**Unit 1: Power Functions**

**DEFINITION:** A **power function** is a function of the form \( f(x) = ax^n \) where \( n \in \mathbb{Z}_{\text{nonneg}} \) (i.e., \( n \) is a nonnegative integer) and \( a \in \mathbb{R} \) (A particular power function will have constants in place of \( a \) and \( n \), leaving \( x \) as the only variable.)

Here is some terminology we will use while studying power functions:

- The number \( n \) is called the **degree** of the power function. (This can also be referred to as the **power** of the power function.)
- The constants \( a \) is called the **coefficient** of the power function.

**EXAMPLE:** Which of the following functions are power functions? For each power function, state the value of the constants \( a \) and \( n \) in the formula \( y = ax^n \).

- \( a. \quad b(x) = 5(x - 3)^4 \)
- \( b. \quad m(x) = 7\sqrt[3]{x} \)
- \( c. \quad l(x) = 3 \cdot 2^x \)
- \( d. \quad s(x) = \sqrt[3]{\frac{7}{x^3}} \)

**SOLUTIONS:**

- **a.** The function \( b(x) = 5(x - 3)^4 \) is not a power function because we cannot write it in the form \( y = ax^n \).

- **b.** The function \( m(x) = 7\sqrt[3]{x} \) is a power function because we can rewrite its formula as \( m(x) = 7 \cdot x^{1/4} \). So \( a = 7 \) is the coefficient and \( n = \frac{1}{4} \) is the degree.

- **c.** The function \( l(x) = 3 \cdot 2^x \) is not a power function because the power is not constant. In fact, \( l(x) = 3 \cdot 2^x \) is an exponential function.
d. Since

\[
\sqrt{x^5} = \sqrt[7]{x^5}
\]

\[
= x^{5/2}
\]

\[
= \sqrt{7} \cdot x^{-5/2}
\]

we see that \( s(x) = \sqrt[7]{x^5} \) can be written in the form \( y = ax^n \) where \( a = \sqrt{7} \) is the coefficient and \( n = -\frac{5}{2} \) is the degree, so \( s \) is a power function.

As is the case with linear functions and exponential functions, given two points on the graph of a power function, we can find the function’s formula.

**EXAMPLE:** Suppose that the points \((1, 81)\) and \((3, 729)\) are on the graph of a function \( f \). Find an algebraic rule for \( f \) assuming that it is ...

- **a.** a linear function.  
- **b.** an exponential function  
- **c.** a power function.

**SOLUTIONS:**

**a.** If \( f \) is a linear function we know that its rule has form \( f(x) = mx + b \). We can use the two given points to solve for \( m \).

\[
m = \frac{729 - 81}{3 - 1} = \frac{648}{2} = 324
\]

So now we know that \( f(x) = 324x + b \). We can use either one of the given points to find \( b \). Let’s use \((1, 81)\):

\[
(1, 81) \quad \Rightarrow \quad f(1) = 81 = 324(1) + b
\]

\[
\Rightarrow \quad b = 81 - 324
\]

\[
\Rightarrow \quad b = -243
\]

Thus, if \( f \) is linear, its rule is \( f(x) = 324x - 243 \).
b. If $f$ is an exponential function we know its rule has form $f(x) = Ca^x$. We can use the two given points to find two equations involving $C$ and $a$:

$$(1, 81) \Rightarrow f(1) = 81 = Ca^1$$

$$(3, 729) \Rightarrow f(3) = 729 = Ca^3.$$ 

In Section II: Unit 2 we solved similar systems of equations by forming ratios. Let’s try a different method here: the substitution method.

Let’s start by solving the first equation for $C$:

$$81 = Ca^1$$

$$\Rightarrow C = \frac{81}{a}$$

Now we can substitute the expression $\frac{81}{a}$ for $C$ in the second equation:

$$729 = Ca^3$$

$$\Rightarrow 729 = \frac{81}{a} \cdot a^3$$

$$\Rightarrow 729 = 81 \cdot a^2$$

$$\Rightarrow \frac{729}{81} = a^2$$

$$\Rightarrow 9 = a^2$$

$$\Rightarrow a = \sqrt{9} = 3$$ (we don't need $\pm \sqrt{9}$ since the base of an exponential function is always positive)

Now that we know what $a$ is, we can use the fact that $C = \frac{81}{a}$ to find $C$:

$$C = \frac{81}{a}$$

$$= \frac{81}{3}$$

$$= 27$$

Thus, if $f$ is exponential, its rule is $f(x) = 27 \cdot 3^x$. 

c. Since \( f \) is a power function we know that its rule has form \( f(x) = ax^n \). We can use the two given points to find two equations involving \( a \) and \( n \):

\[
(1, 81) \quad \Rightarrow \quad f(1) = 81 = a(1)^n
\]

\[
(3, 729) \quad \Rightarrow \quad f(3) = 729 = a(3)^n.
\]

We can use the first equation to immediately find \( n \).

\[
81 = a(1)^n
\]

\[
\Rightarrow \quad a = 81
\]

Now we can find \( n \) by substituting \( a = 81 \) into the second equation:

\[
729 = 81(3)^n
\]

\[
\Rightarrow \quad \frac{729}{81} = 3^n
\]

\[
\Rightarrow \quad 9 = 3^n \quad \text{(note that this could be solved with logarithms if the solution weren't so obvious)}
\]

\[
\Rightarrow \quad n = 2
\]

Thus, if \( f \) is a power function, its rule is \( f(x) = 81x^2 \).

In Figure 1, let’s graph the three functions we’ve found that all pass through the points \((1, 81)\) and \((3, 729)\).

![Figure 1: Graphs of \( y = 324x - 243 \), \( y = 27 \cdot 3^x \), and \( y = 81x^2 \) passing through \((1, 81)\) and \((3, 729)\).](image)
Graphs of Power Functions

The graphs of power functions behave differently for large $x$ values than for small $x$ values.

When $x$ values are greater than 1, the greater the degree of the power function, the faster the outputs grow. In Figure 2 we've graphed six power functions: notice that as the degree increases, the outputs increase more and more quickly. Sometimes mathematicians say, “when $x$ is larger, larger powers dominate smaller powers,” in order to represent this behavior.

![Figure 2: Graphs of $y = x$, $y = x^{3/2}$, $y = x^{2}$, $y = x^{3}$, $y = x^{4}$, and $y = x^{5}$](image)

When $x$ values are between 0 and 1, the graphs of power functions exhibit the opposite behavior: the smaller the degree, the larger the output values. In Figure 3 we've graphed six power functions, emphasizing the interval $(0, 1)$ on the $x$-axis: notice how the linear power function $y = x$ has greater output values than functions of larger power. Sometimes mathematicians say, “when $x$ is small, smaller powers dominate larger powers,” in order to represent this behavior.

![Figure 3: Graphs of $y = x$, $y = x^{3/2}$, $y = x^{2}$, $y = x^{3}$, $y = x^{4}$, and $y = x^{5}$](image)
Comparing Power Functions and Exponential Functions

Power functions and exponential functions are often confused for one another since they both involve exponents. Expressions with exponents consist of a base (i.e., something that’s raised to the exponent) and the exponent: in a power function, the base is a variable and the exponent is a constant (i.e., a number) while in an exponential function, the base is a constant and the exponent is a variable.

**EXAMPLE:** The functions $y = x^3$, $y = 7x^{30}$, and $y = \frac{1}{7}x^{1/3}$ are power functions since they involve a variable raised to a constant – so the base is a variable and the exponent is a constant.

The functions $y = 3^x$, $y = 7 \cdot 30^x$, and $y = \frac{1}{7} \cdot \left(\frac{1}{3}\right)^x$ are exponential functions since they involve a constant raised to a variable – so the base is a constant and the exponent is a variable.

Another important distinction to make between power functions and exponential functions concerns their long-term behavior. The long-term behavior of a function is the behavior when $x$-values are large, and keep getting larger. (We started looking at the long-term behavior of power functions on the previous page when we looked at different power functions when the $x$-values are greater than 1.) Mathematicians often use the notation “$x \rightarrow \infty$” to represent this concept and we often translate this notation with the phrase, “$x$ approaches infinity,” but infinity isn’t actually a “thing” we can approach – it’s just the idea of “get bigger and bigger with no limit on how big we get” – so it’s better communicated as “$x$ increases without bound.”

**EXAMPLE:** In this video we’ll compare the long-term behavior of power functions and exponential functions.

(Here’s a link to the Desmos file used in the graph so that you can investigate on your own: [www.desmos.com/calculator/dknkn1pbnv](http://www.desmos.com/calculator/dknkn1pbnv).)

**Key Point:** Any positive increasing exponential function eventually grows faster than any power function.