), which is \((0, 1)\) we see that the \( y \)-intercept of \( y = n(x) \) is \((0, -7)\). In Figure 4 (below) we’ve graphed \( y = h(x) \) and \( y = n(x) \).
Notice that the number that plays the role of $a$ in the rules for $m(x)$ and $n(x)$ is the $y$-coordinate of the $y$-intercept for both functions:

$$m(x) = 4 \cdot 2^x \quad \Rightarrow \quad a = 4 \quad \Rightarrow \quad y \text{-intercept: } (0, 4)$$

$$n(x) = -7 \cdot 2^x \quad \Rightarrow \quad a = -7 \quad \Rightarrow \quad y \text{-intercept: } (0, -7)$$

We already suggested this in our definition of exponential functions earlier in these notes but let’s put this important fact in a green box:

The $y$-intercept of an exponential function of the form $f(x) = a \cdot b^x$ is $(0, a)$. 

Figure 4: $h(x) = 2^x$ and $n(x) = -7 \cdot 2^x$
THE NUMBER $e$

The symbol "$e$" represents a famous number:

$$e \approx 2.718281828459$$

The number $e$ is similar to $\pi$ in the sense that it is an extremely important irrational number. An irrational number cannot be expressed as the ratio of integers (i.e., $e$ cannot be expressed as a fraction and the decimal expansion for $e$ never establishes a pattern). The only way to express the number $e$ is with the symbol "$e$." (We can approximate it with a decimal, but we cannot express it exactly.)

As we’ll see in Unit 4, the number $e$ is a natural base for exponential functions: the function $f(x) = e^x$ is sometimes represented as $f(x) = \exp(x)$ since it is “The Exponential Function.” So we want to familiarize ourselves with $e$ as the start of our study of exponential functions.
FINDING AN EXPONENTIAL FUNCTION GIVEN TWO POINTS

EXAMPLE: Find the algebraic rule for an exponential function passing through the two points (1, 6) and (3, 24).

SOLUTION:

If we name the desired function \( f \), we can use function notation to translate the given ordered pairs into equations involving \( f \):

\[
(1, 6) \implies f(1) = 6 \\
(3, 24) \implies f(3) = 24
\]

Since we know that the desired function is exponential, we know that it has form

\[
f(x) = C \cdot a^x.
\]

We can now use this definition of \( f \) to build on the equations we found above:

\[
f(1) = C \cdot a^1 = 6 \\
f(3) = C \cdot a^3 = 24
\]

We can use these equations to form equivalent ratios that we can set equal; then we can use the resulting equation to solve for \( a \):

\[
\frac{C \cdot a^3}{C \cdot a} = \frac{24}{6}
\]

\[
\implies \frac{C \cdot a^3}{C \cdot a} = 4
\]

\[
\implies a^2 = 4
\]

\[
\implies a = 2 \quad \text{Note that although both 2 and -2 solve } a^2 = 4, \text{ only 2 can represent } a \text{ since the base of an exponential function is always positive.}
\]

Since \( a = 2 \), we know that the desired function is \( f(x) = C \cdot a^x \). We can use either one of the given ordered pairs to find \( C \).

\[
(1, 6) \implies f(1) = C \cdot 2^1 = 6 \\
\implies C = 3
\]

Therefore, \( f(x) = 3 \cdot 2^x \).
**EXAMPLE:** If \( f \) is an exponential function such that \( f(-3) = \frac{5}{8} \) and \( f(2) = 20 \), find an algebraic rule for \( f \).

**SOLUTION:**

Since we know that the desired function is *exponential*, we know that it has form

\[
f(x) = C \cdot a^x.
\]

We can now use this definition of \( f \) to create equations based on the given information:

\[
f(-3) = \frac{5}{8} = C \cdot a^{-3}
\]

\[
f(2) = 20 = C \cdot a^2
\]

We can use these equations to form equivalent ratios that we can set equal; then we can use the resulting equation to solve for \( a \):

\[
\frac{C \cdot a^2}{C \cdot a^{-3}} = \frac{20}{\frac{5}{8}}
\]

\[
\Rightarrow \frac{C \cdot a^2}{C \cdot a^{-3}} = 20 \cdot \frac{8}{5}
\]

\[
\Rightarrow a^5 = 32
\]

\[
\Rightarrow a = 2
\]

Since \( a = 2 \), we know that the desired function is \( f(x) = C \cdot 2^x \). We now use the fact that \( f(2) = 20 \) to find \( C \).

\[
f(2) = 20 = C \cdot 2^2
\]

\[
\Rightarrow 20 = 4C
\]

\[
\Rightarrow C = 5
\]

Therefore, \( f(x) = 5 \cdot 2^x \).