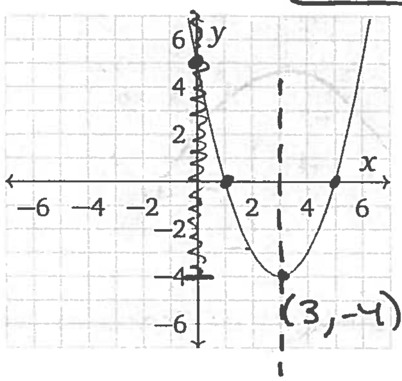
Quadratic Graphs and Vertex Form

$f(x) = a \cdot x^2 + bx + c$ with $a \neq 0$

1. Here is the graph of a quadratic function.



a) Does it open upward or downward?

b) Where is its vertex? "low point" or "high point"

(3,−4)

c) What is its axis of symmetry? a vertical line right down the middle of the parabola

$x = 3$

d) What is the function's domain and range?

anything can be used as input for ax^2+bx+c.

$(-\infty, \infty)$ $[-4, \infty)$

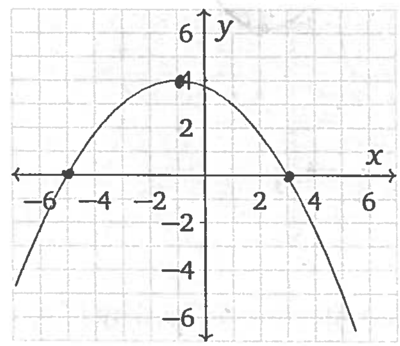
e) What is its vertical intercept?

(0,5)

f) Does the function have horizontal intercepts? If so, what are they?

(1,0) and (5,0)

2. Here is the graph of a quadratic function.



a) Does it open upward or downward?

b) Where is its vertex?

(−1,4)

c) What is its axis of symmetry?

$x = -1$

d) What is the function's domain and range?

$(-\infty, \infty)$ $(-\infty, 4]$

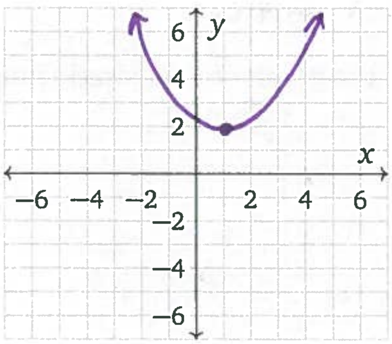
e) What is its vertical intercept?

(0, 3.9) probably not perfectly correct.

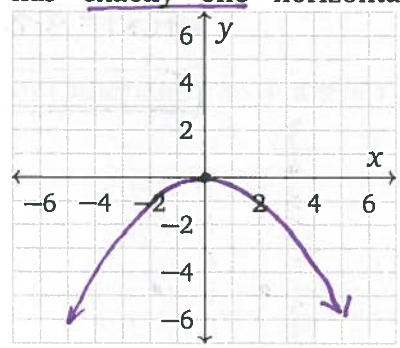
f) Does the function have horizontal intercepts? If so, what are they?

(−5,0) and (3,0)

3. Sketch the graph of a quadratic function that does not have any horizontal intercepts.



4. Sketch the graph of a quadratic function that opens downward and has exactly one horizontal intercept.



5. Let f be a function defined by $f(x) = 4x^2+9x+55$. In the following, it is OK to use a calculator to do arithmetic, but please do not use a calculator to do any graphing.

(h,k)

$\left(\frac{-9}{8}, \frac{799}{16}\right)$

$\approx$

$(-1.125, 49.9375)$

a) Where is the vertex of the graph of f?

$\boxed{h = \frac{-b}{2a}} = \frac{-9}{2\cdot 4} = \frac{-9}{8}$

$h = -1.125$

$\boxed{k = f(h)} = f\left(\frac{-9}{8}\right) = 4\left(\frac{-9}{8}\right)^2 + 9\left(\frac{-9}{8}\right) + 55 = 4\left(\frac{81}{64}\right) - \frac{81}{8} + 55$

$= \frac{81}{16} - \frac{81}{8} + \frac{55}{1}$

$= \frac{81}{16} - \frac{162}{16} + \frac{880}{16}$

$= \frac{799}{16}$

b) In the graph of f, where is the y-intercept?

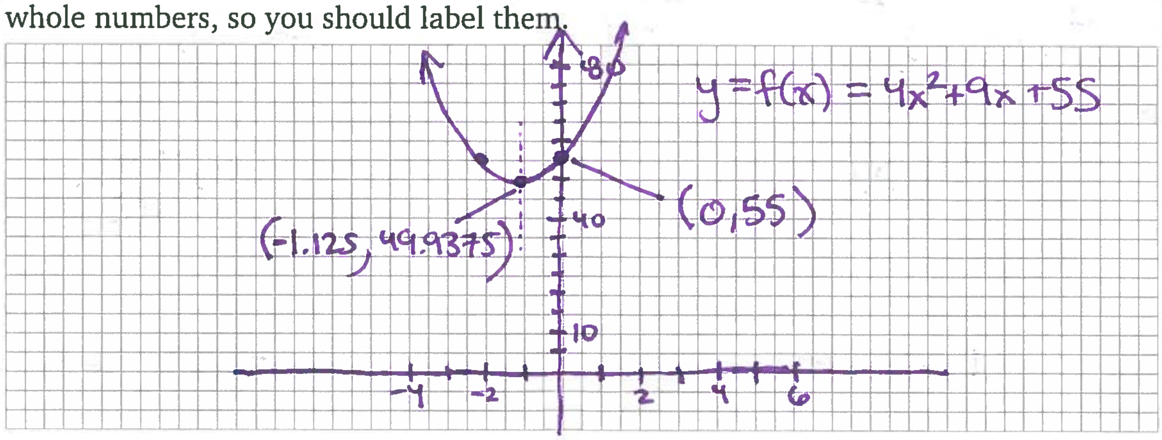
Since $f(0) = 55$, y-intercept is at $(0, 55)$.

$(0, ?)$

c) In the graph of f, are there any x-intercepts, and if so, where are they?

We got started plotting... y-intercept & vertex -- and it became clear there are no x-intercepts.

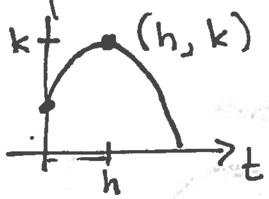
d) Use your answers so far to sketch a graph of f. The "special points" may not be at nice whole numbers, so you should label them.



6. An object was launched from the top of a hill (at 30 feet above sea level) with an upward vertical velocity of 100 feet per second. The height of the object can be modeled by the function $h(t) = -16t^2 + 100t + 30$, where t represents the number of seconds after the launch. Assume the object landed on the ground at sea level. How high did the object get before it started to fall back down? How many seconds did it take to get that high? (It is OK to use a calculator to do arithmetic.)

→ h.... 3.125 seconds

quadratic function



We want k.

$h = \frac{-b}{2a} = \frac{-100}{2(-16)} = \frac{-100}{-32} = \frac{25}{8} = 3.125$

$k = h(3.125) = -16(3.125)^2 + 100(3.125) + 30$
$= 186.25$

vertex at (3.125, 186.25). → It got 186.25 ft above sea level before starting to fall.

7. Where is the vertex of $y = 3(x-9.1)^2 - 3.6$?

$y = a(x-h)^2 + k$ is "vertex form"

Smallest this can get is 0

could only get as low as −3.6.

$y = 3(x-9.1)^2 - 3.6$ (9.1)

→ −3.6 AND −3.6 is the lowest output possible

So the vertex of this is (9.1, −3.6).

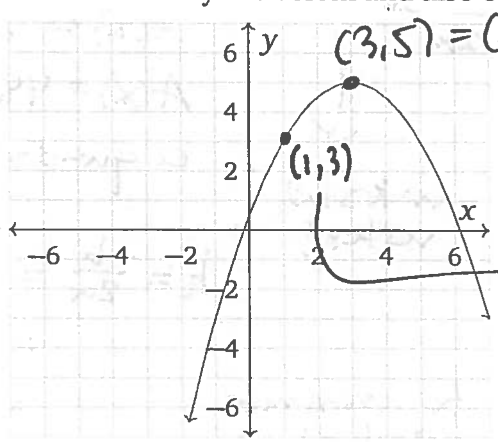
8. Write the vertex form for the quadratic function f whose vertex is (1, 9) and has leading coefficient $a = -8$.

h k

✓

$f(x) = a(x-h)^2 + k$] generic vertex form
$f(x) = -8(x-1)^2 + 9$

9. A graph of a function f is given. Use the graph to write a formula for f in vertex form. You will need to identify the vertex and also one more point on the graph to find the leading coefficient.



$f(1) = 3$

$f(x) = a(x-h)^2 + k$
$f(x) = \overset{?}{a}(x-3)^2 + 5$
$f(1) = a(1-3)^2 + 5 = 3$
$a(-2)^2 + 5 = 3$
$a \cdot 4 + 5 = 3$
$a \cdot 4 = -2$
$a = \frac{-2}{4}$
$a = -\frac{1}{2}$

So $f(x) = -\frac{1}{2}(x-3)^2 + 5$

10. Currently, an artisan can sell 60 handmade dining tables every year at the price of \$900 per table. Each time she raises the price by \$20.00, she sells 1 fewer table per year. Let f be a function where the input is how many times she raises the price by \$20, and the output is how much revenue she takes in. Then

revenue

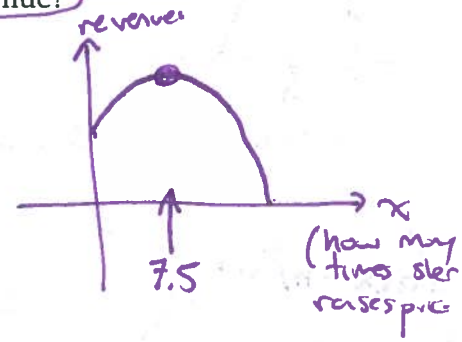
$$f(x) = (\text{price}) \cdot (\text{number sold})$$
$$f(x) = (900 + 20x)\cancel{(60-x)}(60-x)$$
$$f(x) = \cancel{5400 - 780x - 20x^2}$$
$$54000 + 300x - 20x^2.$$

How many times should she raise the price by \$20 to maximize revenue?

$$\text{revenue}(x) = 54000 + 300x - 20x^2$$

Find vertex. $h = \frac{-b}{2a} = \frac{-300}{2(-20)}$

$$= \frac{-300}{-40} = \frac{300}{40}$$
$$= \frac{30}{4} = \frac{15}{2} = 7.5$$



She should raise price 7.5 times..

7 times $\Longrightarrow$ + \$140

8 times $\Longrightarrow$ + \$160

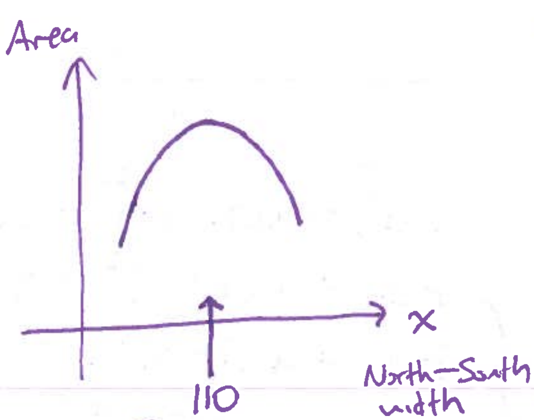
7.5 times $\Longrightarrow$ + \$150.

11. You will build a rectangular sheep enclosure next to a river. There is no need to build a fence along the river, so you only need to build on three sides. You have a total of 440 feet of fence to use. Find the dimensions of the pen such that you can enclose the maximum possible area. One approach is to let x represent the length of fencing that runs perpendicular to the river, and write a formula for a function of x that outputs the area of the enclosure. Then find its vertex and interpret it.

$\Longrightarrow$ Area is

$$A(x) = x(440 - 2x)$$
$$A(x) = 440x - 2x^2$$

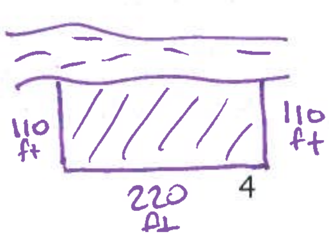
a quadratic function!



unknown vertex!

$$h = \frac{-b}{2a} = \frac{-440}{2(-2)}$$
$$= \frac{-440}{-4}$$
$$= 110.$$

So do like:



to maximize enclosed area.

Section 13.3 Completing the Square

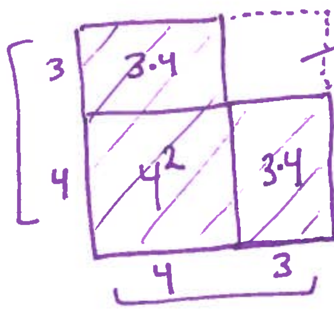
Ex Consider $\underbrace{16}_{\text{a perfect square: } 4^2} + 24 = 16 + 6 \cdot 4$

$= 16 + 2(3 \cdot 4)$

$= 4^2 + \underbrace{2}_{\text{copies}} \underbrace{(3 \cdot 4)}_{3\times 4 \text{ rectangle}}$

Now, 4 divides into 24 ...

visualize with squares and rectangles...



a missing little square... if we added this, we'd get one big square.

Start with $16 + 24$

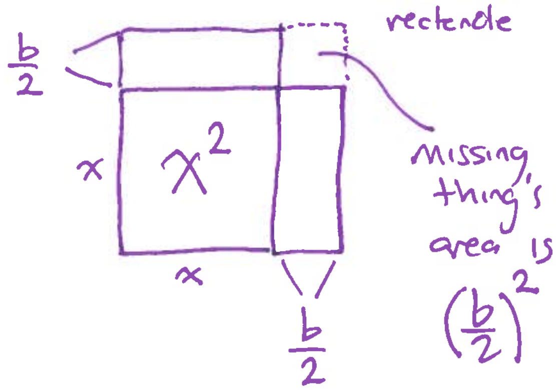
Add 9 to that: $\underbrace{16 + 24}_{\text{not a perfect square}} + 9 = 49$

$= 7^2$

add something to it that "completes the square"

Abstractly:

$$\underbrace{x^2}_{\text{a perfect square}} + b \cdot x = x^2 + \underbrace{2}_{\text{copies}} \underbrace{\left(\frac{b}{2}\right) x}_{\frac{b}{2} \text{ by } x \text{ rectangle}}$$



Start with $x^2 + bx$.

Find $\left(\frac{b}{2}\right)^2$. Add this, and you get a perfect square!

$$x^2 + bx + \left(\frac{b}{2}\right)^2$$

$$= \left(x + \frac{b}{2}\right)^2$$

Completing the Square

1. Use a square root to solve the equation.

a) $(x+5)^2 = 16$

$\sqrt{(x+5)^2} = \sqrt{16}$

$|x+5| = 4$

$x+5 = 4$ or $x+5 = -4$

$x = -1$ or $x = -9$

b) $(r+3)^2 = 13$

$r+3 = \sqrt{13}$ or $r+3 = -\sqrt{13}$

$r = -3+\sqrt{13}$ or $r = -3-\sqrt{13}$

2. Solve the equation. Try to notice that the left side is a perfect square trinomial, and use that to make the equation resemble those from the previous exercise.

a) [redacted] $+ 20x + 100 = 5$

x^2

recognize $(x+10)^2 = 5$

$x+10 = \sqrt{5}$ or $x+10 = -\sqrt{5}$

$x = -10+\sqrt{5}$ or $x = -10-\sqrt{5}$

b) $16t^2 - 24t + 9 = 64$

$(4t-3)^2 = 64$

$4t-3 = 8$ or $4t-3 = -8$

$4t = 11$ or $4t = -5$

$t = \frac{11}{4}$ $t = \frac{-5}{4}$

3. Solve the equation by completing the square.

a) $x^2 + 6x = 55$

could have its square completed.

$\left(\frac{b}{2}\right)^2 = \left(\frac{6}{2}\right)^2$

$= 3^2$

$= 9$

$x^2 + 6x + 9 = 55 + 9$

$(x+3)^2 = 64$

$x+3 = 8$ or $x+3 = -8$

$x = 5$ or $x = -11$

b) $t^2 + 4t = -1$

$\left(\frac{b}{2}\right)^2 = \left(\frac{4}{2}\right)^2$

$= 2^2$

$= 4$

$t^2 + 4t + 4 = 3$

$(t+2)^2 = 3$

$t+2 = \sqrt{3}$ or $t+2 = -\sqrt{3}$

$t = -2+\sqrt{3}$ or $t = -2-\sqrt{3}$

c) $y^2+10y+24=0$

-24

to complete the square...

$y^2+10y=-24$

$\left(\frac{10}{2}\right)^2 = 5^2 = 25$

$y^2+10y+25=1$

$(y+5)^2=1$

$y+5=1$ or $y+5=-1$

$y=-4$ or $y=-6$

d) $z^2+5z-1=0$

$z^2+5z=1$

$\left(\frac{b}{2}\right)^2=\left(\frac{5}{2}\right)^2 = \frac{25}{4} = 6.25$

$z^2+5z+\frac{25}{4}=1+\frac{25}{4}$

$\left(z+\frac{5}{2}\right)^2=\frac{29}{4}$

$z+\frac{5}{2}=\sqrt{\frac{29}{4}}$ or $z+\frac{5}{2}=-\sqrt{\frac{29}{4}}$

$z+\frac{5}{2}=\frac{\sqrt{29}}{2}$ or $z+\frac{5}{2}=-\frac{\sqrt{29}}{2}$

$z=-\frac{5}{2}+\frac{\sqrt{29}}{2}$ or $z=-\frac{5}{2}-\frac{\sqrt{29}}{2}$

4. For each quadratic function given in standard form, give the formula for the function in vertex form. Then state the location of the vertex on the graph of that function.

ax^2+bx+c

$a(x-h)^2+k$

a) $f(x)=x^2+4x-3$

what would it take to complete the square?

$\left(\frac{4}{2}\right)^2=2^2=4$

$=x^2+4x+4-4-3$

$=(x+2)^2-7$

$=1\cdot(x+2)^2-7$

Used C.T.S. to get vertex form...

The vertex is at $(-2,-7)$.

b) $g(x)=4x^2-32x-2$

$=4(x^2-8x)-2$

$\left(\frac{-8}{2}\right)^2=(-4)^2=16$

$=4(x^2-8x+16-16)-2$

$=4((x-4)^2-16)-2$

$=4(x-4)^2-64-2$

$=4(x-4)^2-66$

And the vertex is at $(4,-66)$

5. For each quadratic function given in standard form, give the formula for the function in vertex form. Then state the domain and the range of that function.

a) $f(x)=x^2-14x+43$

$\left(\frac{b}{2}\right)^2=\left(\frac{-14}{2}\right)^2=(-7)^2=49$

$=x^2-14x+49-49+43$

$=(x-7)^2-6$

domain: $(-\infty,\infty)$

range:

$[\#,\infty)$ or $(-\infty,\#]$

upward, since $a=1$, is pos.

range is $[-6,\infty)$

b) $g(x)=-x^2-18x-80$

opens downward!

$=-(x^2+18x)-80$

$=-(x^2+18x+81-81)-80$

$=-((x+9)^2-81)-80$

$=-(x+9)^2+81-80$

$=-(x+9)^2+1$

domain $(-\infty,\infty)$ and range $(-\infty,1]$