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Knots Prime on Many Strings

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[1]under the supervision of Dr. Steven Bleiler

This paper is an investigation of [Bl] by S. A. Bleiler. In the first section we review the definitions and theorems of tangle and manifold theory, presenting the notion of string primality. The second section gives several families of knots which do not factor into prime tangles. Here we present an application of the techniques developed in [Bl]. In the third section we present the new characterization of knot primality and restate its proof. The author would like to thank Dr. Steven Bleiler for his guidance and inspiration. Throughout this work a knot is a link of one component and all work is done in the PL category.

1 Definitions and prerequisite lemmas.

A *tangle (B,t)* is a 3-ball with a finite number of properly embedded disjoint spanning arcs t. These arcs are called *strings.* Two tangles (B_1, t_1), (B_2, t_2) are *equivalent* if there is a homeomorphism of pairs from (B_1, t_1) to (B_2, t_2). Figure 1(i) illustrates a *trivial tangle.* A tangle (B, t) is an (n-string) *untangle* if (B, t) is equivalent to $(D \times I, \{x_1, x_2, ...x_n\} \times I)$, where D is a 2-disc with distinct points $x_1, x_2, ...x_n$ in its interior. Figure 1(ii) illustrates a (2-string) untangle. Using techniques developed by Conway [Co], one relates to a given 2-string untangle a rational number, possibly $\frac{1}{0}$, which characterizes the untangle up to ambient isotopy fixing the boundary. These untangles are thus referred to as *rational tangles.* Figure 2 presents some examples. BE ADVISED that here the sign convention opposite to that in [Co] is used.

Definition. A tangle *(B,t)* is *prime* if:

(1) Any 2-sphere embedded in B which meets the strings transversely in two points bounds a 3-ball in B which meets t in a single unknotted arc.

(2) Any properly embedded disc D which meets t transversely in a single point is such that ∂D bounds a disc in ∂B which also meets the strings in a single point.

(3) No properly embedded disc separates the strings.

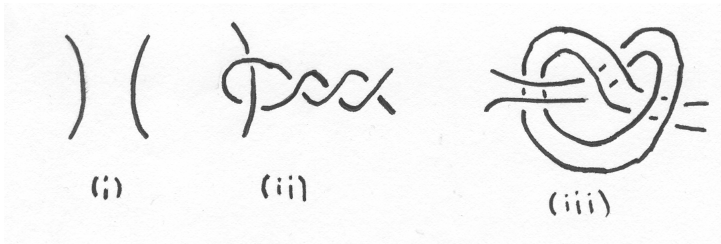


Figure 1: 2-string untangles

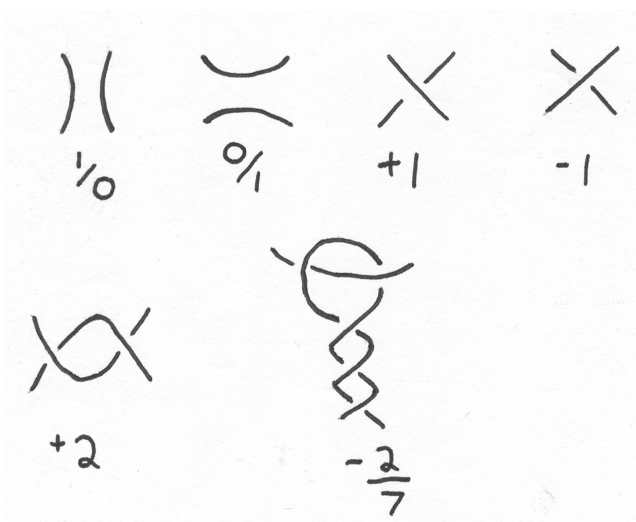


Figure 2: Some rational tangles

Definition. A prime knot K in the 3-sphere is an *n-string composite* if there is an embedded 2-sphere intersecting the knot transversely which separates (S^3,K) into prime n-string tangles. If K is not n-string composite, K is said to be *prime on n-strings.* A knot which is prime on two strings is said to be *doubly prime.* By convention we shall, on occasion, call a composite knot a *1-string composite* and say that a prime knot is *prime on one string.*

We now recall some basic definitions of manifold theory. We hold to the convention throughout that a *surface* is a connected 2-manifold. A 3-*manifold* means a compact, orientable 3-manifold with or without boundary.

Definition. Let M be a 3-manifold. A surface F embedded in M is said to be *properly* embedded if $F \cap \partial M = \partial F$.

Definition. Let F be a surface properly embedded in a 3-manifold M. The surface F is called *2-sided* if there exists an embedding $h : F \times I \to M$ such that $h(x, \frac{1}{2})$=x for each $x \in F$, $h(\partial F \times I) \subset \partial M$, and $h(F \times I)$ is a neighborhood of F in M.

Definition. A surface F properly embedded in a 3-manifold M is said to be *compressible* if either

(i) $F = S^2$ and F bounds a 3-cell in M, or

(ii) there exists a disc $D \subset M$ such that $D \cap F = \partial D$ and ∂D is not homotopically trivial in F. Otherwise, F is called *incompressible.*

Observe, for a 2-sided surface F, distinct from S^2, F is incompressible if and only if the homomorphism $i_* : \pi_1(F) \to \pi_1(M)$ induced by inclusion is injective.

Definition. A properly embedded surface F in a 3-manifold M is said to be *∂-parallel* if F is isotopic, fixing ∂F, to a subsurface of ∂M.

Definition. Let A be a 2-sided annulus in a 3-manifold M. The annulus A is said to be *essential* if A is incompressible and not ∂-parallel.

Definition. A surface F properly embedded in a 3-manifold M is said to be *∂-compressible* if either

(i) F is a disc and F is ∂-parallel to a disc in ∂M, or

(ii) F is not a disc and there exists a disc $D \subset M$ such that $D \cap F = \alpha$ is an arc in ∂D, $D \cap \partial M = \beta$ is an arc in ∂D, with $\alpha \cap \beta = \partial\alpha = \partial\beta$ and $\alpha \cup \beta = \partial D$, and either α does not separate F or α separates F into two components and the closure of neither is a disc. Otherwise, F is called *∂-incompressible.*

Definition. Let V be a solid torus homeomorphic to $S^1 \times D^2$, where $h : S^1 \times D^2 \to V$ is the homeomorphism. A simple closed curve in ∂V that is homotopically trivial in V is called a *meridian* of V. A simple closed curve in ∂V of the form $h(S^1 \times 1)$ is called the *longitude* of V.

Definition. Let K be a PL knot in S^3. Let V be a tubular neighborhood of K and let $\mathring{V}$ denote the interior of V. The 3-manifold with boundary $S^3 - \mathring{V}$ is called the *exterior* of K and is denoted $\mathrm{Ext}(K)$. Another term for knot exterior is *knot manifold.*

Lemma 1.1 (Bl) *A prime knot K in S^3 is doubly prime if and only if the exterior of K does not contain a properly embedded, incompressible, ∂-incompressible, quadruply punctured 2-sphere with boundary components isotopic to meridians.*

Proof. First suppose that K is not doubly prime; that is, a 2-sphere F' exists in S^3 separating (S^3,K) into prime tangles (A,K_A) and (B,K_B). Let E_A and E_B denote the complement of an open regular neighborhood of K_A in A and K_B in B respectively, and let $F = F' \cap E_A = F' \cap E_B = F' \cap \mathrm{Ext}(K)$.

Now to show that F is incompressible in Ext(K), assume F is compressible. Let D be the compressing disc and suppose wlog $D \subset E_A$. The Loop Theorem [Ro, pg. 385] says there is an embedding g: $D \to E_A$, where $[g|_{\partial D}]$ is not trivial in π_1(F). This implies D separates the strings and thus contradicts property (3). Further, to show F is ∂-incompressible assume F is ∂-compressible in Ext(K). Then, there is a 2-disc D in A, say, with $\partial D = \alpha \cup \beta$, where α is an arc in F and β is an arc in ∂E_A-Int(F), not isotopic rel endpoints to a curve in ∂A. Since ∂E_A-Int(F) is two cylindrical tubes parallel to the strings of K_A, β runs from one end of a tube to the other end. We can form a disc which separates the strings of A by taking the sides of a regular neighborhood of D and the complement of the neighborhood in the tube containing β. The separating disc contradicts property (3) of the primality of (A,K_A). Thus F is ∂-incompressible.

Conversely, let F be a surface as in the statement of the lemma. By filling the meridional discs, we get a 2-sphere F' separating (S^3,K) into tangles (A,K_A) and (B,K_B). Using the above notation, both $E_A \cap \partial A$ and $E_B \cap \partial B$ are incompressible. Suppose the tangles do not satisfy property (3) then the separating discs are compressing discs of the surfaces. This contradicts the incompressibility of $E_A \cap \partial A$ and $E_B \cap \partial B$. Therefore, we may conclude the tangles satisfy property (3). For proving property (2) consider the case for one of the tangles, say (A,K_A). Let D be a 2-disc properly embedded in A. Then, D separates A as well as ∂D separates ∂A. If ∂D bounds a disc in ∂A that intersects K_A in two points then as a 2-string tangle K_A must intersect D more than once. Therefore, if D meets K_A transversely in a single point then D can not bound a disc in ∂A which intersects K_A in two points. A similar argument eliminates the other unwanted case. Therefore property (2) holds for (A,K_A) and, by the same argument, (B,K_B). Next, if property (1) fails for A, say, then there is a 3-ball A' in A meeting K_A in a knotted arc. Since K is a prime knot, we obtain the unknot if the knotted arc is replaced by an unknotted arc. Since F deform retracts to a loop space, $\pi_1(F) \cong$

Z * Z * Z. For the unknot K_0, $\pi_1(Ext(K_0)) \cong$ Z. If F is incompressible then the Seifert-Van Kampen Theorem says $\pi_1(F) < \pi_1(Ext(K_0))$, which is impossible. Therefore, F is not an incompressible surface in Ext(K_0). By the Loop Theorem, like before, we find a compressing disc that separates the strings in one of the tangles. This disc will separate the strings if we replace an unknotted segment of a string with a knotted one. Thus either (A,K_A) or (B,K_B) contains a disc separating the strings. Again, the separating disc compresses F in Ext(K), giving a contradiction. □

To continue, we recall two more concepts from manifold theory.

Definition. A 3-manifold M is called *irreducible* if every 2-sphere embedded in M bounds a 3-cell.

Definition. A 3-manifold M is said to be *∂-irreducible* if M is irreducible and ∂M is incompressible in M.

Now let us examine the idea of tangle primality by studying two-fold branched coverings. This is a natural development considering the work done with the two-fold branched coverings of prime knots. In [Li] Lickorish examines the two-fold cover of the 3-ball branched over the strings of a tangle T. Denote this cover by $M(T)$. Lickorish [Li, pg. 327] shows that $M(T)$ is irreducible, ∂-irreducible if and only if T is a prime tangle. Here is a corollary to this characterization.

Lemma 1.2 (Bl) *A prime knot K in the 3-sphere is an n-string composite (n$\geq$ 2) if and only if $M(K)$, the two-fold cover of S^3 branched over K, contains an incompressible closed surface F satisfying the following conditions:*

(1) F is orientable of genus n-1.

(2) F is invariant under the action of the nontrivial covering translation τ and meets the fixed point set of this map in precisely $2n$ *points.*

(3) F separates $M(K)$ into irreducible, ∂-irreducible pieces.

Proof. The surface F is the lift of the separating sphere for the n-string composite. Therefore, F is closed, orientable of genus n-1. The surface F is incompressible, since a compressing disc in $M(K)$ would separate the strings. Since F is symmetric about the branch set, F is invariant under the action of the nontrivial covering translation τ. The above result of Lickorish tell us F separates $M(K)$ into irreducible, ∂-irreducible pieces. □

In particular, Lemma 1.2 says that prime knots whose double-branched covers are irreducible, nonsufficiently large 3-manifolds are prime on n-strings for every positive n.

2 Knots prime on many strings.

Here we reveal the n-string primality of some well known families of knots. Also, we investigate relationships with two-fold branched coverings. The following theorems make vital use of Lemma 1.2.

Theorem 2.1 (Bl) *Two-bridge knots are prime on n-strings for every $n \geq 2$.*

Proof. We present the proof given in [Bl]. Conway [Co] has shown that two-bridge knots and links correspond bijectively with the lens spaces via double-branched coverings. The fundamental group of a lens space is finite and thus cannot contain the group of an orientable surface of positive genus as a subgroup. Since the fundamental group of a two-sided incompressible surface injects, we conclude that the two-fold branched cover of a two-bridge knot does not contain a surface satisfying the conditions of Lemma 1.2. Therefore the two-bridge knots are prime on n-strings for every n≥2. □

For rational knots, we can use the aforementioned correspondence given by Conway to see the Seifert fiber structure of the double-branched cover.

Conway has shown that up to an ambient isotopy fixing the boundary, an untangle may be put into one of the two forms in Figure 3, depending on if n is even or n is odd. By computing the following continued fraction, we associate the rational number $\frac{p}{q}$ to such an untangle:

$$c_n + \cfrac{1}{c_{n-1} + \cfrac{}{\ddots \cfrac{1}{c_2 + \cfrac{1}{c_1}}}} .$$

We consider how the number $\frac{p}{q}$ is interpreted in the double-branched cover. The double-branched cover of an untangle is a solid torus. This leads us to the Dehn surgery description of the double branched cover. Intuitively, twisting on the untangle below affects the Dehn surgery above. From Conway [Co], we obtain a two-bridge knot by replacing a $\frac{1}{0}$ untangle in the unknot with a $\frac{p}{q}$ untangle. As detailed in [Ro], this lifts to $\frac{p}{q}$ Dehn surgery on the unknot in S^3. This is the usual surgery description of $L(p, q)$.

To gain more families of knots prime of many strings, we need to express $L(p, q)$ as a Seifert fiber space over S^2 with one exceptional fiber. When performing surgery on a single fiber of $S^2 \times S^1$ one must not kill it in $S^2 \times S^1$ homotopically. The following lemma gives us the surgery coefficient.

Lemma 2.2 (Bl) *The lens space $L(p,q)$ is obtained by performing $\frac{-q}{p}$ surgery on a fiber of $S^2 \times S^1$.*

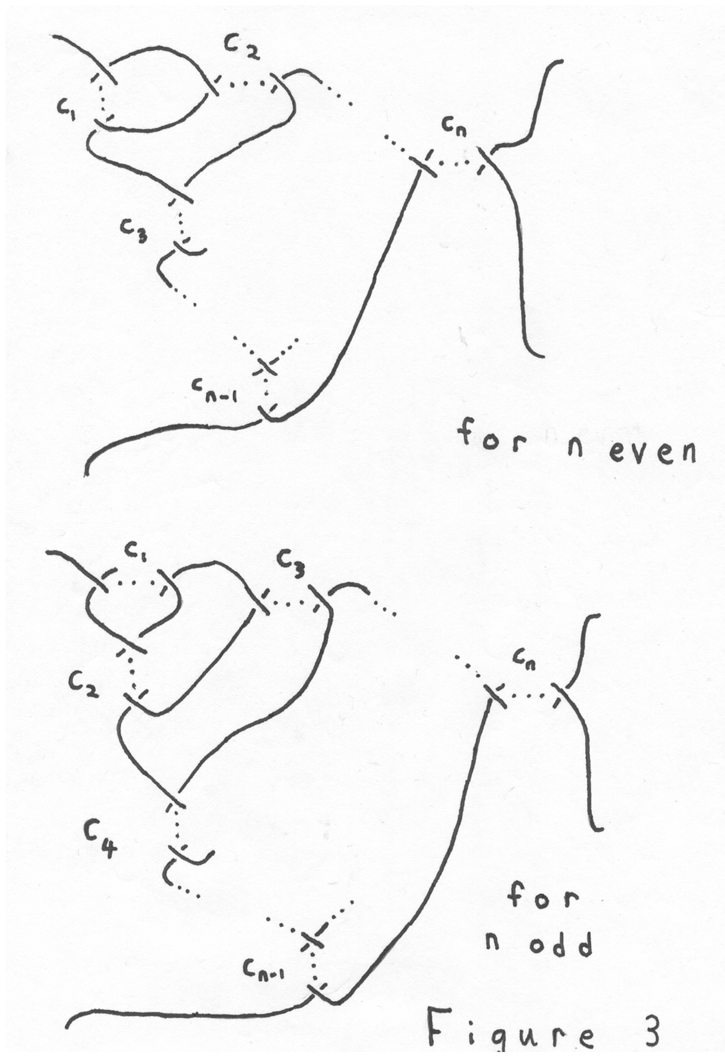


Figure 3: Rational tangle forms

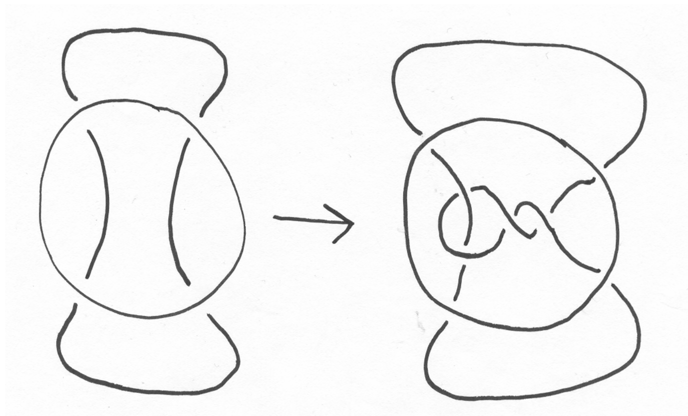


Figure 4: 5/2 untangle surgery on the unknot

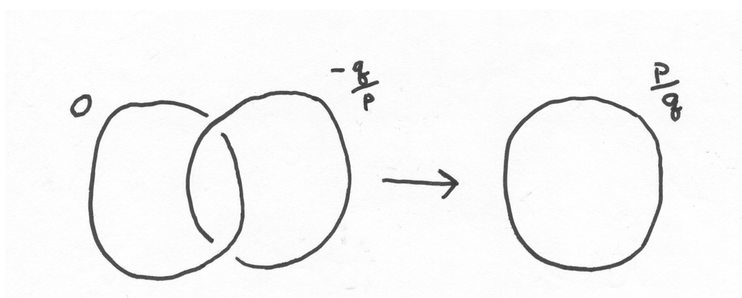


Figure 5: $\frac{-q}{p}$ surgery on a fiber

Proof. Using the Kirby-Rolfsen calculus [Ro], we show the 3-manifold obtained by $\frac{-q}{p}$ surgery on a fiber as surgery on the Hopf link in S^3, Figure 5. A closed regular neighborhood of the component labeled $\frac{-q}{p}$ is a solid torus V_1 in S^3. When we perform $\frac{-q}{p}$ surgery on the core of this solid torus, a meridional disc goes to a disc with boundary a $(p, -q)$ torus knot on ∂V_1. The complement of $\overset{\circ}{V}_1$ in S^3 is a solid torus, call it V_2, with 0-surgery performed on the core. In S^3 a torus knot $(p, -q)$ in ∂V_1 is identified with a torus knot $(-q, p)$ in ∂V_2. By performing 0-surgery on the core of V_2 , we send a longitude to a meridian and a meridian to the negative of a longitude. This changes a $(-q, p)$ torus knot on ∂V_2 into a (p, q) torus knot on the boundary of a trivially fibered solid torus. Therefore, our description satisfies $\frac{p}{q}$ Dehn surgery on the unknot. $\square$

By using Conway's correspondence between lens spaces and rational knots, we can relate double branched coverings to surgery on a single fiber of $S^2 \times S^1$. Lemma 2.2 says $-\frac{q}{p}$ surgery on a fiber of $S^2 \times S^1$ is the double branched cover of S^3 with branch set the knot in S^3 obtained by removing a $\frac{0}{1}$ tangle from the unlink, as illustrated in Figure 4, and replacing it by a $\frac{p}{q}$ untangle surgery, where $\frac{p}{q} \neq \frac{1}{0}$. To prevent killing the homotopy of the fiber, $\frac{p}{q}$ can not be $\frac{1}{0}$. The surgery coefficient in $S^2 \times S^1$ and the untangle surgery description for the replacement untangle are negative reciprocals.

We are ready to consider the next family of knots, the 3-braid *rational pretzel* knots. One defines a 3-braid rational pretzel as a knot with a projection obtained from the unlink by exactly three untangle surgeries, as illustrated by Figure 6. For rational numbers r_1, r_2, r_3 we construct a rational pretzel knot by replacing three $\frac{0}{1}$ untangles with the untangle surgeries corresponding to $-{r_i}^{-1}$, 1≤i≤3. Denote this rational pretzel knot by $P(r_1, r_2, r_3)$. Figure 6 shows the rational pretzel $P(\frac{7}{2}, 3, -2)$.

Theorem 2.3 (Bl) *The (3-braid) rational pretzel knots are prime on n strings for every $n \geq 2$.*

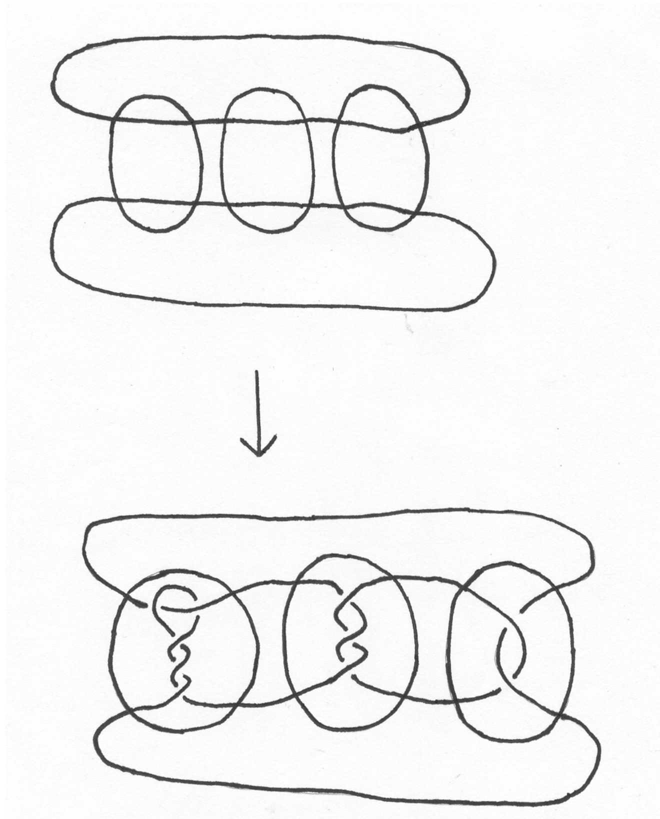


Figure 6: Rational pretzel $P\left(\frac{7}{2}, 3, -2\right)$

Proof. Let M_P be the double-branched cover of the rational pretzel knot $P = P(r_1,r_2,r_3)$. By the arguments above we construct M_P from $S^2 \times S^1$ by taking three distinct fibers f_1, f_2, f_3 and performing the surgery instructions r_1, r_2, r_3 on the corresponding fibers. Suppose $r_1 = \frac{\alpha_1}{\beta_1}$, $r_2 = \frac{\alpha_2}{\beta_2}$, $r_3 = \frac{\alpha_3}{\beta_3}$, where $\alpha_i \in Z$ and $\beta_i \in Z$. This gives M_P as a Seifert fibered space with no more than three exceptional fibers. If the Seifert fibered space M_P has two exceptional fibers, i.e. $|\alpha_3| \leq 1$, then [J-N, pg. 30] says M_P is the lens space $L(p,q)$ with

$$p = det \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix}$$

and

$$q = det \begin{pmatrix} \alpha_1 & {\alpha_2}' \\ \beta_1 & {\beta_2}' \end{pmatrix}$$

where,

$$det \begin{pmatrix} \alpha_2 & {\alpha_2}' \\ \beta_2 & {\beta_2}' \end{pmatrix} = 1.$$

Therefore, if M_P has less than 3 exceptional fibers, then Theorem 2.1 gives us the conclusion. Otherwise, M_P is a Seifert fibered space over S^2 with three exceptional fibers, and Waldhausen [Wa_2, pg. 510] has shown that such a space does not contain a separating incompressible surface. We give a proof of this fact.

The following proof appears in [E-J, pp. 87-89]. We shall construct M_P. Let $M_1 = S^2 \times S^1$ and let $p{:}M_1 \to S^2$ be the natural projection. Choose disjoint discs D_1, D_2, D_3 in S^2 and let $M' = cl(M_1 - \cup_{i=1}^{3} p^{-1}(D_i))$. Note that $p^{-1}(D_i)$ is a solid torus for each i (1≤i≤3). Let b_i=$(\partial D_i) \times 0$ and let h_i=$x_i \times S^1$, where x_i is a point in b_i (1≤i≤3). We construct a manifold M by sewing solid tori ${D_i}' \times S^1$ onto $b_i \times h_i$ in such a way that the curve $(\partial {D_i}') \times 0$ is identified with the curve $b_i^{\alpha_i} h_i^{\beta_i}$ where (α_i,β_i) is a pair of relatively prime integers $(1 \leq i \leq 3)$.

Let M_P be the Seifert fibered 3-manifold constructed from M_1 with exceptional fibers of type (α_1,β_1), (α_2,β_2), (α_3,β_3) corresponding to the solid tori $p^{-1}(D_1)$, $p^{-1}(D_2)$, $p^{-1}(D_3)$ respectively. We may assume $|\alpha_i| \geq 1$ $(1 \leq i \leq 3)$.

The group $\pi_1(M_P)$ has the following presentation

$$\pi_1(M_P) = \langle x, y, h \mid [x,h], [y,h], x^{\alpha_1}h^{\beta_1}, y^{\alpha_2}h^{\beta_2}, (xy)^{\alpha_3}h^{\beta_3}\rangle.$$

Lemma 2.4 (E-J, pg. 87) *The group $H_1(M_P)$ is finite if and only if*

$$\eta = \alpha_1\alpha_2\beta_3 - \alpha_1\beta_2\alpha_3 - \beta_1\alpha_2\alpha_3 \neq 0.$$

Proof. Let $\bar{x}$, $\bar{y}$, and $\bar{h}$ denote, respectively, the classes of x, y, and h in $H_1(M_P)$. Then if $\eta = 0$, the map

$$\varphi : \begin{cases} \bar{x} \to t^{-\beta_1\alpha_2\alpha_3} \\ \bar{y} \to t^{-\alpha_1\beta_2\alpha_3} \\ \bar{h} \to t^{\alpha_1\alpha_2\alpha_3} \end{cases}$$

is a homomorphism of $H_1(M_P)$ into the infinite cyclic group $Z(t)$. By our choice $|\alpha_i| \geq 1$ $(1 \leq i \leq 3)$, it follows that $\varphi(\bar{h}) \neq 1$. Therefore, $H_1(M_P)$ is an infinite group.

In the other direction, observe that $\bar{h}^{\eta} = 1$ in $H_1(M_P)$. Also, we have the relations $\bar{x}^{\alpha_1}\bar{h}^{\beta_2} = 1$ and $\bar{y}^{\alpha_1}\bar{h}^{\beta_2} = 1$ in $H_1(M_P)$. It follows that if $\eta \neq 0$, then $\bar{h}$ and hence both $\bar{x}$ and $\bar{y}$ have finite order in $H_1(M_P)$. Since $\bar{x}$, $\bar{y}$, $\bar{h}$ generate $H_1(M_P)$, we have $H_1(M_P)$ is finite. □

If we let G denote the quotient of $\pi_1(M_P)$ by the smallest normal subgroup of $\pi_1(M_P)$ generated by h. Since h is central in $\pi_1(M_P)$, the subgroup generated by h is normal. Then, G has a presentation of the form

$$G = \langle x, y \mid x^{\alpha_1}, y^{\alpha_2}, (xy)^{\alpha_3}\rangle.$$

In [C-M, pg. 67] we can find that G is finite if and only if $\sum_{i=1}^{3} \frac{1}{\alpha_1} > 1$. Since $|\alpha_i| \neq 1$ $(i = 1, 2, 3)$, we have that G is noncyclic [C-M]. Using the fact that the quotient group of a cyclic group by a cyclic group must be cyclic, we conclude that $\pi_1(M_P)$ cannot be isomorphic to Z.

Next, we wish to show that the manifold M_P is irreducible. If S is a 2-sphere in M_P, then S must separate M_P. For if S does not separate M_P, then $\pi_1(M_P) \cong Z * H$, where $H \cong \pi_1(M_P - S)$. We arrive at this result via the Seifert-Van Kampen Theorem where g is a path homeomorphic to S^1 that intersects S once, g generates Z, and the amalgamating subgroup is $\pi_1(g - S) \cong 1$. Since $\pi_1(M_P) \cong Z * H$, $H_1(M_P)$ is infinite and by Lemma 2.4, $\eta = 0$. Then the map $\varphi : \pi_1(M_P) \to Z$, defined above, represents h as a nontrivial element of Z. Thus, the center of $\pi_1(M_P)$ is nontrivial. It follows that $H = 1$ and $\pi_1(M_P) \cong Z$. But, we have already observed that this cannot occur.

Let $\mathcal{F}$ denote the collection of 2-spheres in M_P which do not bound 3-cells in M_P. We wish to show that $\mathcal{F}$ is empty.

If $\mathcal{F} \neq \emptyset$, then let $S \in \mathcal{F}$ be chosen such that S meets $\cup_{i=1}^{3}({D_i}' \times S^1)$ in a minimal number of components. To continue the proof we consider some well-known results in manifold theory given by [Wa_1].

In the following, a *surface* is a connected compact 2-manifold with or without boundary. Also, surfaces are orientable. A *system of surfaces* F is a finite number of disjoint surfaces and $U(F)$ is a regular neighborhood of F. Again, a 3-*manifold* is a compact, orientable 3-manifold with or without boundary.

Lemma 2.5 ($\mathbf{Wa_1}$, pg. 314) *In a 3-manifold M, let F be a system of surfaces and S be an incompressible 2-sphere. If the number of intersection curves is as small as possible, then $\tilde{S} = S \cap \tilde{M}$ is incompressible where $\tilde{M} = cl(M - U(F))$.*

Lemma 2.6 ($\mathbf{Wa_1}$, pg. 317) *Let M be a solid torus. Let G be an incompressible surface in M,where G is not a 2-disc. Then G is a ∂-parallel annulus.*

Lemma 2.7 ($\mathbf{Wa_1}$, pg. 319) *For $p: M \to B$ an S^1-bundle. If B is not a 2-sphere or projective plane, then M is irreducible.*

Lemma 2.8 ($\mathbf{Wa_1}$, pg. 314) *Let M be a 3-manifold, and let F be a system of surfaces in M or ∂M. Then, F is incompressible if and only if each of the components of F are incompressible.*

Lemma 2.9 ($\mathbf{Wa_1}$, pg. 315) *Let a 3-manifold M be irreducible. Also, let F and G be two systems of surfaces in M, where G is incompressible and is deformed so that $F \cap G$ is a minimal number of curves and arcs. Define $\tilde{M} = cl(M - U(F))$ and $\tilde{G} = G \cap \tilde{M}$. Then $\tilde{G}$ is incompressible in $\tilde{M}$.*

Definition. In a Seifert fibered manifold a subspace is called *vertical* if it is a union of fibers.

Considering $S \in \mathcal{F}$, as above, S is incompressible, since it does not bound a 3-cell. By Lemma 2.7, we have that

$$S \cap \bigcup_{i=1}^{3} (D_i{}' \times S^1) \neq \emptyset.$$

Also, note, for $S' = S \cap M'$, that S' separates M'.

By our choice of S such that $S \cap \cup_{i=1}^{3} (D_i \times S^1)$ has a minimal number of components, Lemma 2.5 and 2.8 together say $S \cap \cup_{i=1}^{3} (D_i{}' \times S^1)$ is incompressible. Now by Lemma 2.6, the components of $S \cap \cup_{i=1}^{3} (D_i{}' \times S^1)$ are either 2-discs or ∂-parallel annuli. However, ∂-parallel discs and annuli would contradict the minimality of the intersection $S \cap \cup_{i=1}^{3} (D_i{}' \times S^1)$. Therefore, all the components of $S \cap \cup_{i=1}^{3} (D_i{}' \times S^1)$ are meridional discs.

Again, since $S \cap \cup_{i=1}^{3} (D_i' \times S^1)$ has a minimal number of components, by Lemma 2.5, $S' = S \cap M'$ is incompressible in M'. To continue we shall need to make use of the following algebraic result.

Lemma 2.10 *Let G be a group that is a nontrivial free product with amalgamation, $A *_C B$, and G has nontrivial center, then the amalgamating subgroup C has nontrivial center.*

Proof. Assume C has trivial center. Let g be an element in the center of G. Also, let C' be the subgroup of A corresponding to C, and let D be the subgroup of B corresponding to C. Then, according to [M-K-M, pg. 201] g has a unique normal form

$$g = hc_1 \cdots c_r,$$

where (i) h is an element, possibly 1, of C';

(ii) c_i is a coset representative of A mod C' or B mod D;

(iii) $c_i \neq 1$;

(iv) c_i and c_{i+1} are not both in A and are not both in B.

The elements $c_1 \cdots c_r$ in the unique normal form for g could have four different forms, that is:

(i) c_1, $c_r \in A$ mod C';

(ii) $c_1 \in A$ mod C', and $c_r \in B$ mod D;

(iii) $c_1 \in B$ mod D, and $c_r \in A$ mod C';

(iv) c_1, $c_r \in B$ mod D.

We present only the first case. Assume c_1, $c_r \in A$ mod C'. So, g has the form $ha_1b_2 \cdots b_{r-1}a_r$, where $a_i \in A$ mod C' and $b_j \in B$ mod D. If h commutes with a_1 then $ha_1b_2 \cdots b_{r-1}a_r = a_1hb_2 \cdots b_{r-1}a_r$. Since g is central, $a_1^{-1}ga_1 = hb_2 \cdots b_{r-1}(a_ra_1)$. Let α be the coset representative of (a_ra_1) in A mod C'. Then, $g = hb_2 \cdots b_{r-1}\alpha$ which contradicts the uniqueness of the normal form. If h does not commute with a_1 then consider if h commutes with $a_1 b_2$ and if so, apply the same argument. If not, continue checking. If h does not commute with $a_1 b_2 \cdots b_{r-1} a_r$ then $a_1 \cdots a_r h \neq h a_1 \cdots a_r$. However, since g is central,

$$a_1 b_2 \cdots b_{r-1} a_r g a_r^{-1} b_{r-1}^{-1} \cdots a_1^{-1} = g,$$

giving us a contradiction. Similar arguments work for the other three cases. Therefore C has a nontrivial center. □

We use Lemma 2.10 with the group $\pi_1(M')$. The group $\pi_1(S')$ is the amalgamating subgroup. Since S does not bound a 3-cell and S' separates M', $\pi_1(M')$ is a nontrivial free product with amalgamation. Therefore, Lemma 2.10 tells us $\pi_1(S')$ has a nontrivial center. This means S' must be either a torus or an annulus. Suppose S' is a torus. Since the torus S' is incompressible, S' surrounds one of the boundary tori. This implies S' is ∂-parallel to a boundary torus and we can isotope S' into a solid torus, making S' compressible. The contradiction forces S' to be an annulus. Since S' separates M', both components of $\partial S'$ are contained in the same component $(b_i \times S^1)$ of $\partial M'$.

The surface S' cannot be vertical, since a regular fiber in $\partial M'$ cannot be at the same time a meridional curve. Therefore, we can fill in the other components of $\partial M'$ to obtain a solid torus and use Lemma 2.6 to get that S' is ∂-parallel to an annulus in $b_i \times S^1$. However, this implies we can isotope S into $D_i' \times S^1$, which gives a 2-sphere in a solid torus which does not bound a 3-cell in the solid torus. This contradiction gives us that the collection $\mathcal{F}$ is empty, and thus M_P is irreducible.

Now our main lemma.

Lemma 2.11 *Let M_P be the 3-manifold described above. Then M_P does not contain a (2-sided) separating incompressible surface.*

Proof. Let F be a (2-sided) separating incompressible surface in M_P that meets

$$\bigcup_{i=1}^{3}(D_i{}' \times S^1)$$

in a minimal number of components. Using Lemma 2.9 and by the same argument as above, we may assume F meets $\cup_{i=1}^{3}(D_i{}' \times S^1)$ in meridional discs. Let $F' = F \cap M'$. The surface F' must separate M'. Since $\pi_1(M')$ has nontrivial center, again by the same arguments as above, we conclude that F' is an annulus and F is a 2-sphere. This gives us the desired contradiction. □

Therefore, our M_P must not contain a separating incompressible surface and, by Lemma 1.2, the rational pretzel knots are prime on n-strings for every $n \geq 2$. □

The family $K(\frac{p}{q})$, as illustrated in Figure 7, is the family of knots or links constructed by inserting the $\frac{p}{q}$ untangle into the space labeled $\frac{p}{q}$. One calls a knot or link with such a projection a $K(\frac{p}{q})$ knot or link respectively. Notice for $K(\frac{p}{q})$ if $\mid q \mid= 1$ then $K(\frac{p}{q})$ is a knot or link according to whether P is odd or even.

Theorem 2.12 *A $K(\frac{p}{q})$ knot is prime on n-strings for every $n \geq 1$.*

Proof. For p not even and $q \neq 0$, $K(\frac{p}{q})$ is a 3-braid rational pretzel knot, the proof follows from Theorem 2.3. □

Lemma 2.13 *The double cover of S^3 branched over the knot $K(\frac{p}{q})$ is obtained by $\frac{p}{q}$ surgery on the right-hand trefoil.*

Proof. As noted above, Conway [Co] has shown the 3-manifold obtained by performing $\frac{p}{q}$ Dehn surgery on the unknot double-branch covers S^3 with

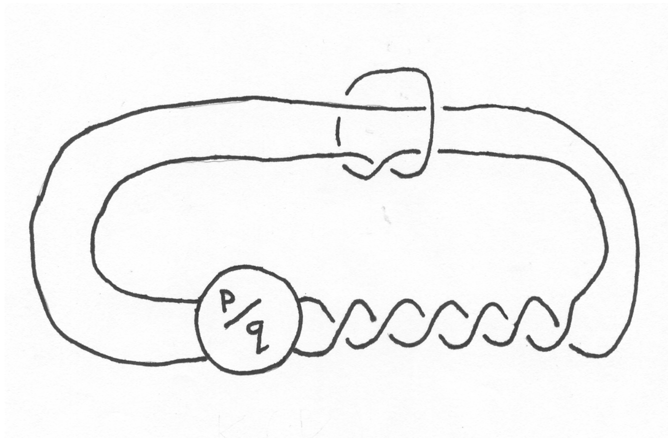


Figure 7: $K\left(\frac{p}{q}\right)$

branch set the knot or link constructed by replacing the $\frac{1}{0}$ tangle in the unknot with the $\frac{p}{q}$ untangle. We shalll derive our proof from this method of untangle surgery.

One can use Conway's technique to demonstrate the 3-manifold obtained by $\frac{p}{q}$ Dehn surgery of the right-hand trefoil double-branch covers the 3-sphere and, thereby, presents the branch set. In order to correctly perform the untangle surgery, one must keep track of the "framing", that is, one must know the preferred longitude of the untangle. The isotopy class of this curve is obtained by projecting both the preferred longitude and its translate under the deck transformation to the boundary of the untangle to be surgered. For the right-hand trefoil, Figure 8 shows the preferred longitude and shows the sequence of isotopies that get us to $K(\frac{p}{q})$. □

Remarks. Now we can see $\frac{p}{q}$ Dehn surgery on the right-hand trefoil double branch covers S^3 with branch set a 3-braid rational pretzel knot, when p is odd and $\frac{p}{q} \neq \frac{1}{0}$. This allows us to analize Dehn surgery on the right-hand trefoil. For instance, $K(\frac{5}{1})$ is the (2,-5) torus knot, so $\frac{5}{1}$ surgery on the right-hand trefoil creates the lens space L(5,1). Also, If one performs $\frac{6}{1}$ surgery, $K(\frac{6}{1})$ is the composition of the Hopf link with the trefoil. This implies the double branched cover is the connected sum of the lens spaces L(2,1) and L(3,1). If we investigate the intersection of a meridian of the regular closed neighborhood of the trefoil after $\frac{p}{q}$ Dehn surgery, and a regular fiber in the complement of the open neighborhood of the trefoil. First notice that the preferred longitude of the trefoil intersects a regular fiber six times. For $\frac{p}{q}$ Dehn surgery, the following formula gives the intersection number between a regular fiber and a meridian:

$$\left| det \begin{pmatrix} p & 6 \\ q & 1 \end{pmatrix} \right|.$$

When $K(\frac{p}{q})$ has three rational tangles that correspond to three exceptional fibers in the double branched cover, we observed in the proof of Theorem

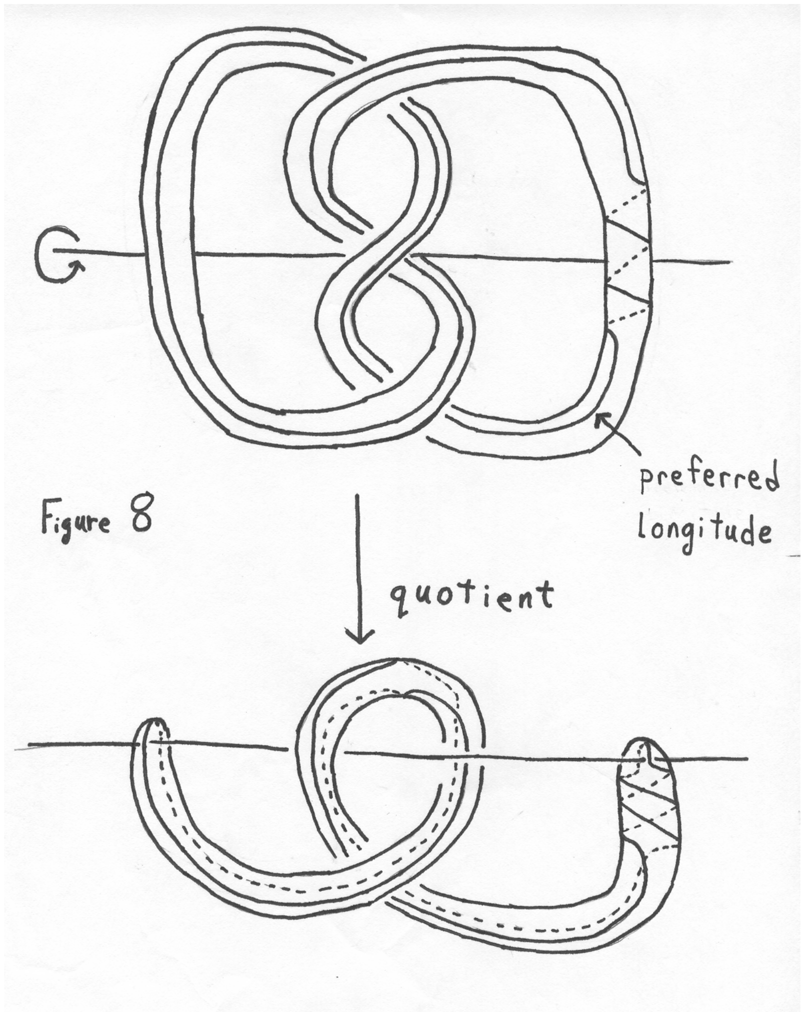


Figure 8: $\frac{p}{q}$ Dehn surgery on the right-hand trefoil

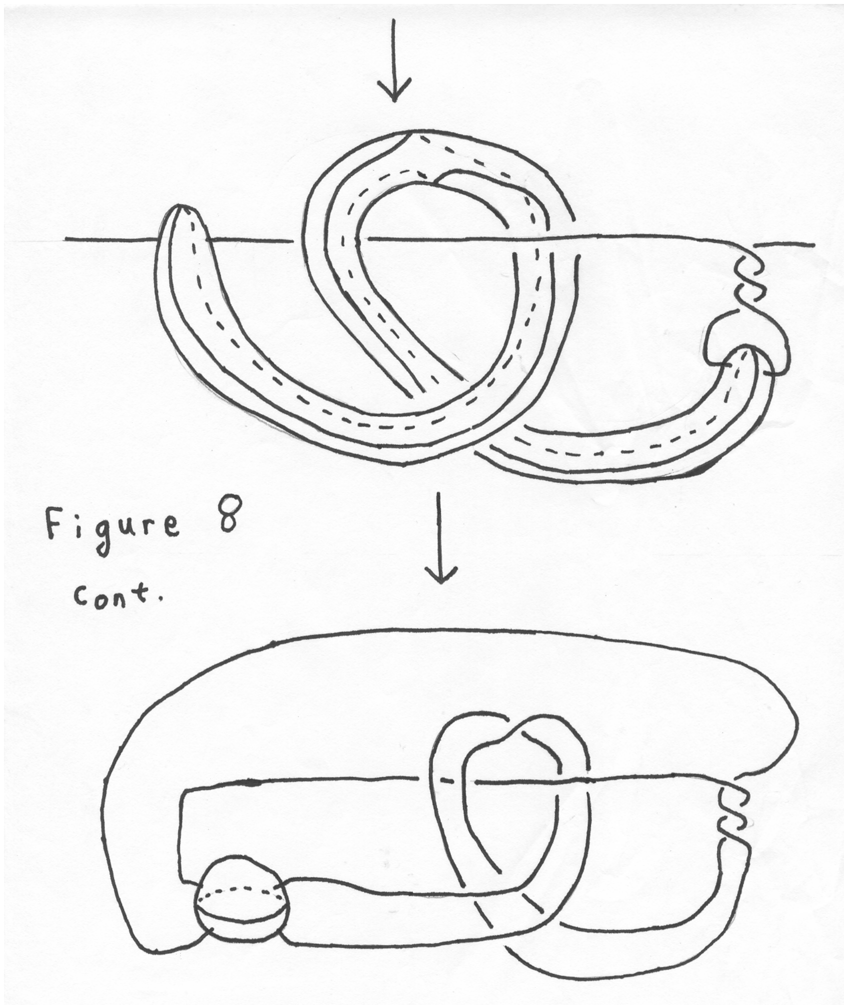
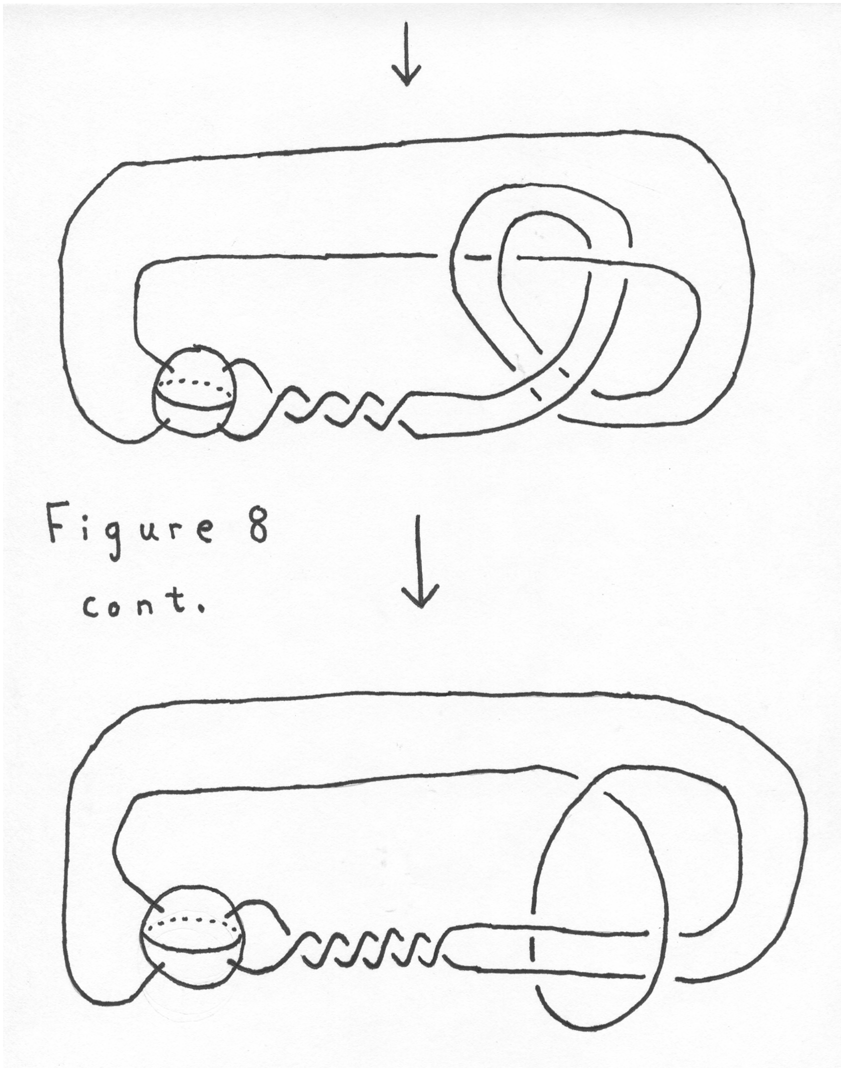


Figure 8: continued



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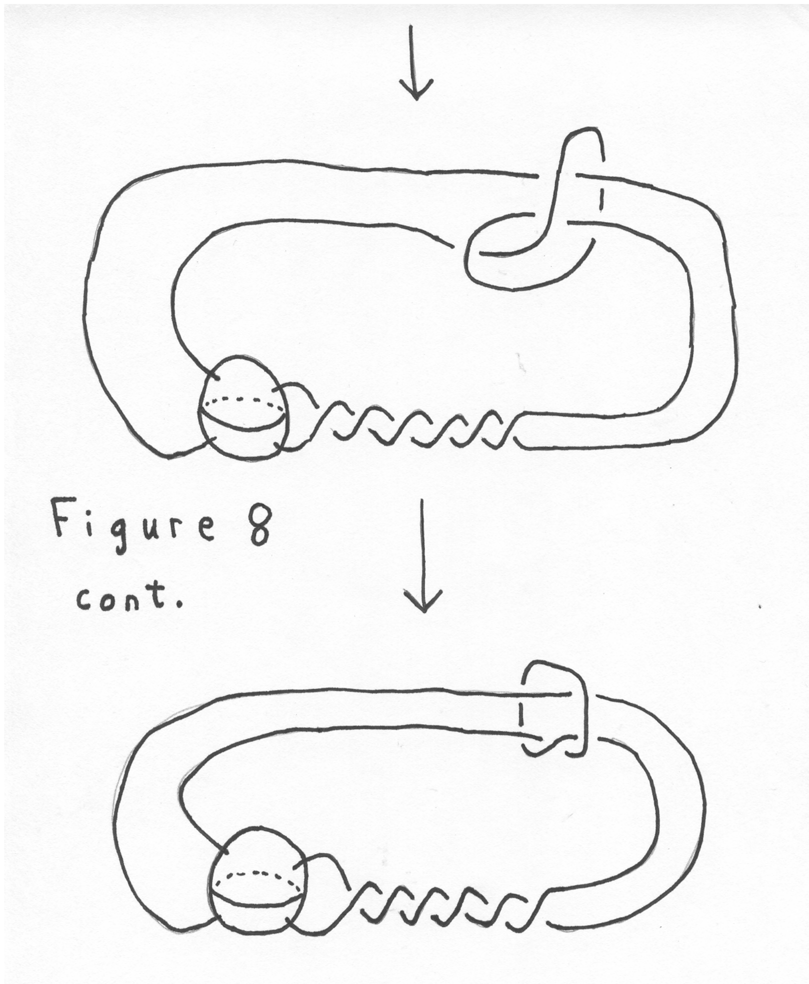

Figure 8: done

2.3 that the fundamental group of the double branched cover, M_P, has a noncyclic quotient group. Therefore, $\pi_1(M_P)$ is noncyclic and $\pi_1(M_P) \neq 1$. When $K(\frac{p}{q})$ is a nontrivial knot, and the double branched cover, M_P, does not have three exceptional fibers, we have seen that M_P is a lens space. For $\frac{p}{q} \neq \frac{1}{0}$, we would like to establish that this lens space is not S^3. Observe, when M_P is a lens space and $\frac{p}{q} \neq \frac{1}{0}$, $K(\frac{p}{q})$ is a rational tangle with value:

$$2 + \frac{3}{3m-1} = \frac{6m+1}{3m-1},$$

where m is a nonzero integer. Therefore, the lens space M_P has nontrivial fundamental group. We conclude that the right-hand trefoil has property P and $\frac{p}{q}$ Dehn surgery on the right-hand trefoil can not produce a counter-example to the Poincare Conjecture.

3 Primality and companionship

In this section we present the characterization of knot primality and make some observations. Let K be a knot in the 3-sphere. Let E be a second knot which is embedded in a solid torus V so that every meridional disc of V meets E nontrivially. Denote a tubular neighborhood of the knot K by V_K. We construct a new knot by mapping V to V_K via a homeomorphism h which takes a meridian of V to a meridian of V_K and the preferred longitude of V to the sum of the preferred longitude of V_K with q meridians. The new knot $h(E)$ is the *q-twist satellite* of K with *embellishment* E. Our original knot K is a *companion* of $h(E)$ and the torus ∂V_K is a *companion torus.* H. Schubert [Sc, pg. 250] has shown that if K is nontrivial and every meridional disc of V meets E in at least two points and if the pair (V, E) does not contain a knotted ball-arc then the knot $h(E)$ is prime.

To obtain the new characterization, we must consider the important class of satellite knots known as the q-twist doubles. One forms these by taking the embellishment the unknot pictured in Figure 9. We obtain a

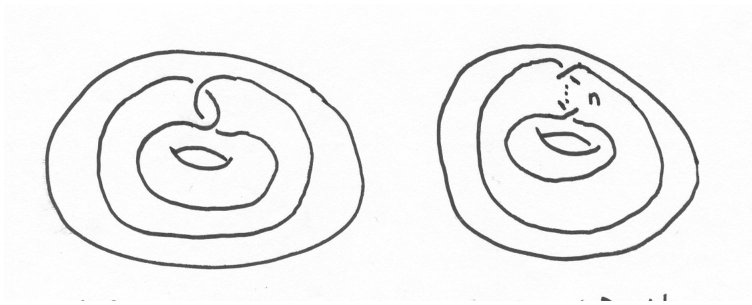

Figure 9: A doubling curve and a generalized doubling curve

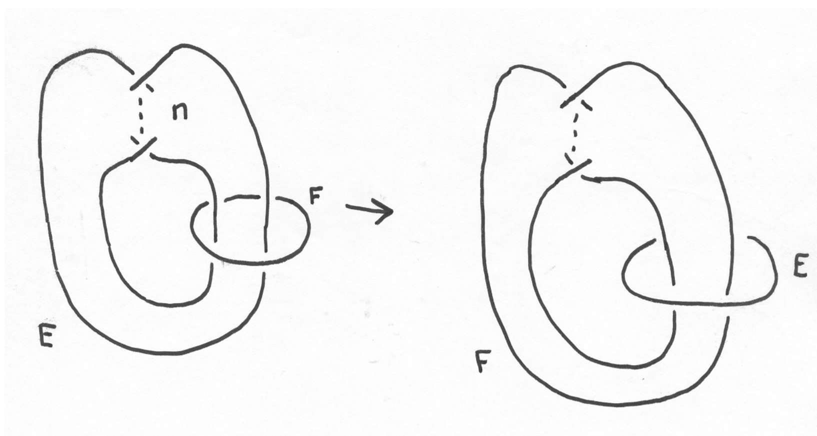

Figure 10: E and F symmetry

generalized double when the unknot is embedded with more than two half-twists, see Figure 9. By looking at generalized doubles, Bleiler [Bl] finds a characterization of knot primality, namely of the companion knot.

Theorem 3.1 (Bl) *A knot K in the 3-sphere is prime if and only if any generalized double of K is doubly prime.*

Proof. Let D_K be a generalized double of K in S^3 and denote its double branched cover $M(D_K)$ by just M. We prove the theorem by using Lemma 1.2 and investigating incompressible tori in M.

The companion torus ∂V_K lifts to two disjoint incompressible tori in M and by cutting M along these tori we obtain three pieces. Two are copies of the exterior of K. The third component, call it L, is the double cover of the solid torus branched over the embellishment E. We obtain L by considering the 3-sphere as the double cover of itself branched over the unknot E. Think of the solid torus V used in the construction of D_K as the exterior of the unknot F, shown in Figure 10. Observe that E and F are symmetric; that is, there is an ambient isotopy of S^3 interchanging E and F. By viewing E as the z-axis, we see that the double cover of V is the (2,2n) torus link exterior with covering translation given by rotation through π about the z-axis, Figure 11. By using the homeomorphism between V and V_K, we can therefore display L as the (2,2n) torus link exterior. There is an easy way to visualize a Seifert fiber structure on L. Take one component of the (2,2n) torus link in S^3 and pull it "straight"; that is, take one point of the component to infinity. The exterior of the "straight" string is a solid torus N with the other string inside. Consider a Seifert fibered solid torus with an exceptional fiber at the core of index n, as in Figure 12, then this solid torus with an open regular neighborhood of a regular fiber removed is homeomorphic to the exterior of the other string in N, i.e. the exterior

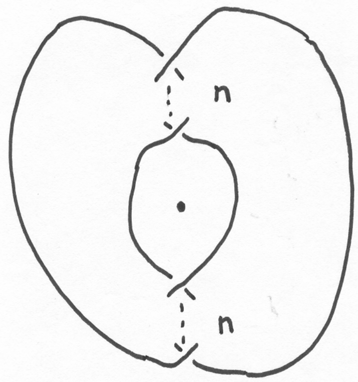

Figure 11: $(2, 2\,n)$ torus link

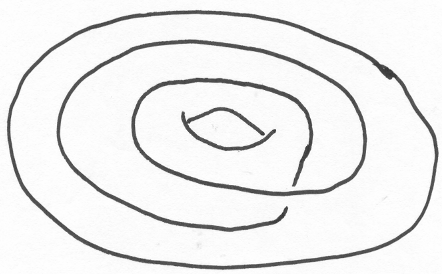

Figure 12: Seifert fibered torus with an exceptional fiber

of the (2,2n) torus link. From this we can see L has a natural Seifert fiber structure over the annulus with a single exceptional fiber of type (n,1).

Assume D_K is 2-composite. Let T be the lift to M of the separating 2-sphere. Lemma 1.2 guarantees that T is an incompressible torus. Via isotopy, we position T to have minimal transverse intersection with the lifts of the companion torus. Since T meets the branch set, $T \cap L$ is nonempty. Suppose T lies entirely in L. Since T is incompressible, we can not find a compressing disc for T, and therefore, T surrounds the removed regular fiber of L. This implies T is ∂-parallel to one of the boundary tori. Therefore, the torus T can be isotoped off L, contradicting the fact $T \cap L$ is nonempty. Therefore, T can not lie entirely in L.

Next we establish the fact that knot exteriors and link exteriors are irreducible and ∂-irreducible 3-manifolds. Let k be a knot in S^3. Let V denote a tubular neighborhood of k. Also, let C be the exterior of k, i.e. $C = S^3 - \mathring{V}$. For S an embedded PL 2-sphere, the Generalized Schönflies Theorem [Ro, pg. 34] for S^3 says S bounds a ball on both sides. Therefore, C is irreducible. We can use a similar argument to show the exterior of a link is irreducible. Now consider the following lemma with the same notation as above.

Lemma 3.2 (Ro, pg. 103) *The knot k is nontrivial if and only if the inclusion homomorphism $i_* : \pi_1(\partial V) \to \pi_1(S^3 - \mathring{V})$ is injective.*

From Lemma 3.2 $i_* : \pi_1(\partial C) \to \pi_1(C)$ is injective, and this implies, by our definition of compressible, that ∂C is incompressible. Thus, C is ∂-irreducible.

To show that a link exterior is ∂-irreducible, we use the negation of Lemma 3.2 and come to the conclusion that one of the components of the link is an unlinked unknot. This contradicts the fact we want our link to be linked. Therefore, link exteriors have incompressible boundary and thus are ∂-irreducible. Since both knot and link exteriors are irreducible and ∂-irreducible, no component of $T \cap L$ or $T\cap$ {knot exteriors} is a disc. By assuming otherwise, we contradict the fact that T has minimal transverse intersection with the lifts of the companion torus. This implies T meets L and the knot exteriors in essential annuli.

We need to examine the isotopy classes of the boundary curves of these essential annuli in the knot exteriors. The work of J. Simon [Si] gives the two following lemmas. We present them here as they appear in [B-Z].

Lemma 3.3 (B-Z, pg. 287) *Let C, W_0, W_1 be knot manifolds, $C = W_0 \cup (A \times [0,1]) \cup W_1$, $W_0 \cap ((A \times [0,1]) \cup W_1) = A \times \{0\}$, either the components of ∂A bound discs in ∂C or the components bound meridional discs in $cl(S^3 - C)$.*

Lemma 3.4 (B-Z, pg. 287) *Let C be a knot manifold in S^3, $C = W_0 \cup W_1$, where W_0 is a cube with a hole, W_1 is a solid torus, and $A = W_0 \cap W_1 = \partial W_0 \cap \partial W_1$ is an annulus. Denote by t_C the core of the solid torus $cl(S^3 - C)$. Assume that $i_* : \pi_1(A) \to \pi_1(W_1)$ is not surjective. Then t_C is a (p,q)-cable knot of the core t_0 of $cl(S^3 - W_0)$, $\mid q \mid \geq 2$.*

Now we present the following results by using a proof presented in [B-Z, pg. 288]. Let A be a component of $T\cap$\{knot exteriors\} and therefore, an essential annulus. Let C be the knot exterior containing A. Since A is essential, the components of ∂A bound annuli in ∂C. Hence, there are submanifolds X_1 and X_2 bounded by tori such that $C = X_1 \cup X_2$, $X_1 \cap X_2 = A$, and by Alexander's Theorem [B-Z, pg. 307], X_i is either a knot manifold or a solid torus.

If X_1 and X_2 are both knot manifolds then, by Lemma 3.3, each component of ∂A bounds a meridional disc in cl$(S^3 - C)$, and a core of cl$(S^3 - C)$ is a composite knot.

Suppose now that X_2 is a solid torus. There is an annulus $B \subset \partial C$ satisfying $A \cup B = \partial X_2$. If the homomorphism $i_* : \pi_1(A) \to \pi_1(X_2)$, induced by inclusion, is not surjective then Lemma 3.4 says a core of $\mathrm{cl}(S^3 - C)$ is a cable knot. Assume that $i_* : \pi_1(A) \to \pi_1(X_2)$ is surjective. Then a simple arc $\beta \subset B$ which leads from one component of ∂B to the other can be extended by a simple arc $\alpha \subset A$ to a simple closed curve $\mu \subset \partial X_2$ which is null-homotopic in the solid torus X_2 and, hence, a meridian of X_2. Since μ intersects each component of ∂A in exactly one point, it follows that A is ∂-parallel. This contradicts the fact that A is essential. Therefore, C is the exterior of a composite knot if ∂A are meridians and C is the exterior of a cable knot, otherwise.

To determine the status of K, we investigate the boundaries of essential annuli in L. Since L is a Seifert fibered manifold, an essential annulus in L is ∂-incompressible. The fundamental group of L has the presentation,

$$\langle c_1, d_1, d_2, h \mid [c_1, h],\ [d_1, h],\ [d_2, h],\ h^b = c_1 \cdot d_1 \cdot d_2 \rangle,$$

where b is an integer. From this we have $N = \langle h \rangle$ is an infinite cyclic, normal subgroup of $\pi_1(L)$. Let A be an essential annulus in L. Then, A is a two-sided, incompressible, ∂-incompressible surface in L. [Ja] gives the following lemmas.

Lemma 3.5 (Ja, pg. 102) *If $N < \pi_1(A)$, then L is homeomorphic to a Seifert fibered manifold via a homeomorphism taking A to a union of fibers.*

Lemma 3.6 (Ja, pg. 103) *If $N \not< \pi_1(A)$ and A does not separate L, then N is central in $\pi_1(L)$, $L = A \times_\varphi S^1$ with fiber A and sewing map φ and φ is periodic.*

We also need a definition.

Definition. In a Seifert fibered manifold a surface is called *horizontal* if it is transverse to all the fibers.

By Lemmas 3.5 and 3.6, an essential annulus in L is vertical or horizontal. For a horizontal surface S in L, the projection $\pi : S \to B$ onto the orbit manifold of the Seifert fibering is a branched covering, with a single branch point of multiplicity n. For this branched covering $\pi : S \to B$, there is a useful formula relating the Euler characteristic of S and B:

$$\chi(B) - \frac{\chi(S)}{n} = 1 - \frac{1}{n} \quad .$$

When S and B are annuli $\chi(S) = \chi(B) = 0$, and the above formula makes it impossible for an essential annulus to be horizontal. Therefore, an essential annulus must be vertical and we can represent the possible essential annuli by *essential arcs* in the orbit manifold of L, as illustrated by Figure 13. We see there are three such arcs up to isotopy.

The essential annuli corresponding to these arcs can be visualized. Two are annuli with both boundary components on a single peripheral torus. Consider Figure 12, if we remove an open regular neighborhood of a regular fiber we get two tori boundary components. One annulus is attached to the "inside" boundary torus. The other annulus is attached to the "outside" boundary torus. The third essential annulus has a boundary component on each peripheral torus. This annulus is the Seifert annulus for the $(2,2n)$ torus link if we orient this link by lifting the orientation from the curve F, as illustrated in Figure 14. This annulus is the lift of a meridional disc of V.

The Seifert annulus is the only one which intersects the branch set. Therefore, the Seifert annulus must be one of the components of $T \cap L$. By taking a boundary component of the Seifert annulus, we get a curve of the ∂V_K via the restriction of h to ∂V. The lift of this curve on the peripheral torus of the exterior of K gives us a curve, call it G, which bounds an essential annulus in the exterior of K. Since the Seifert annulus is the lift of a meridional disc of V, the curve G must be a meridian of K. By the arguments above K is a composite knot.

Conversely, assume K is a composite knot. Then, the exterior of K contains an essential annulus with boundary components meridians. In M we obtain an incompressible torus by taking two copies of the Seifert annulus and a copy of an essential annulus in each knot exterior. The four annuli together give us a torus which satisfies the conditions of Lemma 1.2. Therefore, D_K is 2-composite. □

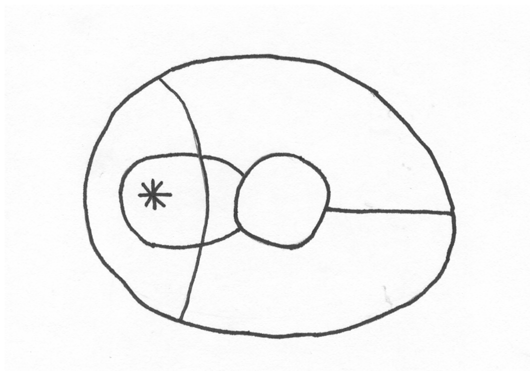

Figure 13: Orbit manifold of L

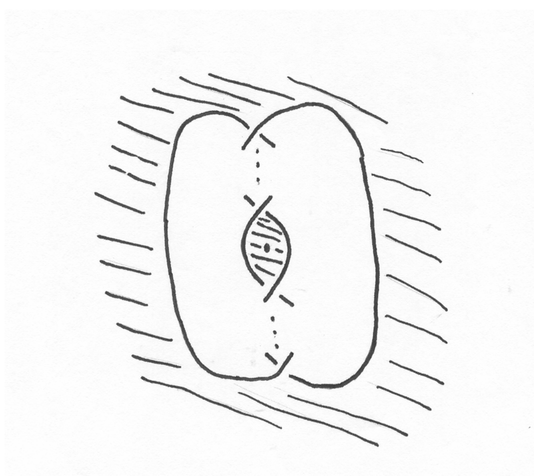

Figure 14: Seifert annulus for the $(2, 2\,n)$ torus link

4 Conclusion

In this paper we reviewed aspects of knot theory and tangle theory, demonstrated n-string primality for some knot families, and applied the techniques of [Bl] to the right-hand trefoil. These techniques are powerful tools for obtaining information about 3-manifolds. We ended our investigation of [Bl] by presenting the new characterization of knot primality.

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