Section II: Exponential and Logarithmic Functions

Module 5: Logarithmic Functions

EXAMPLE: Consider the exponential function $k(x) = 2^x$ which is graphed in Figure 1.

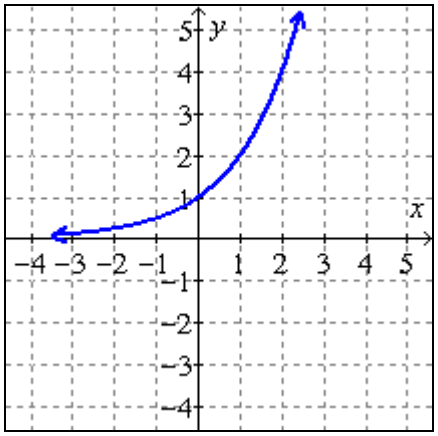

Figure 1: The graph of $y = k(x)$.

Does k have an inverse?

The graph of $k(x) = 2^x$ passes the horizontal line test (no horizontal line crosses the graph of k more than once), which implies that each output (or y-value) comes from exactly one input (or x-value). Thus, k is one-to-one, so k has an inverse function.

Sketch a graph of k^{-1}.

As we learned in Section I: Module 6, the graph of the inverse of a function is the reflection of the function about the line $y = x$. So to graph k^{-1} we need to reflect the graph of k about the line $y = x$.

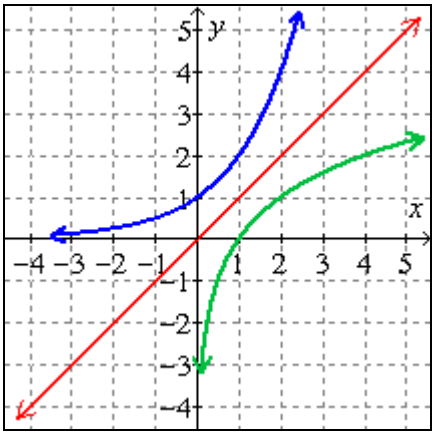

Figure 2: The graphs of $y = k(x)$, $y = k^{-1}(x)$, and $y = x$.

We call the inverses of exponential functions **logarithmic functions**. So the inverse of $k(x) = 2^x$ is called the **base-2 logarithmic function** and is written $k^{-1}(x) = \log_2(x)$. Notice that the **domain** of $k^{-1}(x) = \log_2(x)$ is $[0, \infty)$.

Since all exponential functions have graphs that are similar to that of $k(x) = 2^x$ (specifically, the graphs of **all** exponential functions **pass the horizontal line test**, so all exponential functions are **one-to-one**), we can conclude that **all exponential functions have inverse functions**. The inverse of an exponential function is a **logarithmic function**. The **domain** of a logarithmic function is $(0,\infty)$.

The inverse of the function $f(x) = b^x$ (where $b > 0$) is the function $f^{-1}(x) = \log_b(x)$, the **logarithm of base *b***.

EXAMPLE: If $h(x) = 7^x$, then the inverse of h is the function $h^{-1}(x) = \log_7(x)$.

EXAMPLE: Consider the function $g(x) = 10^x$. Since g is an exponential function, its inverse is a logarithmic function. Both g and g^{-1} are graphed in Figure 3.

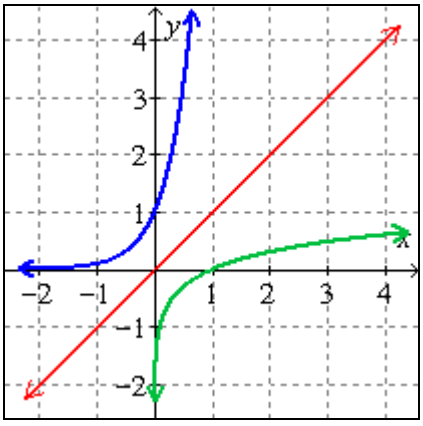

Figure 3: The graphs of $y = g(x)$, $y = g^{-1}(x)$, and $y = x$.

The inverse of $g(x) = 10^x$ is $g^{-1}(x) = \log_{10}(x)$, the **logarithm of base 10**. This is an important and often-used function so it is given a special name, the **common logarithm**, and is denoted $g^{-1}(x) = \log(x)$.

Now let's investigate the relationship between the inputs and outputs of a logarithmic function.

 **EXAMPLE:** Consider $g(x) = 10^x$ and $g^{-1}(x) = \log(x)$. See Tables 1 and 2 below.

Table 1: $g(x) = 10^x$

x	$g(x)$
–2	$\frac{1}{100}$
–1	$\frac{1}{10}$
0	1
1	10
2	100
3	1000

Table 2: $g^{-1}(x) = \log(x)$

x	$g^{-1}(x)$
$\frac{1}{100}$	–2
$\frac{1}{10}$	–1
1	0
10	1
100	2
1000	3

Recall that the inputs become the outputs and the outputs become the inputs when we create the inverse of a function. Since the input into exponential functions (like $g(x) = 10^x$) are exponents, the outputs of logarithmic functions (like $g^{-1}(x) = \log(x)$) are exponents. Thus, it is useful to keep the following sentence in mind while working with logarithms.

> "*The outputs for logarithms are exponents.*"

 **EXAMPLE:** Simplify the following:

a. $\log_2(8)$ **b.** $\log_5(125)$ **c.** $\log_9\left(\frac{1}{3}\right)$

SOLUTIONS:

a.

$$\log_2(8) = \log_2\left(2^3\right)$$
$$= 3$$

b.

$$\log_5(125) = \log_5\left(5^3\right)$$
$$= 3$$

c.

$$\log_9\left(\frac{1}{3}\right) = \log_9\left(3^{-1}\right)$$
$$= \log_9\left(\left(9^{1/2}\right)^{-1}\right)$$
$$= \log_9\left(9^{-1/2}\right)$$
$$= -\frac{1}{2}$$

Recall the exponential function $g(x) = e^x$. Like all exponential functions, $g(x) = e^x$ is one-to-one, so it has an inverse: $g^{-1}(x) = \log_e(x)$. Just as $g(x) = e^x$ is an important function, its inverse is important and, therefore, has a special name: the **natural logarithm**, and is usually denoted $g^{-1}(x) = \ln(x)$.

EXAMPLE: Simplify the following:

a. $\ln\left(e^4\right)$ **b.** $\ln\left(\frac{1}{e}\right)$ **c.** $\ln(e)$

SOLUTIONS:

a.

$\ln\left(e^4\right) = 4$ (since the natural logarithm has base e)

b.

$$\ln\left(\frac{1}{e}\right) = \ln\left(e^{-1}\right)$$
$$= -1$$

c.

$$\ln(e) = \ln\left(e^1\right)$$
$$= 1$$

LOG LAWS

There are a few "laws" that are unique to logarithmic expressions. As you will (hopefully) notice these laws are analogous to the laws of exponents. Let's start with the Log-of-Products Law.

LOG-OF-PRODUCTS LAW:

If $m, n, b \in \mathbb{R}^+$, then $\log_b(m \cdot n) = \log_b(m) + \log_b(n)$.

PROOF: We will prove the log-of-products law by using the common logarithm (i.e., the base 10 logarithm). This law also holds for **all** other bases as well.

Let $x = \log(m)$ and $y = \log(m)$.

Then $m = 10^x$ and $n = 10^y$.

So

$$\begin{aligned} \log(m \cdot n) &= \log\left(10^x \cdot 10^y\right) \\ &= \log\left(10^{x+y}\right) \\ &= x + y \qquad \text{(since the common logarithm has base 10)} \\ &= \log(m) + \log(n) \qquad \text{(since } x = \log(m) \text{ and } y = \log(m)\text{)} \end{aligned}$$

EXAMPLE: $\log(2) + \log(5)$

SOLUTION:

$$\begin{aligned} \log(2) + \log(5) &= \log(2 \cdot 5) \qquad \text{(using the log-of-products law)} \\ &= \log(10) \\ &= 1 \end{aligned}$$

LOG-OF-POWERS LAW:

If $m, b \in \mathbb{R}^+$ and $p \in \mathbb{R}$, then $\log_b\left(m^p\right) = p \cdot \log_b(m)$.

PROOF: We will prove the log-of-powers law by using the common logarithm (i.e., the base 10 logarithm). This law also holds for **all** other bases as well.

Let $x = \log(m)$. Then $m = 10^x$. So

$$\begin{aligned} \log\left(m^p\right) &= \log\left(\left(10^x\right)^p\right) \\ &= \log\left(10^{x \cdot p}\right) \\ &= x \cdot p \\ &= p \cdot \log(m) \qquad \text{(since } x = \log(m)\text{)} \end{aligned}$$

EXAMPLE: Solve $\log_3\left(9^x\right) = 4$ for x.

SOLUTION:

$$\begin{aligned} & \log_3\left(9^x\right) = 4 \\ \Rightarrow\ & x \cdot \log_3(9) = 4 \qquad \text{(using the log-of-powers law)} \\ \Rightarrow\ & x \cdot \log_3\left(3^2\right) = 4 \\ \Rightarrow\ & x \cdot 2 = 4 \\ \Rightarrow\ & x = 2 \end{aligned}$$

LOG-OF-QUOTIENTS LAW:

If $m, n, b \in \mathbb{R}^+$, then $\log_b\left(\frac{m}{n}\right) = \log_b(m) - \log_b(n)$.

PROOF: We will prove the log-of-quotients law by using the common logarithm (i.e., the logarithm of base 10). This law also holds for **all** other bases as well.

Our proof will utilize the log-of-products and log-of-powers laws.

$$\begin{aligned}
\log\left(\frac{m}{n}\right) &= \log\left(m \cdot \frac{1}{n}\right) \\
&= \log(m) + \log\left(\frac{1}{n}\right) \quad \text{(using the log-of-products law)} \\
&= \log(m) + \log\left(n^{-1}\right) \\
&= \log(m) + (-1)\cdot\log(n) \quad \text{(using the log-of-powers law)} \\
&= \log(m) - \log(n)
\end{aligned}$$

EXAMPLE: $\log_3(12) - \log_3(4)$

SOLUTION:

$$\begin{aligned}
\log_3(12) - \log_3(4) &= \log_3\left(\frac{12}{4}\right) \quad \text{(using the log-of-quotients law)} \\
&= \log_3(3) \\
&= 1
\end{aligned}$$

SOLVING LOGARITHMIC EQUATIONS

 **EXAMPLE:** Solve $\log_3(x^2 - 8x) = 2$ for x.

SOLUTION:

$$\log_3(x^2 - 8x) = 2$$

$\Rightarrow \quad x^2 - 8x = 3^2$ (we first translate the logarithmic equation into its "exponential" equivalent)

$\Rightarrow \quad x^2 - 8x - 9 = 0$

$\Rightarrow (x - 9)(x + 1) = 0$

$\Rightarrow \quad x = 9$ or $x = -1$

CHECK:

We need to check our work. It is always a good idea to check, but when solving logarithmic equations it is especially important since the log-laws allow for the possibility that we will find TOO MANY solutions. Sometimes we will do ALL of the math correctly, but still get an incorrect solution! **So it is necessary to check your solutions to logarithmic equations in order to rule out any extraneous solutions.**

$$\log_3(x^2 - 8x) = 2$$

$x = 9 \Rightarrow \log_3((9)^2 - 8(9)) \stackrel{?}{=} 2$

$\Rightarrow \log_3(81 - 72) \stackrel{?}{=} 2$

$\Rightarrow \log_3(9) \stackrel{?}{=} 2$

$\Rightarrow \log_3\left(3^2\right) = 2$

So 9 is a solution.

$$\log_3(x^2 - 8x) = 2$$

$x = -1 \Rightarrow \log_3((-1)^2 - 8(-1)) \stackrel{?}{=} 2$

$\Rightarrow \log_3(1 + 8) \stackrel{?}{=} 2$

$\Rightarrow \log_3(9) \stackrel{?}{=} 2$

$\Rightarrow \log_3\left(3^2\right) = 2$

So -1 is a solution.

Since both 9 and -1 check, the solution set for the equation $\log_3(x^2 - 8x) = 2$ is $\{9, -1\}$

EXAMPLE: Solve $\ln(2x+5)=0$ for x.

SOLUTION:

$$\ln(2x+5)=0$$

$\Rightarrow \quad 2x+5=e^{0}$ (first we translate the logarithmic equation into its "exponential" equivalent)

$\Rightarrow \quad 2x=1-5$

$\Rightarrow \quad x=\dfrac{-4}{2}=-2$

CHECK:

$$\ln(2x+5)=0$$

$x=-2 \Rightarrow \ln(2(-2)+5) \stackrel{?}{=} 0$

$$\ln(-4+5) \stackrel{?}{=} 0$$

$$\ln(1) = 0$$

So -2 is a solution.

Therefore, the solution set for the equation $\ln(2x+5)=0$ is $\{-2\}$.

EXAMPLE: Solve $\log_2(t)+\log_2(t+2)=3$ for t.

SOLUTION:

$$\log_2(t)+\log_2(t+2)=3$$

$\Rightarrow \quad \log_2\big(t(t+2)\big)=3$ (use the log-of-product law to obtain a single logarithmicexpression on left side)

$\Rightarrow \quad t(t+2)=2^3$ (translate the logarithmic equation into its "exponential" equivalent)

$\Rightarrow \quad t^2+2t-8=0$

$\Rightarrow \quad (t+4)(t-2)=0$

$\Rightarrow \quad t=-4 \text{ or } t=2$

CHECK:

$$\log_2(t) + \log_2(t+2) = 3$$

$t = -4 \Rightarrow \log_2(-4) + \log_2(-4+2) \stackrel{?}{} 3$

↑

Negative numbers are not in the domains of logarithmic functions.

So -4 is not a solution.

$$\log_2(t) + \log_2(t+2) = 3$$

$t = 2 \Rightarrow \log_2(2) + \log_2(2+2) \stackrel{?}{} 3$

$\log_2(2) + \log_2(4) \stackrel{?}{} 3$

$1 + 2 = 3$

So 2 is a solution.

Therefore, the solution set for the equation $\log_2(t) + \log_2(t+2) = 3$ is $\{2\}$.

EXAMPLE: Solve $2\log_5(m) = \log_5(2m-1)$ for m.

SOLUTION

This equation has logarithms of the same base on both sides. Since logarithmic functions are one-to-one, once we get isolated logarithmic expressions on both sides of the equation, we can set the "inputs" equal.

$$2\log_5(m) = \log_5(2m-1)$$

$\Rightarrow \log_5\left(m^2\right) = \log_5(2m-1)$ (use the log-of-powers law to obtain an isolated logarithmic expression on the left side)

$\Rightarrow m^2 = 2m-1$ (since the logarithimic functions are one-to-one, the "inputs" must be equal if the outputs are equal)

$\Rightarrow m^2 - 2m + 1 = 0$

$\Rightarrow (m-1)(m-1) = 0$

$\Rightarrow m = 1$

CHECK:

$$2\log_5(m) = \log_5(2m-1)$$

$m = 1 \Rightarrow 2\log_5(1) \stackrel{?}{} \log_5(2(1)-1)$

$2 \cdot 0 \stackrel{?}{} \log_5(1)$

$0 = 0$ So 1 is a solution.

Therefore, the solution set for the equation $2\log_5(m) = \log_5(2m-1)$ is $\{1\}$.

 **EXAMPLE:** Solve $\ln(2) + \ln(3w - 1) = 1$ for w.

SOLUTION

$$\ln(2) + \ln(3w - 1) = 1$$

$$\Rightarrow \quad \ln\big(2(3w - 1)\big) = 1$$ (use the log-of-powers law to obtain an isolated logarithmic expression on the left side)

$$\Rightarrow \quad 2(3w - 1) = e^1$$ (translate the logarithmic equation into its "exponential" equivalent)

$$\Rightarrow \quad 6w - 2 = e$$

$$\Rightarrow \quad 6w = e + 2$$

$$\Rightarrow \quad w = \frac{e + 2}{6}$$

CHECK:

$$\ln(2) + \ln(3w - 1) = 1$$

$$w = \frac{e+2}{6} \Rightarrow \ln(2) + \ln\left(3\left(\frac{e+2}{6}\right) - 1\right) \stackrel{?}{=} 1$$

$$\ln(2) + \ln\left(\frac{e+2}{2} - \frac{2}{2}\right) \stackrel{?}{=} 1$$

$$\ln(2) + \ln\left(\frac{e}{2}\right) \stackrel{?}{=} 1$$

$$\ln\left(2 \cdot \frac{e}{2}\right) \stackrel{?}{=} 1$$

$$\ln(e) = 1$$

So $\frac{e+2}{6}$ is a solution.

Therefore, the solution set for the equation $\ln(2) + \ln(3w - 1) = 1$ is $\left\{\frac{e+2}{6}\right\}$.

CHANGE-OF-BASE FORMULA

Suppose you wanted to estimate $\log_7(13)$. Most calculators generally only have "buttons" for the natural and common logarithms (i.e., the base 10 and base e logarithms). Since $\log_7(13)$ has base 7, we need to change its base in order to estimate it on our calculators. In order to derive the change-of-base formula, let $x = \log_7(13)$ and solve for x only using functions that are easy to approximate on a calculator:

$$\begin{aligned} & x = \log_7(13) \\ \Rightarrow \quad & 7^x = 13 \\ \Rightarrow \quad & \ln\left(7^x\right) = \ln(13) \\ \Rightarrow \quad & x \cdot \ln(7) = \ln(13) \\ \Rightarrow \quad & x = \frac{\ln(13)}{\ln(7)} \end{aligned}$$

So $\log_7(13) = \dfrac{\ln(13)}{\ln(7)}$.

Now we can estimate $\log_7(13)$ since the natural logarithm is programmed into our calculators and we can calculate $\dfrac{\ln(13)}{\ln(7)}$.

$$\begin{aligned} \log_7(13) &= \frac{\ln(13)}{\ln(7)} \\ &\approx 1.318 \end{aligned}$$

There is no reason why we couldn't have used a different logarithm, like the common logarithm, when we solved for x above. Thus, we can obtain the following **change of base formula**.

CHANGE-OF-BASE FORMULA

In general, for any $a, b, x \in \mathbb{R}^+$, $\log_a(x) = \dfrac{\log_b(x)}{\log_b(a)}$.

(This formula is most useful when we take b to be either 10 or e, since we have the common and natural logarithms in our calculators.)

EXAMPLE: Solve $\log_2\left(x^2-2x\right)=3$. Watch the domain!

CLICK HERE FOR THE SOLUTION

EXAMPLE: Solve $\log_2\left(x^2-2x+5\right)-\log_2\left(x+3\right)=2$.

CLICK HERE FOR THE SOLUTION

EXAMPLE: Solve $2\log_3\left(x\right)-\log_3\left(x-2\right)=1$.

CLICK HERE FOR THE SOLUTION

EXAMPLE: Solve $2\log\left(x\right)=\log\left(4\right)+\log\left(x-1\right)$.

CLICK HERE FOR THE SOLUTION

EXAMPLE: Solve $\ln\left(x\right)+\ln\left(x-1\right)=\ln\left(x+1\right)+\ln\left(x+3\right)$.

CLICK HERE FOR THE SOLUTION

EXAMPLE: Solve $x^{\ln(x)} = e^2 x$.

CLICK HERE FOR THE SOLUTION

EXAMPLE: Solve $3^{\log(x)} = x$.

CLICK HERE FOR THE SOLUTION

EXAMPLE: Prove or disprove the formula $\dfrac{\ln(x)}{\ln(y)} = \ln(x) - \ln(y)$.

CLICK HERE FOR THE SOLUTION
